Tôhoku Math. J.
42 (1990), 231–259

CHARACTERIZATIONS OF THE WELL POSED CAUCHY PROBLEM FOR A SYSTEM IN A COMPLEX DOMAIN

MASATAKE MIYAKE

(Received March 27, 1989, revised October 14, 1989)

1. Introduction. The purpose of this paper is to characterize the well posed Cauchy problem for a system of linear partial differential equations in a complex domain. In particular, one of our interests is to investigate matrices of linear partial differential operators normalizable in the time derivative, and another one is to show the necessity of non characteristicness of the initial hyperplane for the well posed Cauchy problem. For that purpose the determinant theory for matrices of linear partial differential operators due to Sato and Kashiwara plays an important role. In particular, characterizations of invertible matrices, which will be give in Section 2, play the most crucial role.

For a (non commutative) unitary ring R, we denote by $M_N(R)$ the set of $N\times N$ matrices with entries in R, and by $GL_N(R)$ the set of invertible matrices in $M_N(R)$.

Let $x=(x_1, \cdots, x_n)=(x_1, x')$ be variables in the complex n-dimensional space $\boldsymbol{C}^n$, $D=(D_1, \cdots, D_n)=(D_1, D')$ be the usual symbol of differentiations, that is, $D_j=\partial/\partial x_j$ $(j=1, \cdots, n)$. Let Ω and p be a domain and a point in $\boldsymbol{C}^n$, respectively. Then we denote by $\mathscr{D}(\Omega)$ (resp. $\mathscr{D}_p$) the non commutative ring of linear partial differential operators with holomorphic coefficients in Ω (resp. at p).

Let $A(x, D)=(A_{ij})\in M_N(\mathscr{D}(\Omega))$ and $\mu=(\mu_1, \cdots, \mu_N)$ be an N-ple of non negative integers. We consider the following Cauchy problem $(A, \mu)_p$ at a point $p=(p_1, p')\in\Omega$:

$$(A, \mu)_p \qquad \begin{aligned} &\sum_{j=1}^{N} A_{ij}(x, D)u_j(x)=f_i(x)\in\mathscr{O}_p, \qquad 1\leq i\leq N, \\ &D_1^k u_j|_{x_1=p_1}=w_{jk}(x')\in\mathscr{O}_{p'}, \qquad 0\leq k<\mu_j, \quad 1\leq j\leq N, \end{aligned}$$

where $\mathscr{O}_p$ (resp. $\mathscr{O}_{p'}$) denotes the germ of holomorphic functions at p (resp. p').

The Cauchy problem $(A, \mu)_p$ is said to be *well posed* if it has a unique solution $\{u_j(x)\}\in\mathscr{O}_p^N$ for any $\{f_i(x)\}\in\mathscr{O}_p^N$ and $\{w_{jk}(x)\}\in\mathscr{O}_{p'}^{|\mu|}$, where $|\mu|=\mu_1+\cdots+\mu_N$.

The Cauchy problem (A, μ) is said to be *well posed in* Ω if $(A, \mu)_p$ is well posed at every point p in Ω.

The following fundamental theorem due to Wagschal motivates the research of this paper. The notions appearing in the theorem will be defined in Section 2 below.

THEOREM 0 [21, Th. 4. 1]. *Let $A(x, D)\in M_N(\mathscr{D}(\Omega))$ be a non degenerate matrix of total order m (≥ 0), with non characteristic initial hyperplane $x_1=p_1$ at $p=(p_1, p')\in\Omega$.*

Then there is at least one μ with $|\mu|=m$ such that the Cauchy problem $(A, \mu)_p$ is well posed.

Here arises a question whether we can give a relation between $A(x, D)$ and μ for the well posed Cauchy problem $(A, \mu)_p$. The purely combinatorial proof given in [21] does not seem to answer this question.

A matrix $A(x, D)\in M_N(\mathscr{D}(\Omega))$ is said to be *reducible to a μ-normal matrix with respect to D_1 in Ω* (resp. *at p*) if there is $P(x, D)\in GL_N(\mathscr{D}(\Omega))$ (resp. $\in GL_N(\mathscr{D}_p)$) such that PA is of μ-normal type with respect to D_1, that is,

$$PA=(D_1^{\mu_j}\delta_{ij}+b_{ij}(x, D)), \qquad \operatorname{order}_{D_1} b_{ij}<\mu_j \quad (1\leq i, j\leq N),$$

where δ_{ij} is Kronecker's delta and $\operatorname{order}_{D_1} b_{ij}$ denotes the order of b_{ij} with respect to D_1.

In this terminology we first have the following theorem.

THEOREM 1. *Let $A(x, D)$ be as in Theorem 0. Then the Cauchy problem (A, μ) is well posed in a neighbourhood of p if and only if $|\mu|=m$ and $A(x, D)$ is reducible in a unique way to a μ-normal matrix with respect to D_1 at p.*

Wagschal constructed such μ that the Cuachy problem (A, μ) is well posed in a neighbourhood of p, and hence Theorem 1 can be applied to his case. However, it must be stressed that the well posedness of $(A, \mu)_p$ does not imply the well posedness of (A, μ) in a neighbourhood of p in general. Such examples will be given in Example 4.3. We shall meet there an example of the well posed Cauchy problem $(A, \mu)_p$ such that it has infinitely many formal power series solutions.

In the above theorem the non characteristicness of the initial hyperplane was assumed apriori, but the following theorem guarantees its necessity for the well posed Cauchy problem in general.

THEOREM 2. *Let $A(x, D)\in M_N(\mathscr{D}(\Omega))$ be a non degenerate matrix of total order m (≥ 0). Then the Cauchy problem (A, μ) with $|\mu|=m$ is well posed in a neighbourhood of $p=(p_1, p')$ only if the initial hyperplane $x_1=p_1$ is not characteristic for $A(x, D)$ at p.*

Although the assumption $|\mu|=m$ seems to be excessive, the author does not know whether we can remove it even for the case $N=1$ (single equation).

Next, we give a characterization of reducible matrices for matrices in $M_N(\mathscr{D}(\Omega))$.

THEOREM 3. *Let $n=2$. Then $A(x, D)\in M_N(\mathscr{D}(\Omega))$ is reducible to a μ-normal matrix with respect to D_1 in Ω if and only if the Cauchy problem $(A, \mu)_p$ has a unique formal solution*

$$\text{(F)} \qquad u_j(x)=\sum_{k=0}^{\infty} u_{jk}(x_2)(x_1-p_1)^k/k!, \qquad u_{jk}(x_2)\in \mathscr{O}_{p_2},$$

($j=1, 2, \cdots, N$) at every point $p=(p_1, p_2)\in\Omega$.

Finally, combining the above theorem with a result by the author [12, Th. 2], we obtain the following:

THEOREM 4. *Assume $n=2$ and that the Cauchy problem $(A, \mu)_p$ has a unique formal solution* (F) *at every point $p \in \Omega$. Then the Cauchy problem (A, μ) is well posed in Ω if and only if $|\mu| = \operatorname{order}_\sigma A$ and $x_1 = p_1$ is not characteristic for $A(x, D)$ at very point $p=(p_1, p_2) \in \Omega$, that is, $(\det_\sigma A)(x, (1, 0)) \neq 0$ in Ω.*

In the above theorem $(\det_\sigma A)(x, \xi)$ $(\in \mathcal{O}(T^*\Omega))$ denotes the determinant of $A(x, D)$ in the sense of Sato and Kashiwara [18], and $\operatorname{order}_\sigma A$ denotes the degree of homogeneous polynomial $\det_\sigma A$ in the fibre variables $\xi \in \boldsymbol{C}^2$, where $T^*\Omega$ is the cotangent space of Ω.

Throughout this paper the determinant theory for matrices of linear partial differential operators plays an essential role, so we give a brief summary of them in Section 2.

In the case $n=2$, we easily obtain results corresponding to Theorems 1 and 2 when the matrix $A(x, D)$ is not assumed to be non degenerate (see Remark 7.3).

We now briefly review relevant results. After the work of Wagschal, the author [13], [14] and Adjamagbo [1] studied the case of ordinary differential equations, and gave a proto-type of the results in this paper. As concerns the partial differential equations, Kitagawa and Sadamatsu [10] developed Wagschal's results and showed a principle of reduction to a normal matrix, which will be used in this paper (see Section 3). After that Sadamatsu [16] studied the case of constant coefficients. Sadamatsu [17] studied the case of variable coefficients, and proved a result similar to Theorem 1 under more restrictive situations. He constructed there an inverse matrix for an invertible matrix in the case $n=2$. His idea will be developed in Sections 5 and 6 for the proof of Theorem 3. Another useful result for our purpose is a characterization of invertible matrices due to Adjamagbo [2] and Andronikov [5] (see Proposition 2.2).

Concerning Theorem 4, Mizohata [15] posed the problem of characterizing Kowalevskian system from the viewpoint of the Cauchy-Kowalevski theorem. In other words, it is the problem of proving the necessity of non characteristicness of the initial hyperplane for the well posed Cauchy problem. It is a fundamental and very important problem, but few results are known on this subject. For a matrix A of μ-normal type with respect to D_1, Mizohata gave a necessary condition for the Cauchy porblem (A, μ) to be well posed in Ω, while the author [12] gave a necessary and sufficient condition in the case $n=2$ (see Theorem 7.2). Theorems 3 and 4 give a complete extension of the author's result in the case where a matrix A is not given as a μ-normal matrix with respect to D_1. These theorems make clear that the notion of characteristics for matrices of partial differential operators must be understood by the determinant of Sato and Kashiwara.

The difficulty of extending Theorems 3 and 4 to general dimension n lies in the treatment of degenerate matrices (see Lemma 5.5, Lemma 6.2 and Theorem 7.2). Our main idea for $n=2$ is to reduce degenerate matrices to non degenerate ones in the category of partial differential operators with meromorphic coefficients, but it is not available in general dimension n.

The author would like to express his thanks to the referees for their careful reading of the manuscript and useful advices.

2. Determinant theory. We give in this section a brief summary of the determinant theory of matrices of linear partial differential operators due to Sato and Kashiwara [18] (see also Hufford [7]), and give characterizations of invertible matrices.

Let $A(x, D)=(A_{ij})\in M_N(\mathscr{D}(\Omega))$ and put

$$m_{ij}=\operatorname{order}_D A_{ij}(x, D)\ (:=\deg_\xi A_{ij}(x, \xi))\,, \tag{2.1}$$

where we define $\operatorname{order}_D A_{ij}=-\infty$ if $A_{ij}\equiv 0$. Then *the total order of* A is defined to be

$$\operatorname{order}_D A=\max_{\sigma\in\mathfrak{S}_N}\sum_{i=1}^{N} m_{i\sigma(i)}\in \boldsymbol{N}\cup\{-\infty\}\,. \tag{2.2}$$

Here $\boldsymbol{N}=\{0, 1, 2, \cdots\}$, $\mathfrak{S}_N$ denotes the permutation group of $\{1, 2, \cdots, N\}$ and we define $l+(-\infty)=-\infty$ for any $l\in\boldsymbol{N}\cup\{-\infty\}$.

A matrix $A(x, D)$ of total order m (≥ 0) is said to be *non degenerate* if

$$m=\deg_\xi(\det A(x, \xi))\quad (\xi\in \boldsymbol{C}^n)\,. \tag{2.3}$$

In this case, *the characteristic polynomial* $a(x, \xi)$ of $A(x, D)$ is defined to be

$$\text{the homogeneous part } a(x, \xi) \text{ of degree } m \text{ of } \det A(x, \xi) \text{ in } \xi\,. \tag{2.4}$$

A hypersurface $S\colon \varphi(x)=0$ (with $\operatorname{grad}\varphi\neq 0$ on S) is said to be *non characteristic for* $A(x, D)$ if

$$a(x, \operatorname{grad}\varphi)\neq 0 \qquad \text{on} \quad S\,. \tag{2.5}$$

The determinant of matrices in the sense of Sato and Kashiwara was defined as an extension of the characteristic polynomial. Let $A(x, D)\in M_N(\mathscr{D}(\Omega))$ and denote by $(\det_\sigma A)(x, \xi)$ the determinant of $A(x, D)$ in the sense of Sato and Kashiwara, which is holomorphic in the cotangent space $T^*\Omega$ of Ω, and is a homogeneous polynomial in the fibre variables $\xi\in\boldsymbol{C}^n$.

THEOREM 2.1 (cf. [18]). *The determinant above is well defined so that it has the following properties*:

(i) $\det_\sigma(AB)=\det_\sigma A\cdot\det_\sigma B$.

(ii) *If* $A(x, D)$ *is non degenerate, then* $\det_\sigma A$ *coincides with* $a(x, \xi)$ *in* (2.4).

(iii) $A(x, D)\in GL_N(\mathscr{D}(\Omega))$ *if and only if* $\det_\sigma A\equiv a(x)\neq 0$ *in* Ω.

(iv) *If* $P(x, D)\in\mathscr{D}(\Omega)$ *commutes with* $A(x, D)$, *then* $\{\sigma(P), \det_\sigma A\}=0$, *where* $\sigma(P)$ *denotes the principal symbol of* $P(x, D)$ *and* $\{\cdot, \cdot\}$ *denotes the Poisson bracket.*

The detailed proofs were given by Andronikov [3].

We define $\operatorname{order}_\sigma A$ by

(2.6) $$\operatorname{order}_\sigma A = \deg_\xi(\det_\sigma A)(x, \xi)$$

and call it *the order of* $A(x, D)$.

As an immediate consequence of this theorem, we can prove that $A(x, D) \in GL_N(\mathscr{D}(\Omega))$ if A has a left or right inverse in $M_N(\mathscr{D}(\Omega))$. Indeed, suppose that A has a left inverse $B \in M_N(\mathscr{D}(\Omega))$, that is, $BA = I_N$. Then we have $\det_\sigma A \cdot \det_\sigma B = 1$. The determinants are holomorphic in $T^*\Omega$ and are homogeneous polynomials in $\xi \in \boldsymbol{C}^n$. Therefore, $\det_\sigma A \equiv a(x) \neq 0$ in Ω.

The next proposition due to Andronikov (see also Adjamagbo [2] and Sadamatsu [17]) gives a characterization of invertible matrices from the viewpoint of the unique solvability of the equation.

PROPOSITION 2.2 (cf. [5]). *Let $P(x, D)$ be an $N \times N'$ matrix of linear partial differential operators with holomorphic coefficients in Ω. If the mapping*

$$\mathscr{O}_p^{N'} \ni u(x) \longmapsto P(x, D)u(x) \in \mathscr{O}_p^N$$

is bijective at every point $p \in \Omega$, then $N = N'$ and $P(x, D) \in GL_N(\mathscr{D}(\Omega))$. The converse is obvious.

The next proposition gives a characterization of invertible matrices containing a variable x_1 as a parameter.

PROPOSITION 2.3. *Let $\Omega = \Omega_1 \times \Omega'$ with $\Omega_1 \subset \boldsymbol{C}^1$ and $\Omega' \subset \boldsymbol{C}_{x'}^{n-1}$. Then $A(x, D') \in GL_N(\mathscr{D}(\Omega))$ if and only if $A(p_1, x', D') \in GL_N(\mathscr{D}(\Omega'))$ for any fixed $p_1 \in \Omega_1$. Moreover, the inverse matrix is obtained in the form $A^{-1} = B(x, D')$.*

This is an immediate consequence of the following lemma, which will be proved in the Appendix.

Lemma 2.4. *Let Ω be as in the above proposition. Let $A(x, D') \in M_N(\mathscr{D}(\Omega))$ and put $\det_\sigma A = a(x, \xi')$ (see Th. 2.1, (iv)).*

(i) *If $a(x, \xi') \equiv 0$, then $\det_{\sigma'} A(p_1, x', D') \equiv 0$ for any $p_1 \in \Omega_1$, where $\det_{\sigma'}$ denotes the determinant for matrices in $M_N(\mathscr{D}(\Omega'))$.*

(ii) *If $a(p_1, x', \xi') \not\equiv 0$, then $\det_{\sigma'} A(p_1, x', D') = a(p_1, x', \xi')$.*

(iii) *If $a(x, \xi') \not\equiv 0$ and $a(p_1, x', \xi') \equiv 0$, then $\operatorname{order}_\sigma A > \operatorname{order}_{\sigma'} A(p_1, x', D')$.*

PROOF OF PROPOSITION 2.3. The necessity follows from Theorem 2.1, (iii) and Lemma 2.4, (ii). Let us prove the sufficiency. Assume that $A(p_1, x', D') \in GL_N(\mathscr{D}(\Omega'))$ for any $p_1 \in \Omega_1$, that is, $\det_{\sigma'} A(p_1, x', D') \equiv a_{p_1}(x') \neq 0$ in Ω'. This assumption and Lemma 2.4, (i) imply $\det_\sigma A \not\equiv 0$, hence $\det_\sigma A \equiv a(x) \not\equiv 0$ by Lemma 2.4, (ii). For any fixed $p_1 \in \Omega_1$, we have $a(p_1, x') \not\equiv 0$. Indeed, if $a(p_1, x') \equiv 0$ for some $p_1 \in \Omega_1$, then Lemma 2.4, (iii) shows that

$$0 = \operatorname{order}_\sigma A > \operatorname{order}_{\sigma'} A(p_1, x', D'),$$

that is, $\det_\sigma A(p_1, x', D') \equiv 0$, which contradicts the assumption. Hence, again, by Lemma 2.4, (ii) we have $a(p_1, x') \equiv a_{p_1}(x') \neq 0$ in Ω' for any $p_1 \in \Omega_1$. Thus we have $A(x, D') \in GL_N(\mathscr{D}(\Omega))$. Next, assume $A^{-1} = \sum_{j=0}^{k} B_j(x, D')D_1^j$. Then the identity $AA^{-1} = I_N$ implies $AB_0 = I_N$ and $AB_j = 0$ $(j = 1, 2, \cdots, k)$. Therefore, $B_j = A^{-1}(AB_j) = 0$ $(j = 1, 2, \cdots, k)$. q.e.d.

The determinant theory relies essentially on the Ore property of $\mathscr{D}_p$, which is also important in the following sections.

PROPOSITION 2.5 (Ore property). *For any non zero two elements $P(x, D)$ and $Q(x, D)$ in $\mathscr{D}_p$, we have*

$$\mathscr{D}_p P \cap \mathscr{D}_p Q \neq \{0\} \quad (\textit{resp. } P\mathscr{D}_p \cap Q\mathscr{D}_p \neq \{0\}),$$

that is, $P(x, D)$ and $Q(x, D)$ have non zero common left (resp. right) multiples.

Kashiwara [8] proved that $\mathscr{D}_p$ is a Noetherian ring without zero divisors, that is, left or right ideals of $\mathscr{D}_p$ satisfy the ascending chain condition. Therefore the above proposition can be proved algebraically (Goldie [6, Th. 1]) or by the method of algebraic analysis (see Andronikov [3] or Schapira [19, Remark 1.3.8, p. 65]).

As an immediate consequence of the Ore property of $\mathscr{D}_p$, we have $\operatorname{order}_\sigma A \geq 0$ if the Cauchy problem $(A, \mu)_p$ is well posed. Indeed, if $\det_\sigma A \equiv 0$, then there exists a non zero left null vector $Q(x, D) = (Q_1, \cdots, Q_N) \in \mathscr{D}_p^N \setminus \{0\}$ of A, that is, $QA = 0$. Hence, the equation $Au = f \in \mathscr{O}_p^N$ has no solutions $u \in \mathscr{O}_p^N$ for such f that $Qf \neq 0$, a contradiction. Therefore, we always assume $\operatorname{order}_D A \geq \operatorname{order}_\sigma A \geq 0$ if the Cauchy problem $(A, \mu)_p$ is well posed.

3. Reduction to normal matrices. Let $A(x, D) = (A_{ij}) \in M_N(\mathscr{D}(\Omega))$ and put $l_{ij} = \operatorname{order}_{D_1} A_{ij}$. Then *the total order of A with respect to D_1* is defined to be

$$\operatorname{order}_{D_1} A = \max_{\sigma \in \mathfrak{S}_N} \sum_{i=1}^{N} l_{i\sigma(i)} \in \boldsymbol{N} \cup \{-\infty\}. \tag{3.1}$$

Suppose $\operatorname{order}_{D_1} A \geq 0$. Then by Volevič's lemma (cf. [12], [20]), there is a system of integers $\{s_i, t_j\}_{i,j=1,2,\cdots,N}$ satisfying

$$l_{ij} \leq t_j - s_i \quad \text{and} \quad \operatorname{order}_{D_1} A = |t| - |s|, \tag{3.2}$$

where $|t| = t_1 + t_2 + \cdots + t_N$, etc. Now we put

$$A_{ij}(x, D) = \sum_{k=0}^{t_j - s_i} a_{ijk}(x, D')D_1^{t_j - s_i - k}$$

and

$$A_0(x, D') = (a_{ij0}(x, D'))_{i,j=1,2,\cdots,N}. \tag{3.3}$$

The purpose of this section is to show the following:

THEOREM 3.1. *Assume* $A_0(x, D') \in GL_N(\mathscr{D}(\Omega))$. *Then at every point* $p \in \Omega$ *the Cauchy problem* $(A, \mu)_p$ *has a unique formal solution* (F) *with* x_2 *in Theorem* 3 *replaced by* x' *if and only if*

(i) $|\mu| = \operatorname{order}_{D_1} A(x, D)$,

(ii) $A(x, D)$ *is reducible in a unique way to a* μ-normal matrix with respect to D_1 *in* Ω.

Since the "if" part is obvious, we have only to prove the "only if" part. Let us fix the notation for the proof.

For an N-ple $\nu = (\nu_1, \cdots, \nu_N)$ of non negative integers, we define two matrices $\Delta^{(\nu)}$ and $\mathscr{L}^{(\nu)}$ by

$$\Delta^{(\nu)} = \operatorname{diag}\{D_1^{\nu_1}, \cdots, D_1^{\nu_N}\} \qquad (N \times N \text{ matrix}), \tag{3.4}$$

$$\mathscr{L}^{(\nu)} = {}^t\begin{pmatrix} 1 & D_1 \cdots D_1^{\nu_1 - 1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1\, D_1 \cdots D_1^{\nu_2 - 1} & 0 \cdots 0 \\ & & & & \\ 0 & \cdots & 0 & 1 & D_1 \cdots D_1^{\nu_N - 1} \end{pmatrix} \tag{3.5}$$

($|\nu| \times N$ matrix), where $\operatorname{diag}\{\alpha_1, \cdots, \alpha_N\}$ denotes the diagonal matrix with the j-th diagonal entry α_j, and ${}^t(\cdot)$ denotes the transposed matrix of $(\cdot)$.

For the Cauchy problem $(A, \mu)_p$, we may assume

$$t_j > \mu_j \quad \text{and} \quad s_i > 0 \qquad \text{for any} \quad i, j = 1, 2, \cdots, N, \tag{3.6}$$

since we can replace $\{s_i, t_j\}$ by $\{s_i + l, t_j + l\}$ ($l \in \boldsymbol{N}$). Applying $\mathscr{L}^{(s)}$ ($s = (s_1, \cdots, s_N)$) to the equation $Au = f$ from the left, we have an equation

$$\mathscr{L}^{(s)} A u = \mathscr{L}^{(s)} f .$$

By the above choice of $\{s_i, t_j\}$, we see that the left hand side of this equation depends only on

$$\mathscr{L}^{(t)} u \cong {}^t({}^t(\mathscr{L}^{(\mu)} u), {}^t(\Delta^{(\mu)} u), {}^t(\mathscr{L}^{(t-\mu-1)} \Delta^{(\mu+1)} u)) ,$$

where $\mu + 1 = (\mu_1 + 1, \cdots, \mu_N + 1)$, etc., and ${}^t(\cdots)$ denotes the transposed vector of $(\cdots)$. Therefore the above equation is rewritten as

$$\mathscr{A}(x, D') \begin{pmatrix} \Delta^{(\mu)} \\ \mathscr{L}^{(t-\mu-1)} \Delta^{(\mu+1)} \end{pmatrix} u + \mathscr{B}(x, D') \mathscr{L}^{(\mu)} u = \mathscr{L}^{(s)} f , \tag{3.7}$$

where $\mathscr{A}(x, D')$ is an $|s| \times (|t| - |\mu|)$ matrix and $\mathscr{B}(x, D')$ is an $|s| \times |\mu|$ matrix. Note that

$$\mathscr{L}^{(\mu)} u \big|_{x_1 = p_1} = (w_{jk}(x'))_{0 \le k < \mu_j,\, 1 \le j \le N}$$

is a known vector by the Cauchy data.

Next, applying $\Delta^{(s)}$ to the equation $Au=f$ from the left, we have

$$A_0(x, D')\Delta^{(t)}u+\mathscr{C}(x, D')\mathscr{L}^{(t)}u=\Delta^{(s)}f \tag{3.8}$$

in view of (3.3). Here $\mathscr{C}(x, D')$ is an $N\times|t|$ matrix. Let

$$u_j(x)=\sum_{k=0}^{\infty}u_{jk}(x')\frac{(x_1-p_1)^k}{k!} \qquad (u_{jk}(x')\in\mathscr{O}_{p'},\ 1\leq j\leq N) \tag{F}$$

be a formal solution of the Cauchy problem $(A, \mu)_p$. Then $\{u_{jk}(x');\ k\geq\mu_j,\ 1\leq j\leq N\}$ are determined by the equations (3.7) and (3.8).

LEMMA 3.2. *If the Cauchy problem $(A, \mu)_p$ has at least one (formal) solution, then the mapping*

$$\mathscr{A}(p_1, x', D'): \mathscr{O}_{p'}^{|t|-|\mu|}\longrightarrow\mathscr{O}_{p'}^{|s|} \tag{3.9}$$

defined by $\mathscr{O}_{p'}^{|t|-|\mu|}\ni U(x')\mapsto\mathscr{A}(p_1, x', D')U(x')\in\mathscr{O}_{p'}^{|s|}$ is surjective. More precisely, we have:

$$|s|\leq|t|-|\mu|, \qquad \textit{that is}, \quad |\mu|\leq\operatorname{order}_{D_1}A(x, D), \tag{3.10}$$

$$\operatorname{rank}\mathscr{A}(p_1, x', D')=|s|, \textit{ that is, } \mathscr{A}(p_1, x', D') \textit{ has at least one non-vanishing minor of degree } |s| \textit{ in the sense of Sato and Kashiwara.} \tag{3.11}$$

PROOF. Restricting the equation (3.7) to $x_1=p_1$, we see that

$$\mathscr{B}(p_1, x', D')\mathscr{L}^{(\mu)}u\big|_{x_1=p_1} \quad\text{and}\quad \mathscr{L}^{(s)}f\big|_{x_1=p_1}$$

are known vectors by the Cauchy data and f. It is obvious that the mapping $\mathscr{O}_p^N\ni f(x)\mapsto\mathscr{L}^{(s)}f\big|_{x_1=p_1}\in\mathscr{O}_{p'}^{|s|}$ is surjective. Hence, the existence of a (formal) solution of $(A, \mu)_p$ implies the surjectivity of (3.9). Let assume the inequality $|s|>|t|-|\mu|$. Then by the Ore property of $\mathscr{D}_{p'}$ (cf. Proposition 2.5), there exists a left null vector

$$Q(x', D')=(Q_1, \cdots, Q_{|s|})\in\mathscr{D}_{p'}^{|s|}\setminus\{0\}$$

of $\mathscr{A}(p_1, x', D')$, i.e., $Q(x', D')\mathscr{A}(p_1, x', D')=0$. Hence, in this case the equation $\mathscr{A}(p_1, x', D')U(x')=F(x')\in\mathscr{O}_{p'}^{|s|}$ has no solutions for such $F(x')$ that $Q(x', D')F(x')\neq0$, a contradiction to the surjectivity of (3.9). Thus we have proved $|s|\leq|t|-|\mu|$. Next, we assume that every minor of degree $|s|$ of $\mathscr{A}(p_1, x', D')$ vanishes. Then again by the Ore property we see the existence of a left null vector of $\mathscr{A}(p_1, x', D')$ as above, which contradicts the surjectivity of (3.9). q.e.d.

The next proposition due to Kitagawa and Sadamatsu plays a crucial role in our proof.

PROPOSITION 3.3 (cf. [10, Prop. 4]). *We consider the Cauchy problem $(A, \mu)_p$ with $|\mu|=\operatorname{order}_{D_1}A(x, D)$. If $\mathscr{A}(x, D')\in GL_{|s|}(\mathscr{D}_p)$ and $A_0(x, D')\in GL_N(\mathscr{D}_p)$, then there exists $P(x, D)\in GL_N(\mathscr{D}_p)$ such that PA is of μ-normal type with respect to D_1. Moreover, the matrix $P(x, D)$ is given by*

$$P(x, D)=\mathscr{Q}\mathscr{A}^{-1}(x, D')\mathscr{L}^{(s)}, \tag{3.12}$$

where $\mathscr{Q}=[I_N \vdots 0]$ *is an* $N\times|s|$ *matrix.*

PROOF. We give here a proof in a form slightly different from that of [10], since it is most fundamental in this paper. Without loss of generality, we may assume $\mathscr{A}(x, D')\in GL_{|s|}(\mathscr{D}(\Omega))$ and $A_0(x, D')\in GL_N(\mathscr{D}(\Omega))$, and hence $\mathscr{A}(p_1, x', D')$ and $A_0(p_1, x', D')$ are invertible for any fixed p_1. These assumptions imply immediately the unique existence of the formal solution (F) of the Cauchy problem $(A, \mu)_p$ at every point $p\in\Omega$. Let $P(x, D)$ be the matrix given by (3.12). Then obviouly PA is of μ-normal type with respect to D_1, that is,

$$PA=\Delta^{(\mu)}+(C_{ij}(x, D)), \qquad \operatorname{order}_{D_1} C_{ij}(x, D)<\mu_j. \tag{3.13}$$

By the above observations, we see that the Cauchy problem for two equations

$$Au=f\in\mathscr{O}_p^N \quad \text{and} \quad PAu=Pf$$

with the same Cauchy data have the same (unique) formal solution (F). Note that the equation $\mathscr{L}^{(t-\mu)}PAu=\mathscr{L}^{(t-\mu)}Pf$ is rewritten in a form similar to (3.7),

$$\tilde{\mathscr{A}}(x, D')\begin{pmatrix}\Delta^{(\mu)}\\ \mathscr{L}^{(t-\mu-1)}\Delta^{(\mu+1)}\end{pmatrix}u+\tilde{\mathscr{B}}(x, D')\mathscr{L}^{(\mu)}u=\mathscr{L}^{(t-\mu)}Pf, \tag{3.14}$$

where $\tilde{\mathscr{A}}$ is a square matrix of size $|s|$ $(=|t|-|\mu|)$ and $\tilde{\mathscr{B}}$ is an $|s|\times|\mu|$ matrix. Since PA is of μ-normal type with respect to D_1, we see that the mapping

$$\tilde{\mathscr{A}}(p_1, x', D'): \mathscr{O}_{p'}^{|s|}\longrightarrow\mathscr{O}_{p'}^{|s|}$$

is injective at every point p' with $p=(p_1, p')$. The surjectivity of is obvious, and hence $\tilde{\mathscr{A}}(p_1, x', D')$ is invertible for any p_1 by Proposition 2.2. Therefore, $\tilde{\mathscr{A}}(x, D')\in GL_{|s|}(\mathscr{D}(\Omega))$ by Proposition 2.3. Now we have the following two equations

$$\begin{pmatrix}\Delta^{(\mu)}\\ \mathscr{L}^{(t-\mu-1)}\Delta^{(\mu+1)}\end{pmatrix}u+\mathscr{A}^{-1}\mathscr{B}\mathscr{L}^{(\mu)}u=\mathscr{A}^{-1}\mathscr{L}^{(s)}f, \tag{3.15}$$

$$\begin{pmatrix}\Delta^{(\mu)}\\ \mathscr{L}^{(t-\mu-1)}\Delta^{(\mu-1)}\end{pmatrix}u+\tilde{\mathscr{A}}^{-1}\tilde{\mathscr{B}}\mathscr{L}^{(\mu)}u=\tilde{\mathscr{A}}^{-1}\mathscr{L}^{(t-\mu)}Pf. \tag{3.16}$$

These equations determine the same coefficients $\{u_{jk}(x'); \mu_j\le k<t_j, 1\le j\le N\}$ in the formal solution (F). By choosing $f=0$, we have $(\mathscr{A}^{-1}\mathscr{B})|_{x_1=p_1}=(\tilde{\mathscr{A}}^{-1}\tilde{\mathscr{B}})|_{x_1=p_1}$ for any p_1, and hence $\mathscr{A}^{-1}\mathscr{B}=\tilde{\mathscr{A}}^{-1}\tilde{\mathscr{B}}$. Therefore,

$$(\mathscr{A}^{-1}\mathscr{L}^{(s)}f)|_{x_1=p_1}=(\tilde{\mathscr{A}}^{-1}\mathscr{L}^{(t-\mu)}Pf)|_{x_1=p_1}$$

for any $f\in\mathscr{O}_p^N$ and any p_1. This implies $\mathscr{A}^{-1}\mathscr{L}^{(s)}=\tilde{\mathscr{A}}^{-1}\mathscr{L}^{(t-\mu)}P$, i.e., $\mathscr{L}^{(s)}=\mathscr{A}\tilde{\mathscr{A}}^{-1}\mathscr{L}^{(t-\mu)}P$. We take an $N\times|s|$ matrix $\mathscr{R}$ such that

$$I_N = \mathscr{R}\mathscr{L}^{(s)} = \mathscr{R}\mathscr{A}\tilde{\mathscr{A}}^{-1}\mathscr{L}^{(t-\mu)}P\,.$$

This proves $P(x, D) \in GL_N(\mathscr{D}(\Omega))$, since $P(x, D)$ has a left inverse $\mathscr{R}\mathscr{A}\tilde{\mathscr{A}}^{-1}\mathscr{L}^{(t-\mu)}$ $\in M_N(\mathscr{D}(\Omega))$. q.e.d.

PROOF OF THEOREM 3.1. As mentioned at the beginning, we only prove the necessity. We assume $\Omega = \Omega_1 \times \Omega'$ ($\Omega_1 \subset \boldsymbol{C}_{x_1}$, $\Omega' \subset \boldsymbol{C}^{n-1}_{x'}$) without loss of generality. From the above proposition, it is sufficient to show that $|\mu| = \mathrm{order}_{D_1} A$ ($= |t| - |s|$) and $\mathscr{A}(x, D') \in GL_{|s|}(\mathscr{D}(\Omega))$. For that purpose it suffices to show the bijectivity of the mapping (3.9) for any fixed $p_1 \in \Omega_1$ by Propositions 2.2 and 2.3. We have only to prove the injectivity by Lemma 3.2.

Let $\{u_{jk}(x');\ \mu_j \leq k < t_j,\ 1 \leq j \leq N\}$ be the coefficients of the formal solution (F), which are determined by the equation (3.7). Then by induction on k, $\{u_{j,t_j+k}(x');\ 1 \leq j \leq N\}$ ($k = 0, 1, 2, \cdots$) are determined uniquely by the relations

$$D_1^k \{A_0(x, D')\Lambda^{(t)}u + \mathscr{C}(x, D')\mathscr{L}^{(t)}u\}\big|_{x_1=p_1} = D_1^k \Lambda^{(s)} f\big|_{x_1=p_1}\,,$$

since $A_0(x, D') \in GL_N(\mathscr{D}(\Omega))$ implies $A_0(p_1, x', D') \in GL_N(\mathscr{D}(\Omega'))$. Therefore the unique existence of the formal solution (F) assures the injectivity of the mapping (3.9).

The uniqueness of the reduction of $A(x, D)$ to a μ-normal matrix with respect to D_1 is proved by the same reasoning as that for $\mathscr{A}^{-1}\mathscr{L}^{(s)} = \tilde{\mathscr{A}}^{-1}\mathscr{L}^{(t-\mu)}P$ in the proof of the above proposition. q.e.d.

4. Proof of Theorems 1 and 2. Theorem 1 is an immediate consequence of the following lemma and Theorem 3.1, since $\mathrm{order}_{D_1} A = \mathrm{order}_D A$ in this case.

LEMMA 4.1. *Let $A(x, D)$ be as in Theorem* 1. *Then the Cauchy problem (A, μ) is well posed in a neighbourhood of p only if the Cauchy problem $(A, \mu)_q$ has a unique formal solution* (F) *at every point q in a neighbourhood of p. The converse is also true.*

PROOF. Let $A(x, D) = (A_{ij})$. We take a system of integers $\{s_i, t_j\}$ such that

$$\mathrm{order}_D A_{ij} \leq t_j - s_i \quad \text{and} \quad \mathrm{order}_D A = |t| - |s|\,. \tag{4.1}$$

We put

$$A_{ij}(x, D) = \sum_{k=0}^{t_j - s_i} a_{ijk}(x, D')D_1^{t_j - s_i - k}\,,$$

$$A_0(x) = (a_{ij0}(x))_{i,j=1,2,\cdots,N}\,,$$

where order $a_{ijk} \leq k$ by (4.1). Then the hyperplane $x_1 = p_1$ is not characteristic at $p = (p_1, p')$ for A if and only if $\det A_0(p) \neq 0$. Since $A_0(x)$ is invertible in a neighbourhood of p, $\mathrm{order}_{D_1} A = \mathrm{order}_D A$ in this case. By the above choice of $\{s_i, t_j\}$, the equation $\Lambda^{(s)}Au = \Lambda^{(s)} f$ ($s_i > 0$, $1 \leq i \leq N$) is rewritten in the form

$$A_0(x)\Delta^{(t)}u+(C_{ij}(x,D))u=\Delta^{(s)}f\,, \tag{4.2}$$

where $\operatorname{order}_D C_{ij}\leq t_j$ and $\operatorname{order}_{D_1} C_{ij}<t_j$. This is a usual system of Cauchy-Kowalevski type, since $A_0(x)$ is invertible in a neighbourhood of p. Every formal solution (F) of the Cauchy problem $(A,\mu)_q$ is also a formal solution of (4.2). Hence it always converges when q varies in a neighbourhood of p. q.e.d.

PROOF OF THEOREM 2. We divide the proof into Steps (1) through (6) under the assumption that the Cauchy problem (A,μ) is well posed in Ω.

(1) $|\mu|=\operatorname{order}_{D_1}A(x,D)$.

Indeed, Lemma 3.2 asserts $|\mu|\leq\operatorname{order}_{D_1}A$. Therefore, the assumption $|\mu|=\operatorname{order}_D A$ $(\geq\operatorname{order}_{D_1}A)$ proves the equality.

Hence the matrix $\mathscr{A}(x,D')$ defined by (3.7) is a square matrix of size $|s|$ $(=|t|-|\mu|)$. By (1), we can take a system of integers $\{s_i,t_j\}$ as in (4.1). Then we have

$$\begin{cases} A_0\equiv A_0(x)\,, \\ \det_{\sigma'}\mathscr{A}(p_1,x',D')\not\equiv 0 \qquad \text{for any} \quad p_1\in\Omega_1\,, \end{cases} \tag{4.3}$$

by Lemma 3.2, where it is assumed that $\Omega=\Omega_1\times\Omega'$.

(2) $\det A_0(x)\not\equiv 0$ in Ω.

Indeed, if $\det A_0(x)\equiv 0$, then we can take a non zero left null vector $q(x)=(q_1,\cdots,q_N)\in\mathscr{O}_p^N\setminus\{0\}$ of $A_0(x)$, that is, $q(x)A_0(x)\equiv 0$. Without loss of generality, we may assume that $s_1\leq s_2\leq\cdots\leq s_N$ and $q_1(x)\not\equiv 0$. We define a matrix $Q(x,D_1)$ by

$$Q(x,D_1)=\begin{pmatrix} q_1 & q_2D_1^{s_2-s_1} & \cdots\cdots & q_ND_1^{s_N-s_1} \\ 0 & 1 & 0\cdots\cdots 0 & \\ 0 & 0 & 1\quad 0\cdots 0 & \\ & & \ddots\quad\ddots & \\ 0 & \cdots\cdots\cdots & 0 \quad 1 & \end{pmatrix}.$$

Then $Q(x,D_1)$ is invertible at a point where $q_1\neq 0$. By this construction of $Q(x,D_1)$, we easily see that

$$\operatorname{order}_{D_1}QA<\operatorname{order}_{D_1}A=|\mu|\,.$$

Since the Cauchy problem $(QA,\mu)_r$ is well posed at a point r where $Q\in GL_N(\mathscr{D}_r)$, the above inequality is a contradiction.

(3) $\det_{\sigma'}\mathscr{A}(p_1,x',D')\equiv\alpha_{p_1}(x')$ $(\not\equiv 0)$ for such p_1 that $\det A_0(p)\neq 0$, where $p=(p_1,p')$.

Indeed, since the hyperplane $x_1=p_1$ is not characteristic for $A(x,D)$ at such a point p and the Cauchy problem (A,μ) is well posed in Ω, the mapping

$$\mathscr{A}(p_1,x',D')\colon \mathscr{O}_{q'}^{|s|}\longrightarrow\mathscr{O}_{q'}^{|s|}$$

is bijective at every point q' in a neighbourhood of p'. Therefore by Proposition 2.2,

we see that $\det_{\sigma'}\mathscr{A}(p_1, x', D') \equiv \alpha_{p_1}(x')$ and it does not vanish in a neighbourhood of p'.

(4) $\det_{\sigma}\mathscr{A}(x, D') \equiv \alpha(x) \not\equiv 0$ and $\alpha(p_1, x') \not\equiv 0$ for any $p_1 \in \Omega_1$.

Indded, if $\det_{\sigma}\mathscr{A} \equiv 0$, then $\det_{\sigma'}\mathscr{A}(p_1, x', D') \equiv 0$ for any $p_1 \in \Omega_1$ by Lemma 2.4, (i), which contradicts (4.3). Moreover, Lemma 2.4, (ii) and 3) imply that $\det_{\sigma}\mathscr{A} \equiv \alpha(x) \not\equiv 0$. Next, by (4.3) and Lemma 2.4, (iii) we get $\alpha(p_1, x') \not\equiv 0$ for any $p_1 \in \Omega_1$. Indeed, if $\alpha(p_1, x') \equiv 0$ for some p_1, then

$$0 = \operatorname{order}_{\sigma}\mathscr{A} > \operatorname{order}_{\sigma'}\mathscr{A}(p_1, x', D'),$$

that is, $\det_{\sigma'}\mathscr{A}(p_1, x', D') \equiv 0$, which contradicts (4.3).

(5) $\alpha(x) \neq 0$ in Ω.

Indeed, if $\alpha(p) = 0$ at some $p = (p_1, p')$, then $\mathscr{A}(p_1, x', D')$ has an inverse matrix $\mathscr{A}^{-1} = \mathscr{B}(x', D')$ with singular coefficients at p'. Therefore the mapping (3.9) is not surjective. Indeed, the equation $\mathscr{A}(p_1, x', D')U(x') = F(x') \in \mathscr{O}_{p'}^{|s|}$ has no solutions in $\mathscr{O}_{p'}^{|s|}$ for such $F(x')$ that $\mathscr{B}(x', D')F(x')$ is singular at p'.

(6) $\det A_0(x) \neq 0$ in Ω. Therefore the initial hyperplane $x_1 = p_1$ is not characteristic for $A(x, D)$ at every point $p = (p_1, p') \in \Omega$.

Indeed, note that 5) shows $\mathscr{A}(p_1, x', D') \in GL_{|s|}(\mathscr{D}(\Omega'))$ for any p_1, and hence the mapping (3.9) is bijective at every point $p' \in \Omega'$. Therefore the well posedness of the Cauchy problem (A, μ) in Ω implies that the mapping

$$A_0(p_1, x') : \mathscr{O}_{p'}^N \longrightarrow \mathscr{O}_{p'}^N$$

is surjective at every point $p' \in \Omega'$ for any $p_1 \in \Omega_1$, which implies immediately $\det A_0(x) \neq 0$ in Ω.

Thus we have proved Theorem 2.

The proof of Lemma 4.1 implies immediately the following:

PROPOSITION 4.2. *Let $A(x, D)$ be as in Theorem 1. Then the Cauchy problem $(A, \mu)_p$ is well posed if and only if the mapping* (3.9) *is bijective. In this case we have $|\mu| = m$.*

PROOF. We prove only the fact $|\mu| = m$. We already know the inequality $|\mu| \leq m$ by Lemma 3.2. If $|\mu| < m$, then $\mathscr{A}(p_1, x', D')$ is an $|s| \times (|t| - |\mu|)$ matrix with $|s| < |t| - |\mu|$. By the Ore property of $\mathscr{D}_{p'}$ there exists a right null vector $R(x', D') \in \mathscr{D}_{p'}^{|t|-|\mu|} \setminus \{0\}$ of $\mathscr{A}(p_1, x', D')$, that is, $\mathscr{A}(p_1, x', D')R = 0$. Hence by a choice of $f(x') \in \mathscr{O}_{p'}$ such that $Rf \neq 0$, we obtain a contradiction to the injectivity of the mapping (3.9). q.e.d.

As mentioned in the Introduction, we give here examples which show that the well posedness of $(A, \mu)_p$ does not imply the well posedness of (A, μ) in a neighbourhood of p.

EXAMPLE 4.3. Let $n = 2$ and consider

$$A(x, D) = \begin{pmatrix} D_1 & a(x, D_2) \\ b(x, D_2) & D_1 \end{pmatrix} \quad (\text{order } a + \text{order } b = 2), \tag{4.4}$$

where a and b are differential operators with holomorphic coefficients in a neighbourhood of the origin O of $\boldsymbol{C}^2$. Obviously, A is non degenerate of $\operatorname{order}_D A=2$ and $x_1=p_1$ is not characteristic for any p_1. We consider the Cauchy problem $(A, \mu_0)_p$ with $\mu_0=(2, 0)$. In this case, we can take $\{s_i, t_j\}$ in (3.2) by

$$s_1=2\,,\quad s_2=1\,,\quad t_1=3\,,\quad t_2=2\,.$$

Then we have

$$A_0(x, D_2)=\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}\in GL_2(\mathscr{D}_O)\,,\qquad \mathscr{A}(x, D_2)=\begin{pmatrix}0 & a & 0\\ 1 & [D_1, a] & a\\ 0 & 0 & 1\end{pmatrix},$$

where $[D_1, a]=D_1a-aD_1$ is the commutator of D_1 and a. It is easily seen that $\det_\sigma \mathscr{A}=-\sigma(a)(x, \xi_2)$, where $\sigma(a)$ denotes the principal symbol of a. Therefore, $\mathscr{A}\in GL_3(\mathscr{D}_O)$ if and only if $a\equiv a(x)\neq 0$ in a neighbourhood of the origin, and in this case the Cauchy problem (A, μ_0) is well posed in a neighbourhood of the origin. Let us consider the Cauchy problem $(A, \mu_0)_p$ at a fixed point $p=(p_1, p_2)$. Note that the bijectivity of the mapping (3.9) is equivalent to the bijectivity of the mapping

$$a(p_1, x_2, D_2)\colon \mathscr{O}_{p_2}\longrightarrow \mathscr{O}_{p_2}\,. \tag{4.5}$$

Hence, the Cauchy problem $(A, \mu_0)_p$ is well posed if and only if the mapping (4.5) is bijective by the above proposition. Now we consider the following three cases of operator a, in each of which the Cauchy problem $(A, \mu_0)_O$ is well posed.

(1) $a=x_1D_2+1$: In this case, the Cauchy problem $(A, \mu_0)_p$ is well posed if and only if $p_1=0$, and when $p_1\neq 0$ the Cauchy problem has infinitely many solutions. It is obvious.

(2) $a=x_2D_2+1$: In this case, $(A, \mu_0)_p$ is well posed if and only if $p_2=0$, and when $p_2\neq 0$ the Cauchy problem has infinitely many solutions. Note that the bijectivity of the mapping (4.5) at $p_2=0$ is obvious, since $x_2=0$ is a regular singular point of a as an ordinary differential operator.

(3) $a=x_2^2D_2^2-D_2+1$: In this case, the Cauchy problem $(A, \mu_0)_p$ is well posed if and only if $p_2=0$, and when $p_2\neq 0$ the Cauchy problem has infinitely many solutions. Since the operator a has an irregular singular point $x_2=0$, the situation is more complicated than Case 2. We denote by $\hat{\mathscr{O}}_O$ the set of formal power series of x_2. By Malgrange [11], for the mappings

$$a\colon \mathscr{O}_O\longrightarrow \mathscr{O}_O\,,\quad a\colon \hat{\mathscr{O}}_O\longrightarrow \hat{\mathscr{O}}_O\,,$$

have indices

$$\chi(a; \mathscr{O}_O):=\dim_{\boldsymbol{C}} \operatorname{Ker} a-\operatorname{codim}_{\boldsymbol{C}} \operatorname{Im} a=0\quad (\operatorname{Ker} a=\{0\})\,,$$
$$\chi(a; \hat{\mathscr{O}}_O)=1\quad (\operatorname{Ker} a\cong \boldsymbol{C})\,.$$

Therefore, they are surjective. These observations prove that the Cauchy problem $(A, \mu_0)_p$ with $p_2=0$ has infinitely many formal power series solution $u(x)=\sum_{k=0}^{\infty} u_k(x_2)(x_1-p_1)^k/k!$ with $u_k(x_2)\in\hat{\mathcal{O}}_O^2$, but has a unique holomorphic solution.

5. Equivalent extension of matrices. Let $\mathscr{D}_M(\Omega)$ denote the set of linear partial differential operators with meromorphic coefficients in Ω. The following proposition is useful in proving Theorem 3.

PROPOSITION 5.1. *Let $A(x, D)\in M_N(\mathscr{D}(\Omega))$ and assume that the Cauchy problem $(A, \mu)_p$ has a unique formal solution* (F) *at every point $p\in\Omega$. If $P(x, D)\in GL_N(\mathscr{D}_M(\Omega))$ and PA is of μ-normal type with respect to D_1, then $P(x, D)\in GL_N(\mathscr{D}(\Omega))$.*

PROOF. Let Z be the analytic set where the singularities of the coefficients of P are located. We first consider the case $PA\in M_N(\mathscr{D}_p)$ $(p\in Z)$. In this case, we choose $f(x)\in\mathcal{O}_p^N$ so that Pf is singular at p, or more precisely, so that Pf does not have formal series expansion as (F). Then the Cauchy problem $(A, \mu)_p$ for the equation $Au=f$ has no formal solutions (F), since $PA\in M_N(\mathscr{D}_p)$. Next, we consider the case where PA has singular coefficients at $p\in Z$. In this case, we consider the Cauchy problem $(A, \mu)_p$ for the homogeneous equation,

$$\begin{aligned} &Au=0\,, \\ &D_1^k u_j\big|_{x_1=p_1}=w_{jk}(x')\in\mathcal{O}_{p'} \quad (0\le k<\mu_j,\ 1\le j\le N)\,. \end{aligned} \tag{5.1}$$

Then the coefficients $\{u_{jk}(x');\ k\ge\mu_j,\ 1\le j\le N\}$ in the formal solution (F) are obtained uniquely from the equation $PAu=0$ by induction on k $(\ge\mu_j)$. Since PA is of μ-normal type with respect to D_1, we may assume PA to be written in the form

$$PA=D_1 I_m-B(x, D')\,, \qquad B(x, D')\in M_m(\mathscr{D}_M(\Omega))\,,$$

where $m=|\mu|$.

We consider the case where $Z\not\supset\{x_1=p_1\}$. We set

$$B(x, D')=\sum_{k=0}^{\infty}(x_1-p_1)^k B_k(x', D')$$

and put $k_0=\min\{k;\ B_k \text{ has singular coefficients at } p'\}$. Then by choosing the Cauchy datum $w_0(x')=(w_{jk}(x'))_{0\le k<\mu_j,\,1\le j\le N}$ so that $B_{k_0}(x', D')w_0(x')$ is singular at p', we see that the Cauchy problem $(A, \mu)_p$ has no formal solutions (F), a contradiction. Hence we may assume that $Z\subset\{x_1=p_1\}$ and that $B(x, D')$ has holomorphic coefficients at $x_1\neq p_1$ in a neighbourhood of p. Therefore we can write

$$B(x, D')=\sum_{k=-l}^{\infty}(x_1-p_1)^k B_k(x', D')$$

for some l (≥ 1) with $B_{-l}\neq 0$. Hence by choosing the Cauchy datum $w_0(x')$ so that

$B_{-l}w_0(x')\neq 0$, we see that the Cauchy problem $(A, \mu)_p$ has no formal solutions (F), a contradiction. Thus we have proved $P(x, D)\in M_N(\mathscr{D}(\Omega))\cap GL_N(\mathscr{D}_M(\Omega))$. In order to prove $P^{-1}\in M_N(\mathscr{D}(\Omega))$, we note that the Cauchy problem $(PA, \mu)_p$ has a unique formal solution (F) at every point $p\in\Omega$. Hence $P^{-1}\in M_N(\mathscr{D}(\Omega))$ in the same way as that for the first case. q.e.d.

DEFINITION 5.2. Let $B(x, D)\in M_l(\mathscr{D}_M(\Omega))$. Then a matrix $C(x, D)\in M_l(\mathscr{D}_M(\Omega))\cup M_{l+1}(\mathscr{D}_M(\Omega))$ is called an equivalent extension of $B(x, D)$ if with some $R\in SL_l(\mathscr{D}_M(\Omega))$

$$C(x, D)=R(x, D)B(x, D) \tag{5.2}$$

or

$$C(x, D)=\left(\begin{array}{c|c} RB & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ \hline * \cdots * & 1 \end{array}\right) \qquad (*\in\mathscr{D}_M(\Omega)) . \tag{5.3}$$

Moreover, $E(x, D)\in M_{l+k}(\mathscr{D}_M(\Omega))$ $(k\geq 0)$ is said to be equivalent to $B(x, D)$ if E is obtained by repeated use of equivalent extensions from B.

Here $SL_l(\mathscr{D}_M(\Omega))$ denotes the set of unimodular matrices, that is, it is the subset of $GL_l(\mathscr{D}_M(\Omega))$ which is generated by matrices of the form $I_l+a\Theta_{ij}$ $(i, j=1, \cdots, l,\ i\neq j,\ a\in\mathscr{D}_M(\Omega))$, where $\Theta_{ij}=(\delta_{ik}\cdot\delta_{rj})_{k,r=1,\cdots,l}$. It is obvious that $\det_\sigma B=\det_\sigma E$ for equivalent matrices B and E, since $\det_\sigma R=1$ for any $R\in SL_l(\mathscr{D}_M(\Omega))$.

Let $E(x, D)\in M_{N+l}(\mathscr{D}_M(\Omega))$ $(l\geq 0)$ be equivalent to $A(x, D)\in M_N(\mathscr{D}(\Omega))$. If the Cauchy problem $(A, \mu)_p$ is well posed or has a unique formal solution (F) at every point $p\in\Omega$, then the Cauchy problem $(E, \bar{\mu})_p$ is well posed or has a unique formal solution (F) at a generic point $p\in\Omega$, where $\bar{\mu}=(\mu_1, \cdots, \mu_N, 0, \cdots, 0)$ an $(N+l)$-ple of integers.

PROPOSITION 5.3. *Assume $n\leq 2$. Then for any matrix $A(x, D)\in M_N(\mathscr{D}_M(\Omega))$ with $\det_\sigma A\not\equiv 0$, there exists a non degenerate matrix $E(x, D)\in M_{N+l}(\mathscr{D}_M(\Omega))$ $(l\geq 0)$ equivalent to $A(x, D)$. In the case $n=1$, we can take $l=0$.*

PROPOSITION 5.4. *Assume $n\leq 2$. Then $A(x, D)\in GL_N(\mathscr{D}_M(\Omega))$ if and only if $\det_\sigma A\equiv a(x)\not\equiv 0$ in Ω.*

To prove above propositions, it is sufficient to consider the case of degenerate matix A, that is,

$$\operatorname{order}_D A(x, D)>\operatorname{order}_\sigma A(x, D) . \tag{5.4}$$

Let $A(x, D)=(A_{ij})$ and take a system of integers $\{s_i, t_j\}$ such that

$$\operatorname{order}_D A_{ij}\leq t_j-s_i , \qquad \operatorname{order}_D A=|t|-|s| . \tag{5.5}$$

Let $a_{ij0}(x, D)$ be the homogeneous part of order $t_j - s_i$ of A_{ij}, and put

$$A_0(x, D) = (a_{ij0}(x, D)) . \tag{5.6}$$

Then (5.4) is equivalent to

$$\det A_0(x, \xi) \equiv 0 . \tag{5.7}$$

The following lemma due to Sadamatsu is important in our argument.

LEMMA 5.5. (cf. [17, Prop. 2]). *Assume $n=2$. If the condition (5.7) is satisfied, then there is a matrix $B(x, D) \in M_{N+l}(\mathscr{D}_M(\Omega))$ $(l \geq 0)$ equivalent to $A(x, D)$ such that*

$$\operatorname{order}_\sigma A \leq \operatorname{order}_D B < \operatorname{order}_D A .$$

Proposition 5.3 is an immediate consequence of this lemma. Since the "only if" part of Proposition 5.4 is trivial, we have to prove the "if" part. We first note that the assumption $\det_\sigma A \equiv a(x) \not\equiv 0$ implies that A has an inverse matrix A^{-1} of partial differential operators with singular coefficients in Ω. Hence we have to show $A^{-1} \in M_N(\mathscr{D}_M(\Omega))$. For that purpose we use the following:

LEMMA 5.6. *Let $E(x, D) \in M_{N+l}(\mathscr{D}_M(\Omega))$ be equivalent to $A(x, D)$. If there exists $Q(x, D) \in GL_{N+l}(\mathscr{D}_M(\Omega))$ such that QE is of $\bar{\mu}$-normal type with respect to D_1, then there exists $P(x, D) \in GL_N(\mathscr{D}_M(\Omega))$ such that PA is of μ-normal type with respect to D_1.*

PROOF. It is sufficient to prove our lemma in the case where E is an equivalent extension of A. When $E = RA$ with $R \in SL_N(\mathscr{D}_M(\Omega))$, there is nothing to prove. Otherwise, since QE is of $(\mu_1, \cdots, \mu_N, 0)$-normal type with respect to D_1, it is written as

$$QE = \overset{N}{\overbrace{}} \left\{ \left(\begin{array}{c|c} Q_{11} & Q_{12} \\ \hline Q_{21} & Q_{22} \end{array} \right) \right. \left(\begin{array}{c|c} RA & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline * \cdots * & 1 \end{array} \right) = \left(\begin{array}{c|c} B & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline * \cdots * & 1 \end{array} \right) ,$$

where $* \in \mathscr{D}_M(\Omega)$, $R \in SL_N(\mathscr{D}_M(\Omega))$ and B is of μ-normal type with respect to D_1. By the above expression, we have $Q_{12} = {}^t(0, \cdots 0)$ and $Q_{22} = 1$. Therefore, $P = Q_{11} R \in GL_N(\mathscr{D}_M(\Omega))$ and $PA = B$. q.e.d.

THE PROOF OF PROPOSITION 5.4 CONTINUED. Let $B(x, D) = (B_{ij}) \in M_{N+l}(\mathscr{D}_M(\Omega))$ be a non degenerate matrix equivalent to $A(x, D)$. Hence $0 = \operatorname{order}_\sigma B = \operatorname{order}_D B = \operatorname{order}_\sigma A$. We may assume $B_{jj} \equiv B_{jj}(x) \not\equiv 0$ $(j = 1, 2, \cdots, N+l)$ by a suitable change of rows and columns, respectively. Therefore, for any $1 \leq i_1 < i_2 < \cdots < i_k \leq N+l$ $(1 \leq k \leq N+l)$ it holds that

$$\sum_{j=1}^{k} \operatorname{order}_D B_{i_j i_{\sigma(j)}} \leq 0 \qquad \text{for any} \quad \sigma \in \mathfrak{S}_k .$$

Then by Volevič's lemma (cf. [12], [20]), there exists a system of integers $\{t_j\}_{j=1}^{N+l}$ such that

$$\operatorname{order}_D B_{ij} \leq t_j - t_i \qquad (i, j = 1, 2, \cdots, N+l) .$$

Without loss of generality we may assume $t_1 \leq t_2 \leq \cdots \leq t_{N+l}$. Hence $B(x, D)$ is written in the form

$$B(x, D) = \operatorname{triang}\{C_{11}(x), \cdots, C_{rr}(x)\} ,$$

i.e., B is a blockwise triagular matrix with the j-th diagonal block $C_{jj}(x)$ of square matrix with entries of meromorphic functions in Ω. Since $\pm \det_\sigma A = \det_\sigma B = \prod_{j=1}^{r} \det C_{jj}(x) \not\equiv 0$, each $C_{jj}(x)$ has an inverse matrix $C_{jj}^{-1}(x)$ with meromorphic functions as entries. This implies $B(x, D) \in GL_{N+l}(\mathscr{D}_M(\Omega))$. Since $B^{-1}B = I_{N+l}$ is nothing but a $(0, \cdots, 0)$-normal matrix with respect to D_1, we have $A(x, D) \in GL_N(\mathscr{D}_M(\Omega))$ by Lemma 5.6. q.e.d.

6. Proof of Theorem 3. Throughout this section, it is always assumed that $n = 2$. By using the results in the previous section, we prove Theorem 3 by reducing the Cauchy problem $(A, \mu)_p$ to $(E, \bar{\mu})_p$ for a matrix $E(x, D) \in M_{N+l}(\mathscr{D}_M(\Omega))$ equivalent to $A(x, D)$ to which Theorem 3.1 or Proposition 3.3 is applicable.

Since the necessity of Theorem 3 is obvious, we prove the sufficiency. Hence, in what follows, we assume that the Cauchy problem $(A, \mu)_p$ has a unique formal solution (F) at every point $p \in \Omega$.

Recall the matrices $A_0(x, D_2)$ and $\mathscr{A}(x, D_2)$ defined by (3.3) and (3.7), respectively. By Lemma 3.2 we have

$$|\mu| \leq \operatorname{order}_{D_1} A(x, D) \qquad (= |t| - |s|) , \tag{6.1}$$

$$\operatorname{rank} \mathscr{A}(p_1, x_2, D_2) = |s| \qquad \text{for any} \quad p_1 \ (p = (p_1, p_2) \in \Omega) . \tag{6.2}$$

By the definition of $\operatorname{order}_{D_1} A$, it always holds that $\operatorname{order}_D A_0 \geq 0$.

LEMMA 6.1. *Assume that the Cauchy problem $(A, \mu)_p$ has a unique formal solution* (F) *at every point $p \in \Omega \setminus Z$, where Z is an analytic set. Then we have*:

(i) *If* $|\mu| = \operatorname{order}_{D_1} A$ *and* $\operatorname{order}_\sigma \mathscr{A} \geq 1$, *then* $\det_\sigma A_0 \equiv 0$.

(ii) *If* $|\mu| < \operatorname{order}_{D_1} A$, *then* $\det_\sigma A_0 \equiv 0$.

PROOF. (i) The condition $|\mu| = \operatorname{order}_{D_1} A$ implies that $\mathscr{A}$ is a square matrix of size $|s|$. Suppose $\det_\sigma A_0 \not\equiv 0$. We put $\det_\sigma \mathscr{A} = \alpha(x)\xi_2^m$ $(\alpha(x) \not\equiv 0, \ m \geq 1)$ and put $\det_\sigma A_0 = a(x)\xi_2^l$ $(a(x) \not\equiv 0, \ l \geq 0)$. Take a point $p = (p_1, p_2) \in \Omega \setminus Z$ such that $\alpha(p) \neq 0$ and $a(p) \neq 0$. Then by Lemma 2.4 we have $\det_{\sigma_2} \mathscr{A}(p_1, x_2, D_2) = \alpha(p_1, x_2)\xi_2^m$ and $\det_{\sigma_2} A_0(p_1, x_2, D_2) = a(p_1, x_2)\xi_2^l$. Therefore, by a result of the author [14, Th. 5], for the mappings

$$\mathscr{A}(p_1, x_2, D_2) : \mathscr{O}_{p_2}^{|t|-|\mu|} \longrightarrow \mathscr{O}_{p_2}^{|s|} \qquad (|t| - |\mu| = |s|) , \tag{6.3}$$

$$A_0(p_1, x_2, D_2) : \mathscr{O}_{p_2}^{N} \longrightarrow \mathscr{O}_{p_2}^{N} , \tag{6.4}$$

we have $\dim_{\mathcal{C}} \operatorname{Ker} \mathscr{A} = m$ (≥ 1), $\operatorname{codim}_{\mathcal{C}} \operatorname{Im} \mathscr{A} = 0$ and (6.4) is surjective. These imply that the Cauchy problem $(A, \mu)_p$ has infinitely many formal solution (F), a contradiction.

(ii) Suppose $\det_\sigma A_0 \not\equiv 0$. Hence we may assume that the mapping (6.4) is surjective at a generic point $p \in \Omega$. Note that $\mathscr{A}(x, D_2)$ is an $|s| \times (|t| - |\mu|)$ matrix with $|s| < |t| - |\mu|$. By the assumption (6.2) we can conclude that the mapping (6.3) is surjective and $\dim_{\mathcal{C}} \operatorname{Ker} \mathscr{A} = \infty$ at a generic point $p \in \Omega$. Indeed, put $\mathscr{A} = (\alpha_{ij}(x, D_2))$ and assume that

$$\det_{\sigma_2}(\alpha_{ij}(p_1, x_2, D_2))_{i,j=1,\cdots,|s|} \not\equiv 0$$

without loss of generality. Then by Proposition 5.3 there exists $P(x_2, D_2) \in GL_{|s|}(\mathscr{D}_M(\Omega_2))$ such that

$$P(x_2, D_2)(\alpha_{ij}(p_1, x_2, D_2))_{i,j=1,2,\cdots,|s|}$$

is a non degenerate matrix, where it is assumed that $\Omega = \Omega_1 \times \Omega_2$. This observation leads us to the assertion. Hence we obtain a contradiction as in (i). q.e.d.

LEMMA 6.2. *Assume* $\det_\sigma A_0 \equiv 0$ *and put* $\operatorname{order}_D A_0 = n \geq 0$. *Then there is a matrix* $E(x, D)$ *equivalent to* $A(x, D)$ *such that*:

(i) *If* $n > 0$, *then either* $\operatorname{order}_{D_1} E < \operatorname{order}_{D_1} A$ *or*
$\operatorname{order}_{D_1} E = \operatorname{order}_{D_1} A$ *and* $\operatorname{order}_D E_0 < n$
holds, where $E_0(x, D_2)$ *is defined similarly to* $A_0(x, D)$.

(ii) *If* $n = 0$, *then* $\operatorname{order}_{D_1} E < \operatorname{order}_{D_1} A$.

PROOF. (i) We put $n_{ij} = \operatorname{order}_D a_{ij0}(x, D_2)$, where $A_0(x, D_2) = (a_{ij0})$. We take a system of integers $\{p_i, q_j\}$ such that

$$n_{ij} \leq q_j - p_i \quad \text{and} \quad n = |q| - |p| . \tag{6.5}$$

We denote by $\mathring{a}_{ij0}(x, \xi_2)$ the principal symbol of a_{ij0} of order $q_j - p_i$, and put

$$\mathring{A}(x, D) = (\mathring{a}_{ij0}(x, D_2) D_1^{t_j - s_i}) .$$

The assumption $\det_\sigma A_0 \equiv 0$ implies

$$\det(\mathring{a}_{ij0}(x, \xi_2)) \equiv 0 \quad \text{and} \quad \det \mathring{A}(x, \xi) \equiv 0 .$$

Let $l(x, \xi_2) = (l_1, \cdots, l_N)$ be a non zero left null vector of $(\mathring{a}_{ij0}(x, \xi_2))$, where $l_i(x, \xi_2)$ are monomials in ξ_2 with meromorphic coefficients. We can take it so that at least one of l_i is a non zero function of x. By a suitable change of rows, we may assume $s_1 \geq s_2 \geq \cdots \geq s_N$. Then

$$r(x, \xi) = (l_1 \xi_1^{s_1 - s_N}, \cdots, l_{N-1} \xi_1^{s_{N-1} - s_N}, l_N)$$

is a left null vector of $\mathring{A}(x, \xi)$. We set

$$i_0 = \max\{i;\ l_i \equiv l_i(x) \not\equiv 0\} . \tag{6.6}$$

We may assume $l_{i_0} \equiv 1$, since we consider the problem in the category of $\mathscr{D}_M(\Omega)$.

We first consider the case where $i_0=\max\{i;\, l_i\not\equiv 0\}$. Let us define a lower triangular matrix $R(x,\xi)$, which is different from the identity matrix only the i_0-th row, by

$$R(x,\xi)=\ {}^{i_0}\!\rangle\begin{pmatrix} 1 & & & & & & & \\ 0 & 1 & & & & 0 & & \\ & & \ddots & & & & & \\ * & \cdots & \cdots & 1 & 0 & \cdots & 0 & \\ & & & & 1 & & 0 & \\ & & 0 & & & \ddots & & \\ & & & & & & 1 & \end{pmatrix}, \tag{6.7}$$

where the i_0-th row is given by

$$(*\cdots 1\ 0\cdots 0)=(l_1\xi_1^{s_1-s_{i_0}},\ \cdots,\ l_{i_0-1}\xi_1^{s_{i_0-1}-s_{i_0}},\ 1,\ 0,\ \cdots,\ 0)\,.$$

In our case, this is also a left null vector of $\mathring{A}(x,\xi)$. Since $R(x,D)\in SL_N(\mathscr{D}_M(\Omega))$ and $\operatorname{order}_{D_1}A>\operatorname{order}_{D_1}RA$, RA is the desired equivalent extension of A.

Next, we consider the other case, i.e., $i_0<\max\{i;\, l_i\not\equiv 0\}$. By the definitions of $r(x,\xi)$ and $R(x,\xi)$, we have

$$r(x,\xi)R^{-1}(x,\xi)=(0,\ \cdots,\ 0,\ \xi_1^{s_{i_0}-s_N},\ l_{i_0+1}\xi_1^{s_{i_0+1}-s_N},\ \cdots,\ l_N). \tag{6.8}$$

We put $R(x,\xi)\mathring{A}(x,\xi)=(\tilde{a}_{ij}(x,\xi))$. Then we have

$$\tilde{a}_{ij}=\mathring{a}_{ij0}(x,\xi_2)\xi_1^{t_j-s_i} \qquad (i\neq i_0)\,,$$

$$\tilde{a}_{i_0j}=b_{i_0j}(x,\xi_2)\xi_1^{t_j-s_{i_0}} \qquad (j=1,2,\cdots,N)\,,$$

where $\deg_{\xi_2}b_{i_0j}=q_j-p_{i_0}$ or $-\infty$, and $(b_{i_01},\cdots,b_{i_0N})\not\equiv 0$, because the vector $(*\cdots 1\ 0\cdots 0)$ in the definition of $R(x,\xi)$ is not a left null vector of $\mathring{A}(x,\xi)$. Note that $r(x,\xi)R^{-1}(x,\xi)$ is a left null vector of $R(x,\xi)\mathring{A}(x,\xi)$.

In the following, it suffices to consider the case where

$$\operatorname{order}_{D_1}R(x,D)A(x,D)=\operatorname{order}_{D_1}A(x,D) \quad \text{and}$$

$$\operatorname{order}_D(RA)_0(x,D_2)=\operatorname{order}_D A_0(x,D_2)\,.$$

Since $l_i(x,\xi_2)$ $(i>i_0)$ has the factor ξ_2, we see that $b_{i_0j}(x,\xi_2)$ $(j=1,\cdots,N)$ also has the factor ξ_2 (see (6.8)). Therefore, $\tilde{a}_{i_0j}$ is written as

$$\tilde{a}_{i_0j}=\xi_2 c_{i_0j}(x,\xi_2)\xi_1^{t_j-s_{i_0}} \qquad (j=1,\cdots,N)\,.$$

We define an equivalent extension $B(x,D)$ of $A(x,D)$ by

$$B(x,D)=\left(\begin{array}{c|c} & 0 \\ R(x,D)A(x,D) & \vdots \\ & 0 \\ \hline * \cdots\cdots * & 1 \end{array}\right) \qquad (R(x,D)\in SL_N(\mathscr{D}_M(\Omega)))\,,$$

where

$$(*\cdots*)=(c_{i_01}(x, D_2)D_1^{t_1-s_{i_0}}, \cdots, c_{i_0N}(x, D_2)D_1^{t_N-s_{i_0}}) .$$

Let

$$C(x, D)=\ {}_{i_0}\rangle \overbrace{\begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & 0 & & \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & -D_2 \\ & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & 0 & & & 1 & \end{pmatrix}}^{N+1} B(x, D) \in M_{N+1}(\mathscr{D}_M(\Omega)) .$$

Then it is easy to check that $\mathrm{order}_{D_1}C=\mathrm{order}_{D_1}A$. By taking a system of integers $\{s_i', t_j'\}_{i,j=1,\cdots,N+1}$ defined by

$$s_i'=s_i, \quad t_j'=t_j\ (i, j=1, \cdots, N, i\neq i_0), \quad s_{i_0}'=s_{i_0}+1 ,$$

$$s_{N+1}'=s_{i_0} \quad \text{and} \quad t_{N+1}'=s_{i_0}+1 ,$$

we easily see that $\mathrm{order}_D C_0(x, D_2)=\mathrm{order}_D A_0(x, D_2)$. More precisely, by taking a system of integers $\{p_i', q_j'\}$ defined by

$$p_i'=p_i, \quad q_j'=q_j \quad (i, j=1, \cdots, N, i\neq i_0), \quad p_{N+1}'=p_{i_0}+1 ,$$

$$q_{N+1}'=p_{i_0}'+1 , \quad \text{where } p_{i_0}' \text{ is given suitably,}$$

we see that

$$\mathring{C}(x, \xi)=\ {}_{i_0}\rangle \left(\begin{array}{ccc|c} & \mathring{a}_{ij0}(x, \xi_2)\xi_1^{t_j-s_i} & & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ 0 & \cdots\cdots\cdots\cdots & 0 & \begin{matrix}-\xi_2\\ 0\end{matrix} \\ & \mathring{a}_{ij0}(x, \xi_2)\xi_1^{t_j-s_i} & & \begin{matrix}\vdots\\ 0\end{matrix} \\ \hline * & \cdots\cdots\cdots\cdots & * & 0 \end{array}\right) ,$$

where

$$(*\cdots*)=(c_{i_01}(x, \xi_2)\xi_1^{t_1-s_{i_0}}, \cdots, c_{i_0N}(x, \xi_2)\xi_1^{t_N-s_{i_0}}) .$$

Therefore, $\det \mathring{C}(x, \xi)\equiv 0$ and the left null vector of $\mathring{C}(x, \xi)$ is given by

$$(0, \cdots, \overset{i}{\check{0}}, \frac{l_{i_0+1}}{\xi_2}\xi_1^{s_{i_0+1}-s_N}, \cdots, \frac{l_N}{\xi_2}, \xi_1^{s_{i_0}-s_N}) .$$

Hence, by repeating these operations, we finally obtain a matrix equivalent to $A(x, D)$ to which the first case is applicable.

(ii) It is obvious, since in this case $\operatorname{order}_{D_1} E = \operatorname{order}_{D_1} A$ and $\operatorname{order}_D E_0 < 0$ does not occur (see the remark stated before Lemma 6.1). q.e.d.

PROOF OF THEOREM 3. If the Cauchy problem $(A, \mu)_p$ has a unique formal solution (F) at every point $p \in \Omega$, then by repeated use of Lemmas 6.1 and 6.2, we finally obtain an equivalent matrix $E(x, D)$ to which Theorem 3.1 or Proposition 3.3 is applicable in the category of $\mathscr{D}_M(\Omega)$. Hence by the results in Section 5, we see that A is reducible to a μ-normal matrix in the category of $\mathscr{D}(\Omega)$, which proves the sufficiency. The necessity is obvious. q.e.d.

7. Proof of Theorem 4. Thruoghout this section, we assume $n=2$. By Theorem 3 it suffices to prove the theorem for a μ-normal matrix $A(x, D)$, that is,

$$\text{(7.1)} \qquad \begin{cases} A = (A_{ij}) = (D_1^{\mu_j}\delta_{ij} + a_{ij}(x, D)) , \\ a_{ij} = \sum_{k=1}^{\mu_j} a_{ijk}(x, D_2) D_1^{\mu_j - k} . \end{cases}$$

Without loss of generality, we may assume $\mu_j \geq 1$ $(j=1, \cdots, N)$ and introduce an unknown function $U(x)$ by

$$\text{(7.2)} \qquad U(x) = {}^t(u_1, \cdots, D_1^{\mu_1-1}u_1, \cdots, u_N, \cdots, D_1^{\mu_N-1}u_N) .$$

For simplicity, we restrict ourselves to the case $N=2$. It is obvious that the Cauchy problem (A, μ) is well posed in Ω if and only if the following Cauchy problem is well posed in Ω:

$$\text{(7.3)} \qquad \begin{aligned} &LU \equiv (D_1 I_m + B(x, D_2))U(x) = F(x) \in \mathcal{O}_p^m , \\ &U\big|_{x_1=p_1} = U_0(x) \in \mathcal{O}_{p_2}^m , \end{aligned}$$

where $m = |\mu|$ and

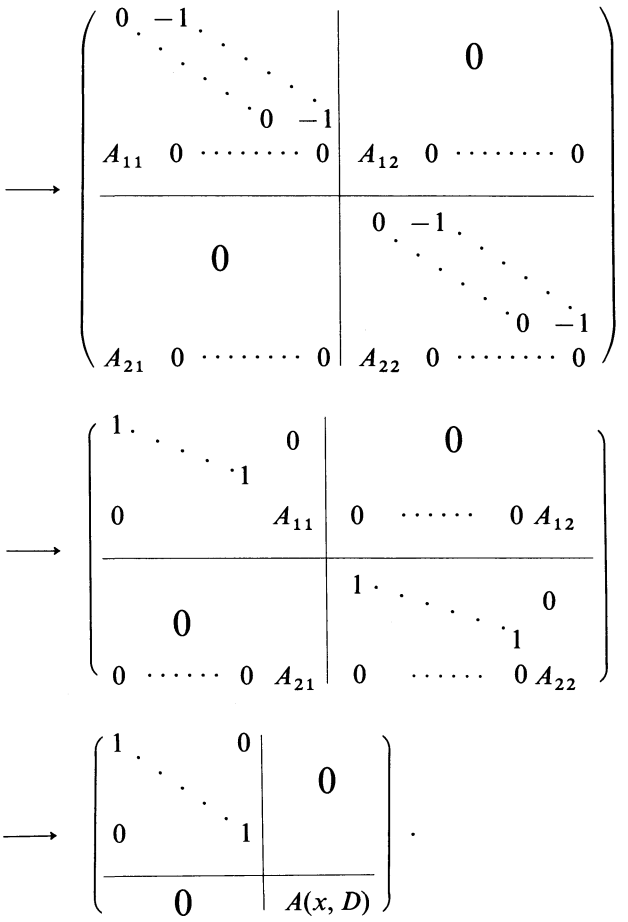
$$B(x, D_2)= \left(\begin{array}{c|c} \begin{matrix} 0 & -1 & & \\ & \ddots & \ddots & \\ & & 0 & -1 \\ a_{11\mu_1} & \cdots & & a_{111} \end{matrix} & \begin{matrix} & 0 & \\ & & \\ a_{12\mu_2} & \cdots & a_{121} \end{matrix} \\ \hline \begin{matrix} & 0 & \\ & & \\ a_{21\mu_1} & \cdots & a_{211} \end{matrix} & \begin{matrix} 0 & -1 & & \\ & \ddots & \ddots & \\ & & 0 & -1 \\ a_{22\mu_2} & \cdots & & a_{221} \end{matrix} \end{array}\right) \begin{matrix} \} \mu_1 \\ \} \mu_2 \end{matrix}$$

$\underbrace{\qquad}_{\mu_1}$ $\underbrace{\qquad}_{\mu_2}$

LEMMA 7.1. $\det_\sigma A = \det_\sigma L$.

PROOF. Note that the determinant is invariant by operating unimodular matrices ($\in SL_m(\mathscr{D}(\Omega))$) from the left or right to $L(x, D)$. We have the following sequence of matrices by operating unimodular matrices from the left or right:

$$L \longrightarrow \left(\begin{array}{c|c} \begin{matrix} 0 & -1 & & \\ & \ddots & \ddots & \\ & & 0 & -1 \\ A_{11} & * & \cdots & * \end{matrix} & \begin{matrix} & 0 & \\ & & \\ A_{12} & * & \cdots & * \end{matrix} \\ \hline \begin{matrix} & 0 & \\ & & \\ A_{21} & * & \cdots & * \end{matrix} & \begin{matrix} 0 & -1 & & \\ & \ddots & \ddots & \\ & & 0 & -1 \\ A_{22} & * & \cdots & * \end{matrix} \end{array}\right) \quad (* \in \mathscr{D}(\Omega))$$



q.e.d.

We now recall a terminology used in Mizohata [15] or [12]. We set $B(x, D_2)=(B_{ij})$. Then the matrix $L=D_1 I_m+B(x, D_2)$ is said to be *Kowalevskian in Volevič's sense* if there is a system of integers $\{t_j\}_{j=1}^m$ such that

$$\text{order}_D B_{ij} \leq t_j - t_i + 1 , \qquad i, j = 1, \cdots, m . \tag{7.4}$$

In this case the matrix $L(x, D)$ is non degenerate and the hyperplane $x_1=p_1$ is not characteristic for $L(x, D)$ at every point $p \in \Omega$. Precisely, $\det_\sigma L$ is equal to the homogeneous part of degree m in ξ of $\det L(x, \xi)$, and has the form

$$\det_\sigma L = \sum_{j=0}^{m} l_j(x, \xi_2)\xi_1^{m-j} \qquad \text{with} \quad l_0 \equiv 1 . \tag{7.5}$$

The following theorem by the author is fundamental.

THEOREM 7.2 (cf. [12, Th. 2]). *The Cauchy problem* (7.3) *is well posed in* Ω *if and only if there exists* $P(x, D_2) \in GL_N(\mathscr{D}_M(\Omega))$ *such that* $P^{-1}LP$ *is Kowalevskian in Volevič's sense.*

PROOF OF THEOREM 4. The "only if" part is obvious by the above theorem and Lemma 7.1. Indeed, in the above theorem $\det_\sigma L = \det_\sigma P^{-1}LP$. Let us prove the "if" part of the theorem. Let $E(x, D) \in M_{m+l}(\mathscr{D}_M(\Omega))$ be a non degenerate matrix equivalent to $A(x, D)$. Then the hyperplane $x_1 = p_1$ is not characteristic at a generic point $p = (p_1, p_2) \in \Omega$. At such a point, the unique existence of the formal solution (F) of the Cauchy problem $(A, \mu)_p$ implies the unique existence of the formal solution (F) of the Cauchy problem $(E, \tilde{\mu})_p$. Moreover, the formal solution (F) always converges by Lemma 4.1. Therefore the Cauchy problem $(A, \mu)_p$ is well posed at a generic point $p \in \Omega$. This observation concludes that the Cauchy problem (A, μ) is well posed in Ω (see the proof of Theorem 7.2 in [12, Section 3]). q.e.d.

AN ALTERNATIVE PROOF OF THE SUFFICIENCY. We first note that the proof below does not depend on the dimension n. We shall prove the well posedness of the Cauchy problem (7.3) instead of the Cauchy problem (A, μ). Let $\mathfrak{M}$ be a system in the sense of Kasiwara [8], [9] defined by

$$\mathfrak{M} = \mathscr{D}^m / \mathscr{D}^m L , \tag{7.6}$$

where $\mathscr{D}$ denote the sheaf of a linear partial differential opertors with holomorphic coefficients on Ω. We denote by $\mathrm{Ch}(\mathfrak{M})$ the characteristic variety of $\mathfrak{M}$, i.e.,

$$\mathrm{Ch}(\mathfrak{M}) = \{(x, \xi) \in T^*\Omega;\ (\det_\sigma L)(x, \xi) = 0\} ,$$

(see Andronikov [3], [4]). By the assumption of the sufficiency, the hyperplane $x_1 = p_1$ is not characteristic for $\mathfrak{M}$ at every point $p = (p_1, p') \in \Omega$. Let Y be a hyperplane in Ω defined by $x_1 = p_1$, and

$$f:\ Y \ni x' \mapsto (p_1, x') \in \Omega .$$

Then the system $\mathfrak{M}_Y$ on Y induced by $\mathfrak{M}$ is given by

$$\mathfrak{M}_Y := \mathscr{O}_Y \otimes_{f^{-1}\mathscr{O}} f^{-1}\mathfrak{M} \cong \bigoplus_{j=1}^{m} \mathscr{D}_Y(1 \otimes e_j) , \tag{7.7}$$

where $\mathscr{O}_Y$ (resp. $\mathscr{D}_Y$) denotes the sheaf of holomorphic functions (resp. linear partial differential operators with holomorphic conefficients) on Y, and $\{e_j = (0, \cdots, 0, \overset{j}{\check{1}}, 0, \cdots, 0)\}_{j=1}^m$ are the usual generator of $\mathfrak{M}$. Note that the last isomorphism follows from the fact that L is of $(1, \cdots, 1)$-normal matrix with respect to D_1.

The Cauchy-Kowalevski-Kashiwara theorem (cf. [8], [9], [19]) implies the following isomorphisms of sheaves:

$$\mathscr{H}om_{\mathscr{D}}(\mathfrak{M}, \mathcal{O})\big|_Y \cong \mathscr{H}om_{\mathscr{D}_Y}(\mathfrak{M}_Y, \mathcal{O}_Y) \cong \mathcal{O}_Y^m , \tag{7.8}$$

$$\mathscr{E}xt^j_{\mathscr{D}}(\mathfrak{M}, \mathcal{O})\big|_Y \cong \mathscr{E}xt^j_{\mathscr{D}_Y}(\mathfrak{M}_Y, \mathcal{O}_Y) = 0 \qquad (j \geq 1) , \tag{7.9}$$

(see [4]). $\mathscr{E}xt^1_{\mathscr{D}}(\mathfrak{M}, \mathcal{O})\big|_Y = 0$ means that the equation

$$L(x, D)U(x) = F(x) \in \mathcal{O}_p^m \quad (p \in Y)$$

has always solutions $U(x) \in \mathcal{O}_p^m$ for any $F(x)$, since $\det_\sigma L \not\equiv 0$, (see [4]). Since

$$\mathscr{H}om_{\mathscr{D}}(\mathfrak{M}, \mathcal{O}) \cong \{U(x) \in \mathcal{O}^m;\ LU = 0\} ,$$

the first isomorphism in (7.8) is given by

$$\{U(x) \in \mathcal{O}^m;\ LU = 0\} \ni U(x) \mapsto U(p_1, x') \in \mathcal{O}_Y^m$$

by (7.7), which gives a correspondence between the solution of the homogeneous equation and its Cauchy data. q.e.d.

REMARK 7.3. When $n=2$, the results corresponding to those in Theorems 1 and 2 hold without the non degeneracy assumption on the matrix $A(x, D)$.

(1) In Theorem 1, the assumptions imposed on $A(x, D)$ should be replaced by $(\det_\sigma A)(p, (1, 0)) \neq 0$ and $\operatorname{order}_\sigma A = m$.

(2) In Theorem 2, the assumption imposed on $A(x, D)$ should be replaced by $\operatorname{order}_\sigma A = m$ (≥ 0).

These can be proved in the same way as Theorems 3 and 4. We have only to note that Proposition 5.1 holds if we replace the assumption of the unique existence of formal solution (F) by the well posedness of the Cauchy problem (A, μ) in Ω, too.

Appendix. Proof of Lemma 2.4. In order to prove Lemma 2.4, we need to employ the determinant theory in the category of microdifferential operators as in [3], [4] and [18].

Let $\Omega = \Omega_1 \times \Omega'$ as in the lemma and let $U \subset T^*\Omega$ be an open set. Let $\mathscr{E}'(U)$ be the set of microdifferential operators not depending on D_1 defined in U. Then Lemma 2.4 is a special case of the following:

PROPOSITION A.1. *Let* $A(x, D') = (A_{ij}) \in M_N(\mathscr{E}'(U))$ *and put* $\det_\sigma A = a(x, \xi')$ $(\in \mathcal{O}(U))$. *Then we have*:

(i) *If* $a \equiv 0$, *then* $\det_{\sigma'} A(p_1, x', D') \equiv 0$ *for any* $p_1 \in \Omega_1$.

(ii) *If* $a(p_1, x', \xi') \not\equiv 0$, *then* $\det_{\sigma'} A(p_1, x', D') \equiv a(p_1, x', \xi')$.

(iii) *If* $a \not\equiv 0$ *and* $a(p_1, x', \xi') \equiv 0$, *then*

$$\operatorname{order}_\sigma A > \operatorname{order}_{\sigma'} A(p_1, x', D') .$$

PROOF. Note first that it suffices to prove the assertions outside analytic sets. Therefore we ignore these sets and we take a point p_1 as $p_1 = 0$ in the following.

The assertions are obvious when $N=1$. In general, we define an integer r_{ij} for

$A_{ij} \not\equiv 0$ by

$$\sigma(A_{ij})(x, \xi') = x_1^{r_{ij}} g_{ij}(x, \xi'), \qquad g_{ij}|_{x_1=0} \not\equiv 0,$$

and put $r = \min\{r_{ij};\ i, j = 1, \cdots, N,\ A_{ij} \not\equiv 0\}$ and denote it by $r = r(A)$. We may assume $r_{11} = r(A)$ without loss of generality.

When $r_{11} = 0$, $A_{11}(x, D')$ is invertible as a microdifferential operator at a generic point in $U \cap \{x_1 = 0\}$, and hence we may regard A_{11} as invertible in $\mathscr{E}'(U)$. Obviously, A_{11}^{-1} does not depend on D_1. For $i, j = 2, \cdots, N$, let

$$P_i(x, D') = -A_{i1}(x, D')A_{11}^{-1}(x, D'),$$

$$Q_j(x, D') = -A_{11}^{-1}(x, D')A_{1j}(x, D').$$

Then we have

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ P_2 & 1 & & \\ \vdots & & \ddots & \\ P_N & & & 1 \end{pmatrix} A \begin{pmatrix} 1 & Q_2 & \cdots & Q_N \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{pmatrix} = \left(\begin{array}{c|c} A_{11} & 0 \cdots 0 \\ \hline 0 & \\ \vdots & B(x, D') \\ 0 & \end{array}\right),$$

with $B(x, D') \in M_{n-1}(\mathscr{E}'(U))$.

Let us consider the case $r(A) > 0$. By Späth-Weierstrass' division theorem (see, for example, [19, Th. 2.2.1]), we have for $i, j = 2, \cdots, N$

$$A_{i1} = -B_i(x, D')A_{11} + R_i(x, D'), \qquad R_i = \sum_{k=0}^{r-1} R_{ik}(x', D')x_1^k,$$

$$A_{1j} = -A_{11}C_j(x, D') + S_j(x, D'), \qquad S_j = \sum_{k=0}^{r-1} S_{kj}(x', D')x_1^k.$$

Then we have

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ B_2 & 1 & & \\ \vdots & & \ddots & \\ B_N & & & 1 \end{pmatrix} A \begin{pmatrix} 1 & C_2 & \cdots & C_N \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{pmatrix} = \left(\begin{array}{c|c} A_{11} & S_2 \cdots S_N \\ \hline R_2 & \\ \vdots & * \\ R_N & \end{array}\right) = B(x, D').$$

In the above transformation of matrices, it holds that $r(A) > r(B)$, $\det_\sigma A = \det_\sigma B$ and $\det_{\sigma'} A(0, x', D') = \det_{\sigma'} B(0, x', D')$. By repeating the above transformations, we finally obtain matrices $P(x, D')$ and $Q(x, D')$ in $SL_N(\mathscr{E}'(U))$ such that

$$PAQ = \operatorname{diag}\{T_1, \cdots, T_k, 0, \cdots, 0\}, \qquad T_j(x, D') \in \mathscr{E}'(U),$$

for some k $(1 \leq k \leq N)$. By this construction of P and Q, we have

$$(PAQ)|_{x_1=0} = P|_{x_1=0} \cdot A|_{x_1=0} \cdot Q|_{x_1=0},$$

and $P(0, x', D')$ and $Q(0, x', D')$ belong to $SL_N(\mathscr{E}(U'))$, where $U' \subset T^*\Omega'$.

$\det_\sigma A \equiv 0$ is equivalent to $k < N$, which implies (i). When $k = N$, we have

$$\det_\sigma A = \prod_{j=1}^{N} \sigma(T_j) \quad \text{and} \quad \det_{\sigma'} A(0, x', D') = \prod_{j=1}^{N} \sigma'(T_j(0, x', D')),$$

which proves (ii) and (iii). q.e.d.

DIRECT PROOF OF LEMMA 2.4 FOR $n=2$. In this case we can prove the lemma without using microdifferential operators. Let $\Omega = \Omega_1 \times \Omega_2$ $(\Omega_i \subset \boldsymbol{C}_{x_i},\ i=1, 2)$ and $A(x, D_2) = (A_{ij}) \in M_N(\mathscr{D}(\Omega))$. If $A(x, D_2)$ is non degenerate or $\operatorname{order}_D A = -\infty$, our assertions are obvious. Otherwise, set $m_{ij} = \operatorname{order}_D A_{ij}$ and $\operatorname{order}_D A = m \geq 0$. Take a system of integers $\{s_i, t_j\}$ so that

$$m_{ij} \leq t_j - s_i \quad \text{and} \quad m = |t| - |s|.$$

We put

$$A_{ij} = \sum_{k=0}^{t_j - s_i} a_{ijk}(x) D_2^{t_j - s_j - k} \quad \text{and} \quad A_0(x) = (a_{ij0}(x)).$$

Then $A(x, D_2)$ is degenerate if and only if $\det A_0(x) \equiv 0$. Let $l(x) = (l_1(x), \cdots, l_N(x))$ be a non zero left null vector of $A_0(x)$ with holomorphic entries in a neighbourhood of the origin, where it is assumed that $O \in \Omega$. We define an integer r_i for $l_i \not\equiv 0$ by

$$l_i(x) = x_1^{r_i} \tilde{l}_i(x), \qquad \tilde{l}_i(0, x_2) \not\equiv 0,$$

and put $r = \min\{r_i;\ i=1, \cdots, N,\ l_i \not\equiv 0\}$. Let $r_{i_0} = r$. Then

$$L(x) = \left(\frac{l_1(x)}{l_{i_0}(x)}, \cdots, \overset{i_0}{\overset{\vee}{1}}, \cdots, \frac{l_N(x)}{l_{i_0}(x)}\right) = (L_1, \cdots, L_N)$$

also is a left null vector of $A_0(x)$ and its entries are meromorphic functions. By the above construction, $L_i(0, x_2)$ $(i=1, \cdots, N)$ are also meromorphic functions in x_2. Without loss of generality we may assume $s_1 \geq s_2 \geq \cdots \geq s_N$. We define a matrix $P(x, D_2)$ by

$$P(x, D_2) = \begin{matrix} \\ \\ i_0 \rangle \\ \\ \\ \end{matrix} \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & \ddots & & \\ L_1 & \cdots & 1 & \cdots & L_N \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix} \operatorname{diag}\{D_2^{s_1 - s_N}, D_2^{s_2 - s_N}, \cdots, 1\}.$$

Then

$$\det_\sigma P = \xi_2^{|s|-Ns_N}, \qquad \operatorname{order}_D PA \leq \operatorname{order}_D A + (|s| - Ns_N) - 1,$$

$$\det_\sigma A = \det_\sigma PA \cdot \xi_2^{Ns_N-|s|} \quad \text{and}$$

$$\det_{\sigma_2} A(0, x_2, D_2) = \det_{\sigma_2}(PA)(0, x_2, D_2) \cdot \xi_2^{Ns_N-|s|}.$$

Therefore, when $\det_\sigma A \not\equiv 0$, by continuing the above operations, we can reduce the problem to the non degenerate case. We have only to note that, when $\det_\sigma A \equiv 0$, we stop the above operations in at most $m+1$ times ($m = \operatorname{order}_D A$). q.e.d.

References

[1] K. Adjamagbo, Problème de Cauchy non-charactéristique pour le système générale d'équations différentielles linéaires, C. R. Acad. Sci. Paris 294 (1982), 159–162.

[2] K. Adjamagbo, Sur les systèmes linéaires d'équations aux dérivées partielles localement inversibles, C. R. Acad. Sci. Paris 299 (1984), 863–866.

[3] E. Andronikov, Le déterminant selon M. Sato et M. Kashiwara, Séminaire Vaillant, Université de Paris VI, 1979–80.

[4] E. Andronikov, Systèmes déterminés d'équations aux dérivées partielles, ibid., 1981–82; C. R. Acad. Sci. Paris 293 (1981), 257–260.

[5] E. Andronikov, Sur une question de S. Mizohata, C. R. Acad. Sci. Paris 299 (1984), 923–926.

[6] A. W. Goldie, The structure of prime rings under ascending chain conditions, Proc. London Math. Soc. 8 (1958), 589–608.

[7] G. Hufford, On the characteristic matrix of a matrix of differential operators, J. Differential Equations 1 (1965), 27–38.

[8] M. Kashiwara, Algebraic study of systems of partial differential equations (in Japanese), Master's thesis, Univ. Tokyo, 1971.

[9] M. Kashiwara, Systems of microdifferential equations, Birkhäuser, Boston-Basel-Stuttgart, 1983.

[10] K. Kitagawa and T. Sadamatsu, Sur le problème de Cauchy pour un système d'équations aux dérivées partielles, J. Math. Kyoto Univ. 21 (1981), 85–103.

[11] B. Malgrange, Remarques sur les points singuliers des équations différentielles, Enseignement Math. 20 (1974), 147–176.

[12] M. Miyake, On Cauchy-Kowalevski's theorem for general system, Publ. Res. Inst. Math. Sci. 15 (1979), 315–337.

[13] M. Miyake, Remarks on the formulations of the Cauchy problem for general system of ordinary differential equations, Tôhoku Math. J. 32 (1980), 79–89.

[14] M. Miyake, On the determinant of matrices of ordinary differential operators and an index theorem, Funkcial. Ekvac. 26 (1983), 155–171.

[15] S. Mizohata, On Kowalevskian systems, Russian Math. Surveys 29 (1974), 223–235.

[16] T. Sadamatsu, On the Cauchy-Kowalevski theorem for general system of differential equations with constant coefficients, Bull. Soc. Roy. Sci. Liège 50 (1981), 107–114.

[17] T. Sadamatsu, On the Cauchy-Kowalevski theorem for general system of differential equations, J. Math. Kyoto Univ. 24 (1984), 593–609.

[18] M. Sato and M. Kashiwara, The determinant of matrices of pseudodifferential operators, Proc. Japan Acad. 51 (1975), 17–19.

[19] P. SCHAPIRA, Microdifferential systems in the complex domain, Springer-Verlag, Berlin-Heiderberg-New York-Tokyo, 1985.
[20] L. R. VOLEVIČ, On general systems of differential equations, Soviet Math. Dokl. 1 (1960), 458–461.
[21] C. WAGSCHAL, Diverses formulations du problème de Cauchy pour un système d'équations aux dérivées partielles, J. Math. Pures Appl. 53 (1974), 51–70.

DEPARTMENT OF MATHEMATICS
COLLEGE OF GENERAL EDUCATION
NAGOYA UNIVERSITY
NAGOYA 464
JAPAN