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POINCARÉ SERIES FOR DISCRETE MOEBIUS GROUPS ACTING ON THE UPPER HALF SPACE

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Abstract. Consider the Poincaré series of order t for a discrete Moebius group acting on the n-dimensional upper half-space. If the point at infinity is a horocyclic limit point or a Garnett point, then the series diverges for any positive number t. If the point at infinity is an ordinary point or a cusped parabolic fixed point, then the series converges for any t which is greater than $n-1$. If the point at infinity is an atom for the Patterson-Sullivan measure, then the series converges for any t which is equal to or greater than the critical exponent of the group.

1. Discrete Moebius groups. Let R^n and $\overline{R^n}$ be the n-dimensional Euclidean space and its one-point compactification, respectively. We use the notation $x=(x_1, \ldots, x_n) \in R^n$ and when matrices act on x, we treat x as a column vector. The subspace $H^n=\{x \in R^n \mid x_n>0\}$ of R^n is a model for the hyperbolic n-space and supports a metric ρ derived from the differential $d\rho=|dx|/dx_n$. We call H^n the n-dimensional upper half-space.

The (full) Moebius group $M(\overline{R^n})$ is the group of Moebius transformations of $\overline{R^n}$, which is generated by inversions in spheres and reflections in planes. Moebius transformations are classified into three conjugacy classes in $M(\overline{R^n})$ as follows. An element in $M(\overline{R^n})$ is said to be loxodromic if it is conjugate to a transformation of the form

$$\gamma(x)=\lambda T x\,, \tag{1.1}$$

where $\lambda>0$, $\lambda \neq 1$, and $T \in O(n)$, the group of $n \times n$-orthogonal matrices, and parabolic if it is conjugate to a transformation of the form

$$\gamma(x)=Tx+a\,, \tag{1.2}$$

where $T \in O(n)$, $a \in R^n$ and $Ta=a \neq 0$. A non-trivial element is said to be elliptic if it is neither loxodromic nor parabolic.

By $\gamma'(x)$ we denote the Jacobian matrix of $\gamma \in M(\overline{R^n})$ at $x \in R^n$. For $\gamma \in M(\overline{R^n})$ the chain rule implies that $\gamma'(x)$ can be written as $\gamma'(x)=vT(x)$ with $v>0$ and $T \in O(n)$. We denote by $|\gamma'(x)|$ this positive number v and call it the linear magnification of γ at x. For $\gamma \in M(\overline{R^n})$ with $\gamma(\infty) \neq \infty$ the set $I(\gamma)=\{x \in R^n \mid |\gamma'(x)|=1\}$ is an $(n-1)$-sphere with center $\gamma^{-1}(\infty)$. The sphere $I(\gamma)$ is called the isometric sphere of γ. The action of γ on

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$\overline{R^n}$ is the composite of an inversion in $I(\gamma)$, followed by a Euclidean isometry. For $x \in \overline{R^n}$ we denote by x^* the image of the inversion of x in the unit sphere centered at the origin. Let $\gamma \in M(\overline{R^n})$ be an arbitrary element which does not fix ∞. Then γ can be written uniquely in the form

$$\gamma(x) = \lambda T(x-a)^* + b\,, \tag{1.3}$$

where $\lambda > 0$, $T \in O(n)$ and $a, b \in R^n$. In this expression $\lambda^{1/2}$ is the radius of the isometric sphere $I(\gamma)$ of γ and $a = \gamma^{-1}(\infty)$ (resp. $b = \gamma(\infty)$) is the center of $I(\gamma)$ (resp. $I(\gamma^{-1})$). If $\gamma \in M(\overline{R^n})$ fixes ∞, then γ is written uniquely as a similarity of the form

$$\gamma(x) = \lambda T x + a\,, \tag{1.4}$$

where $\lambda > 0$, $T \in O(n)$ and $a \in R^n$.

Denote by $M(H^n)$ the subgroup of $M(\overline{R^n})$ which keeps the subspace H^n of $\overline{R^n}$ invariant. Let Γ be a discrete subgroup of $M(H^n)$. A point $\xi \in R^{n-1} = \partial H^n$, the boundary of H^n, is a limit point for Γ if there exist an infinite sequence of $\gamma_m \in \Gamma$ and a point $x \in \mathrm{cl}(H^n)$, the closure of H^n, such that $\gamma_m(x) \to \xi$ as $m \to \infty$. The set of all limit points for Γ is the limit set $\Lambda(\Gamma)$. The set $\Omega(\Gamma) = \mathrm{cl}(H^n) - \Lambda(\Gamma)$ is called the region of discontinuity of Γ.

Points of the boundary $\partial H^n = \overline{R^{n-1}}$ are classified into three kinds of subsets as follows. A point $\xi \in \overline{R^{n-1}}$ is a horocyclic limit point for Γ if for every $x \in H^n$ there exist a sequence $\{\gamma_m\} \subset \Gamma$ and an element $h \in M(H^n)$ such that $h(\xi) = \infty$ and $\tau\{h\gamma_m h^{-1}(x)\} \to \infty$ as $m \to \infty$, where $\tau(y)$ is the n-th coordinate of $y \in H^n$. The set of horocyclic limit points for Γ is called the horocyclic limits set $H(\Gamma)$. The horocyclic limit set $H(\Gamma)$ contains every loxodromic fixed point of Γ. A point $\xi \in \overline{R^{n-1}}$ is a Dirichlet point for Γ if for every $x \in H^n$ there exist elements $\gamma_0 \in \Gamma$ and $h \in M(H^n)$ such that $h(\xi) = \infty$ and $\tau(h\gamma_0 h^{-1}(x)) \geqq \tau(h\gamma h^{-1}(x))$ for every $\gamma \in \Gamma$. The set of Dirichlet points for Γ is denoted by $D(\Gamma)$ and is said to be the Dirichlet set for Γ. We say a point $\xi \in \overline{R^{n-1}}$ to be a Garnett point for Γ if there exist $x \in H^n$, a sequence $\{\gamma_m\} \subset \Gamma$, a transformation $h \in M(H^n)$ and a positive number r such that $h(\xi) = \infty$, $\tau(h\gamma h^{-1}(x)) < r$ for all $\gamma \in \Gamma$ and $\tau(h\gamma_m h^{-1}(x)) \uparrow r$ as $m \to \infty$. The set of Garnett points for Γ is denote by $Q(\Gamma)$. These three subsets $H(\Gamma)$, $D(\Gamma)$ and $Q(\Gamma)$ are invariant under the action of Γ. Note that the boundary $\partial H^n = \overline{R^{n-1}}$ can be written in the disjoint union as $\overline{R^{n-1}} = H(\Gamma) \cup D(\Gamma) \cup Q(\Gamma)$.

2. Cusped parabolic fixed points. Let Γ be a discrete subgroup of $M(H^n)$. For $x \in \mathrm{cl}(H^n)$, the subgroup $\Gamma_x = \{\gamma \in \Gamma \mid \gamma(x) = x\}$ of Γ is called the stabilizer of x. Suppose that Γ' is a subgroup of Γ. Then a subset X of $\mathrm{cl}(H^n)$ is said to be precisely invariant under Γ' in Γ, if $\gamma(X) = X$ for all $\gamma \in \Gamma'$ and $\gamma(X) \cap X = \varnothing$ for all $\gamma \in \Gamma - \Gamma'$. A parabolic fixed point $\xi \in \overline{R^{n-1}}$ of Γ is called a cusped parabolic fixed point of Γ if either

(1) Γ_ξ has rank $n-1$ (in this case the quotient space $[\overline{R^{n-1}} - \{\xi\}]/\Gamma_\xi$ is compact.), or

(2) there exist $h \in M(H^n)$ and $d>0$ such that $h(\xi)=\infty$ and $h^{-1}[R^{n-1}-\{R^k \times B^{n-k-1}(d)\}]$ is precisely invariant under Γ_ξ in Γ, where k $(1 \leqq k \leqq n-2)$ is the rank of Γ_ξ and $B^{n-k-1}(d)=\{x \in R^{n-k-1} \mid |x|<d\}$. (in this case $[\overline{R^{n-1}}-\{\xi\}]/\Gamma_\xi$ is not compact.)

Examples of non-cusped parabolic fixed points are known. See Apanasov [2] and Ohtake [6]. Denote by $C(\Gamma)$ the set of cusped parabolic fixed points of Γ.

First of all we prove the following:

LEMMA 1. *Suppose that Γ is a non-elementary discrete subgroup of $M(H^n)$ and ∞ is a cusped parabolic fixed point of Γ. Then there exists a compact set $K \subset R^{n-1} \subset \partial H^n$ such that if an element $\gamma \in \Gamma$ does not fix ∞ then $h_1\gamma h_2(\infty) \in K$ and $(h_1\gamma h_2)^{-1}(\infty) \in K$ for some $h_1, h_2 \in \Gamma_\infty$.*

PROOF. Let γ be an arbitrary element in $\Gamma-\Gamma_\infty$. Then γ can be written uniquely in the form $\gamma(x)=\lambda T(x-a)^*+b$, where $\lambda>0$, $T \in O(n)$ and $a, b \in R^{n-1}$.

First we deal with the case where Γ_∞ has rank $n-1$. Since Γ_∞ acts on R^{n-1}, there exists a compact fundamental set K for Γ_∞ in R^{n-1}. We can choose elements $h_1, h_2 \in \Gamma_\infty$ so that $h_1^{-1}(a), h_2(b) \in K$. Put $h_i(x)=U_i x+c_i$ for $i=1, 2$. Then by simple calculation we see $(h_2\gamma h_1)(x)=U_2\{\lambda T(U_1x+c_1-a)^*+b\}+c_2=\lambda U_2TU_1(x-U_1^{-1}a+U_1^{-1}c_1)^*+U_2b+c_2=\lambda U_2TU_1(x-h_1^{-1}(a))^*+h_2(b)$. Note $(h_2\gamma h_1)(\infty)=h_2(b) \in K$, $(h_2\gamma h_1)^{-1}(\infty)=h_1^{-1}(a) \in K$ and we have the required result.

Next we suppose that the rank of Γ_∞ is at most $n-2$. Conjugating Γ by a suitable transformation in $M(H^n)$, we may assume that $R^{n-1}-\{R^k \times B^{n-k-1}(d)\}$ is precisely invariant under Γ_∞ in Γ for some $d>0$. We can choose a compact set S in R^k so that $\bigcup_{h \in \Gamma_\infty} h(S \times B^{n-k-1}(d))=R^k \times B^{n-k-1}(d)$. Put $K=S \times \mathrm{cl}(B^{n-k-1}(d))$. Then we see that K is a compact subset of R^{n-1} and $\bigcup_{h \in \Gamma_\infty} h(K) \supset R^k \times B^{n-k-1}(d)$. Since $\gamma^{\pm 1}(\infty) \in \Lambda(\Gamma)-\{\infty\} \subset R^k \times B^{n-k-1}(d)$, we deduce that $\gamma^{-1}(\infty)=a$ and $\gamma(\infty)=b$ belong to $R^k \times B^{n-k-1}(d)$. Thus we can choose $h_1, h_2 \in \Gamma_\infty$ so that $h_1^{-1}(a), h_2(b) \in K$. By an argument similar to that in the former case we have the required result. q.e.d.

For $t>0$ define a subset H_t of H^n by $H_t=\{x \in H^n \mid \tau(x)>t\}$. We denote by R_γ the radius of isometric sphere of $\gamma \in \Gamma-\Gamma_\infty$. Suppose that ∞ is a cusped parabolic fixed point of Γ. If Γ_∞ has rank $n-1$, then Γ_∞ contains a free abelian normal subgroup of rank $n-1$. So Γ_∞ contains a translation. Hence the set $R(\Gamma)=\{R_\gamma \mid \gamma \in \Gamma-\Gamma_\infty\}$ has a positive finite supremum r. Note that any element of Γ_∞ keeps H_t invariant for any $t>0$. Hence the set H_r is precisely invariant under Γ_∞ in Γ. (See Ohtake [6, Corollary 1] or Wielenberg [9, Proposition 4].) If the rank of Γ_∞ is less than $n-1$, then there exist $d>0$ such that $R^{n-1}-\{R^k \times B^{n-k-1}(d)\}$ is precisely invariant under Γ_∞ in Γ. We easily see that for any $\gamma \in \Gamma-\Gamma_\infty$ the center of isometric sphere $I(\gamma)$ of γ is contained in $R^k \times B^{n-k-1}(d)$. If $\sup R(\Gamma)=\infty$, then $R^k \times B^{n-k-1}(s)$ is not precisely invariant under Γ_∞ in Γ for any $s>0$. So $\sup R(\Gamma)$ is positive finite. Take an arbitrary $t>\sup R(\Gamma)$. Then H_t is precisely invariant under Γ_∞ in Γ. Hence we have the following:

LEMMA 2. *Suppose that ∞ is a cusped parabolic fixed point of a discrete subgroup Γ of $M(H^n)$. Then there exists a positive constant t such that the subspace H_t of H^n is precisely invariant under Γ_∞ in Γ.*

By using Lemma 1 and Lemma 2, we show the next result which is announced in Nicholls [4] without proof.

THEOREM 3. *If Γ is a discrete subgroup of $M(H^n)$, then $C(\Gamma)\subset D(\Gamma)$.*

PROOF. Assume the contrary. Let $\xi\in\overline{R^{n-1}}$ be a cusped parabolic fixed point of Γ. Conjugating Γ by an element of $M(H^n)$, we may set $\xi=\infty$. Note that $\overline{R^{n-1}}=\partial H^n$ is decomposed into a disjoint union as $\overline{R^{n-1}}=H(\Gamma)\cup D(\Gamma)\cup Q(\Gamma)$. Assume that ∞ is a horocyclic limit point for Γ. Then for any point $x\in H^n$ there exists a sequence $\{\gamma_m\}$ of Γ such that $\tau(\gamma_m(x))\uparrow\infty$ as $m\to\infty$. Since $\gamma_m\in M(H^n)$ and $\gamma_m(\infty)\neq\infty$, we may put $\gamma_m(x)=\lambda_m T_m(x-a_m)^*+b_m$, where $\lambda_m>0$, $T_m\in O(n)$ and $a_m, b_m\in R^{n-1}\subset\partial H^n\subset\overline{R^n}$ for all m. Then by elementary calculation we have

$$\tau(\gamma_m(x))=\lambda_m\tau(x)/|x-a_m|^2\,. \tag{2.1}$$

Note that the denominators on the right hand side of (2.1) are bounded away from zero. Hence we see $\lambda_m\to\infty$ as $m\to\infty$. It contradicts Lemma 2, so we have $\infty\notin H(\Gamma)$. Next we assume $\infty\in Q(\Gamma)$. Then there exist $x\in H^n$, $r>0$ and $\{\gamma_m\}\subset\Gamma-\Gamma_\infty$ such that $\tau(\gamma(x))<r$ for all $\gamma\in\Gamma$ and $\tau(\gamma_m(x))\uparrow r$ as $m\to\infty$. If $\sup\{|a_m|\,|\,m=1,2,\dots\}=\infty$, then $\sup\{\lambda_m\,|\,m=1,2,\dots\}=\infty$ and it contradicts Lemma 2. Hence there exists a compact set $K_1\subset R^{n-1}$ so that $a_m\in K_1$ for every m. Now from the proof of Lemma 1 there exist a compact set $K_2\subset R^{n-1}$ and a sequence $\{h_m\}\subset\Gamma_\infty$ so that $(h_m\gamma_m)(\infty)=h_m(b_m)\in K_2$ for every m. Note that $(h_m\gamma_m)^{-1}(\infty)=a_m\in K_1$ for every m. Put $K=K_1\cup K_2$. Then we have $(h_m\gamma_m)^{\pm1}(\infty)\in K$ for every m. Noting $R_{h_m\gamma_m}=R_{\gamma_m}$, we can put $(h_m\gamma_m)(x)=\lambda_m U_m(x-\alpha_m)^*+\beta_m$, where $\alpha_m, \beta_m\in K$ and $U_m\in O(n)$ for every m. So if $\{h_m\gamma_m\}$ contains infinitely many distinct elements, there exist a subsequence $\{h_{m_j}\gamma_{m_j}\}$ of $\{h_m\gamma_m\}$, $\lambda>0$, $\alpha, \beta\in K$ and $U\in O(n)$ such that $(h_{m_j}\gamma_{m_j})(x)\to\lambda U(x-\alpha)^*+\beta$ as $j\to\infty$. It contradicts the discreteness of Γ. So it suffices to show that $\{h_m\gamma_m\}$ contains a subsequence consisting of infinitely many distinct elements. Assume the contrary. Then there exists $\{g_j\,|\,j=1,\dots,k\}=\Gamma'\subset\Gamma$ so that $h_m\gamma_m\in\Gamma'$ for all m. Hence we have $h_m\gamma_m=g_j$ for some j and for infinitely many m. It follows that $\tau(\gamma_m(x))=\tau(h_m^{-1}g_j(x))$ is constant for infinitely many m. It cannot occur. Hence we have $\infty\in D(\Gamma)$. q.e.d.

3. Atoms for the Patterson-Sullivan measure.

In this section we summarize some properties of the Patterson-Sullivan measure. For definitions and details see Nicholls [5] and Patterson [8].

Let Γ be a discrete subgroup of $M(H^n)$. For $x, y\in H^n$ and $t>0$, consider the series of the form

$$g(t, x, y) = \sum_{\gamma \in \Gamma} \exp\{-t\rho(x, \gamma(y))\}. \tag{3.1}$$

The critical exponent $\delta = \delta(\Gamma)$ of Γ is defined by $\delta = \inf\{t > 0 \mid g(t, x, y) < \infty\}$. It is well-known that the divergence or convergence of $g(t, x, y)$ does not depend on x, y and $\delta \leqq n-1$.

For $x \in H^n$ let M_x be the collection of positive finite measures on $\mathrm{cl}(H^n)$ with the base point x. Here each $\mu_x \in M_x$ is obtained by weak convergence of sequences of measures derived from the series (3.1) and is said to be the Patterson-Sullivan measure with the base point x. We summarize the properties of this measure in the following. Any measure μ_x belonging to M_x satisfies

(3.2) μ_x is supported on the limit set $\Lambda(\Gamma)$.

(3.3) For any $x, z \in H^n$, μ_x and μ_z are absolutely continuous with respect to each other and the Radon-Nikodym derivative is $\{d\mu_x/d\mu_z\}(\zeta) = \{P(x, \zeta)/P(z, \zeta)\}^\delta$, where $\zeta \in \partial H^n$ and $P(x, \zeta)$ is the Poisson kernel on H^n.

(3.4) For any Borel set E of $\mathrm{cl}(H^n)$ and any $\gamma \in \Gamma$, we have $\mu_x(\gamma^{-1}(E)) = \mu_{\gamma(x)}(E)$.

A point $\xi \in \overline{R^{n-1}}$ is said to be an atom for $\mu_x \in M_x$ if $\mu_x(\xi) > 0$. The set of atoms is denoted by

(3.5) $A(\Gamma) = \{\xi \in \overline{R^{n-1}} \mid \mu_x(\xi) > 0$ for some $\mu_x \in M_x$ and some $x \in H^n\}$.

It is known that $A(\Gamma) \subset D(\Gamma) \cap \Lambda(\Gamma)$ and $A(\Gamma) \cap C(\Gamma) = \emptyset$. (See Bowditch [3] and Nicholls [5].) It is obvious $\Omega(\Gamma) \cap \overline{R^{n-1}} \subset D(\Gamma)$. So we deduce from Theorem 3 and (3.4) the following:

Proposition 4. *Let Γ be a discrete subgroup of $M(H^n)$. Then the three sets $\Omega(\Gamma) \cap \overline{R^{n-1}}$, $C(\Gamma)$ and $A(\Gamma)$ are disjoint, invariant subsets of $D(\Gamma)$ under Γ.*

It is not known whether the set $D(\Gamma) - [\{\Omega(\Gamma) \cap \overline{R^{n-1}}\} \cup C(\Gamma) \cup A(\Gamma)]$ is empty or not.

4. Poincaré series. For $\gamma \in M(H^n)$ we calculate the linear magnification $|\gamma'(x)|$ of γ. If $\gamma(\infty) = \infty$, then γ is a similarity of the form (1.4) and we easily see $|\gamma'(x)| = \lambda$. If $\gamma(\infty) \neq \infty$, then chain rule implies $|\gamma'(x)| = \lambda/|x-a|^2$ from (1.3). Hence (2.1) implies the following:

Lemma 5. *For $\gamma \in M(H^n)$ and $x \in H^n$ we have $\tau(\gamma(x)) = \tau(x)|\gamma'(x)|$.*

Let Γ be a non-elementary discrete subgroup of $M(H^n)$ with the critical exponent δ. Suppose that ∞ is not fixed by any loxodromic element of Γ. We denote by S a system of left coset representatives of $\Gamma_\infty \backslash \Gamma$. For $x \in H^n$ and $t > 0$, we consider the Poincaré series of the form

$$(4.1)\qquad \Theta(x, t)=\sum_{\gamma\in S}|\gamma'(x)|^t .$$

Since Γ_∞ does not contain loxodromic transformations, we see $|h'(x)|=1$ for every $h\in\Gamma_\infty$ and every $x\in H^n$. So for every $\gamma\in\Gamma$ and $h\in\Gamma_\infty$ chain rule implies $|(h\gamma)'(x)|=|h'(\gamma(x))||\gamma'(x)|=|\gamma'(x)|$ for every $x\in H^n$. Hence we have the following:

LEMMA 6. *Let Γ be a non-elementary discrete subgroup of $M(H^n)$. Suppose that Γ_∞ does not contain loxodromic elements of Γ. Then the value of the series* (4.1) *does not depend on the choice of coset representatives.*

Next we suppose that Γ_∞ contains a loxodromic element h. We may assume that ∞ is an attractive fixed point of h. So h can be written in the form $h(x)=\lambda Tx+\alpha$, where $\lambda>1$, $T\in O(n)$ and $\alpha\in R^{n-1}$. Since $|h'(x)|=\lambda$ for any $x\in H^n$, we see $|(h^m\gamma)'(x)|=\lambda^m|\gamma'(x)|$ for any $\gamma\in S$ and any integer m. So it follows that $|(h^m\gamma)'(x)|\to\infty$ (resp. 0) as $m\to\infty$ (resp. $-\infty$). Note that $h^m\gamma$ and γ belong to the same coset. Hence we conclude that the value of $\Theta(x, t)$ may be finite or infinite according to the choice of a system of coset representatives. From now on we consider the series $\Theta(x, t)$ only in the case where Γ_∞ does not contain loxodromic transformations.

The purpose of this section is to prove the following theorem.

THEOREM 7. *Let Γ be a non-elementary discrete subgroup of $M(H^n)$. Suppose that ∞ is not fixed by loxodromic elements of Γ. Then the following hold*:

(1) *If $\infty\in H(\Gamma)$, then $\Theta(x, t)=\infty$ for all $x\in H^n$ and all $t>0$.*

(2) *If $\infty\in Q(\Gamma)$, then $\Theta(x, t)=\infty$ for all $x\in H^n$ and all $t>0$.*

(3) *If $\infty\in\Omega(\Gamma)\cup C(\Gamma)$, then $\Theta(x, t)<\infty$ for all $x\in H^n$ and all $t>n-1$.*

(4) *If $\infty\in A(\Gamma)$, then $\Theta(x, t)<\infty$ for all $x\in H^n$ and all $t\geqq\delta$.*

PROOF. First we show (1). By the definition of horocyclic limit points, there exists a sequence $\{\gamma_m\}$ of Γ such that $\tau(\gamma_m(x))\uparrow\infty$ as $m\to\infty$ for all $x\in H^n$. Any $\gamma\in\Gamma_\infty$ is written uniquely in the form

$$(4.2)\qquad \gamma(x)=Tx+a ,$$

where $T\in O(n)$ and $a\in R^{n-1}\subset\partial H^n$. Since $|\gamma_m'(x)|=\tau(\gamma_m(x))/\tau(x)$ from Lemma 5, we get $|\gamma_m'(x)|\uparrow\infty$ as $m\to\infty$. Assume that for some j, k ($j<k$), γ_j and γ_k belong to the same coset. Then we have $\gamma_j=h\gamma_k$ for some $h\in\Gamma_\infty$. So we have $|\gamma_j'(x)|=|(h\gamma_k)'(x)|=|\gamma_k'(x)|$ by (4.2). It contradicts the definition of horocyclic limit points. Hence any two elements of $\{\gamma_m\}$ belong to distinct cosets. So it follows that $\Theta(x, t)\geqq\sum_{m=1}^{\infty}|\gamma_m'(x)|^t=\infty$ for all $x\in H^n$ and $t>0$ and we have the required result.

Next we deal with (2). Since $\infty\in Q(\Gamma)$, by Lemma 5 there exist $y\in H^n$, $\alpha>0$ and $\{\gamma_m\}\subset\Gamma$ such that $|\gamma_m'(y)|\uparrow\alpha$ as $m\to\infty$ and $|\gamma'(y)|<\alpha$ for all $\gamma\in\Gamma$. Since Γ_∞ cannot contain loxodromic elements, we may assume that any element γ_m can be written in the form $\gamma_m(x)=\lambda_m T_m(x-a_m)^*+b_m$. Let x be an arbitrary point in H^n. Since $|\gamma_m'(x)|/|\gamma_m'(y)|=|y-a_m|^2/|x-a_m|^2$, there exists a positive constant c such that

$|\gamma'_m(x)|/|\gamma'_m(y)|>c$ for all m. By an argument similar to that in the case $\infty\in H(\Gamma)$, we see that any two elements of $\{\gamma_m\}$ belong to distinct consets. Hence we have $\Theta(x, t)\geqq\sum_{m=1}^{\infty}|\gamma'_m(x)|^t\geqq c^t\sum_{m=1}^{\infty}|\gamma'_m(y)|^t=\infty$ for all $x\in H^n$, all $t>0$ and we prove (2).

Suppose that ∞ is a cusped parabolic fixed point of Γ. Take and fix an arbitrary point $x_0\in H^n$. Hence we put $S=\{\gamma_m\}$. Now we show the following. By taking a suitable system of coset representatives $S'=\{\eta_m\}$, we see that there exist a compact set $K\subset R^{n-1}$ and $\beta>0$ so that $\eta_m(B(x_0, \alpha))\subset K\times(0, \beta)$ for all m and a sufficiently small $\alpha>0$, where $B(x_0, \alpha)=\{y\in H^n\mid\rho(x_0, y)<\alpha\}$ and $(0, \beta)$ is an open interval.

Suppose that Γ_∞ has rank $n-1$. Let T be a compact fundamental set for Γ_∞ on R^{n-1}. Then for each m there exist $h_m\in\Gamma_\infty$ and $\beta_0>0$ such that $h_m\gamma_m(x_0)\in T\times(0, \beta_0)$. By putting $\eta_m=h_m\gamma_m$ we easily see that $S'=\{\eta_m\}$ is a system of coset representatives of $\Gamma_\infty\backslash\Gamma$. Hence for a sufficiently small $\alpha>0$ there exist a compact set K $(\supset T)$ in R^n and β $(>\beta_0)$ so that $\eta_m(B(x_0, \alpha))\subset K\times(0, \beta)$ for all m. If k, the rank of Γ_∞, is less than $n-1$, there exists $d>0$ so that $R^{n-1}-\{R^k\times B^{n-k-1}(d)\}$ is precisely invariant under Γ_∞ in Γ. Note that the Euclidean distance between the plane $R^k\times\{(0, \dots, 0)\}$ in R^n and $\gamma_m(x_0)$ remains bounded for every m. Since the quotient of R^k by the restricted action of Γ_∞ is compact, there exist $\gamma_m\in\Gamma_\infty$, $\beta_0>0$ and a compact fundamental set T for Γ_∞ on R^k so that $h_m\gamma_m(x_0)\in T\times B^{n-k-1}(\beta_1)\times(0, \beta_0)$ for some $\beta_1>0$ and every m. Put $\eta_m=h_m\gamma_m$. Hence for sufficiently small $\alpha>0$ there exist a compact set K $(\supset T\times B^{n-k-1}(\beta_1))$ in R^{n-1} and a positive number β $(>\beta_1)$ so that $\eta_m(B(x_0, \alpha))\subset K\times(0, \beta)$.

Here we put $B_m=\eta_m(B(x_0, \alpha))$. We may set $\eta_m(x)=\lambda_m T_m(x-a_m)^*+b_m$. If x_0 is not fixed by any non-trivial element of Γ, then for a sufficiently small $\alpha>0$, $\{B_m\}$ is the set of disjoint balls. But if x_0 is fixed by an elliptic element of order p, then all B_m overlap p times. Take a positive number N and consider the integral

$$I=\sum_{m=1}^{N}\int\cdots\int_{B_m}(x_n)^t\frac{dx_1\cdots dx_n}{(x_n)^n}. \tag{4.3}$$

Then there exists a positive constant c_1 which depends only on x_0 and Γ such that

$$I\leqq c_1\int\cdots\int_{K\times(0,\alpha)}(x_n)^{t-n}dx_1\cdots dx_n. \tag{4.4}$$

Note that if $t-n>-1$, then the right hand side of (4.4) converges to a positive number M which does not depends on N. In each integral of (4.3), we make the change of variable $x=\eta_m(y)$. By simple calculation and Lemma 5, we see $|\eta'_m(y)|=\lambda_m/|y-a_m|^2=\tau(x)/\tau(y)$ for every m. Using this equality we get from (4.3)

$$I=\sum_{m=1}^{N}\int\cdots\int_{B(x_0,\alpha)}|\eta'_m(y)|^t(y_n)^t\frac{dy_1\cdots dy_n}{(y_n)^n}, \tag{4.5}$$

where $y=(y_1, \dots, y_n)$. Since $|\eta'_m(y)|/|\eta'_m(x_0)|=|x_0-a_m|^2/|y-a_m|^2$, then for every m there exists a positive number c_2 which depends on x_0, α and Γ such that

$|\eta'_m(y)|/|\eta'_m(x_0)| \geqq c_2$ for all $y \in B(x_0, \alpha)$. From (4.5) it follows that

$$(4.6) \qquad I \geqq c_2 \sum_{m=1}^{N} |\eta'_m(x_0)|^t \int \cdots \int_{B(x_0, \alpha)} (y_n)^{t-n} dy_1 \cdots dy_n \geqq c_3 \sum_{m=1}^{N} |\eta'_m(x_0)|^t ,$$

where c_3 is a positive constant which does not depend on N. Hence by (4.3), (4.4) and (4.6) we have

$$(4.7) \qquad \sum_{m=1}^{N} |\eta'_m(x_0)|^t \leqq M/c_3 ,$$

for $t > n-1$. Note that the right hand side of (4.7) does not depend on N. Since x_0 is an arbitrary point in H^n, we get

$$(4.8) \qquad \sum_{m=1}^{\infty} |\eta'_m(x)|^t < \infty$$

for every $x \in H^n$ and $t > n-1$. Since η_m and γ_m belong to the same coset $\Gamma_\infty \eta_m$, we have $|\eta'_m(x)| = |\gamma'_m(x)|$ for every m. Hence we get $\Theta(x, t) = \sum_{\gamma \in S} |\gamma'_m(x)|^t < \infty$ for every $x \in H^n$ and $t > n-1$ if $\infty \in C(\Gamma)$.

Next suppose that ∞ is an ordinary point. For any $x_0 \in H^n$ and any m there exist positive numbers α, β and a compact set K in R^{n-1} such that $\gamma_m(B(x_0, \alpha)) \subset K \times (0, \beta)$. By an argument similar to that in the case $\infty \in C(\Gamma)$, it follows that $\Theta(x, t) < \infty$ for every $x \in H^n$ and $t > n-1$, and the statement (3) is proved.

Finally we show (4). Suppose that ∞ is an atom for a measure $\mu_x \in M_x$. It suffices to show $\Theta(x, \delta) < \infty$ for every $x \in H^n$. Note that for $\gamma_i, \gamma_j \in S$ we have $\gamma_i^{-1}(\infty) = \gamma_j^{-1}(\infty)$ if and only if $i = j$. Hence by (3.4) we get $\sum_{\gamma \in S} \mu_{\gamma(x)}(\infty) = \sum_{\gamma \in S} \mu_x(\gamma^{-1}(\infty)) \leqq \mu_x(\mathrm{cl}(H^n)) < \infty$. By (3.3) we see

$$(4.9) \qquad \sum_{\gamma \in S} \mu_{\gamma(x)}(\infty) = \sum_{\gamma \in S} \left[\frac{P(\gamma(x), \infty)}{P(x, \infty)} \right]^\delta \mu_x(\infty) .$$

Note that the Poisson kernel $P(x, \infty)$ for the upper half-space is given by $P(x, \infty) = \tau(x)$ for all $x \in H^n$. Since ∞ is an atom for μ_x, the value $\mu_x(\infty)$ is positive finite. Put $c = \mu_x(\infty)$. Then the right hand side of (4.9) is $c \sum_{\gamma \in S} [\tau(\gamma(x))/\tau(x)]^\delta = c \sum_{\gamma \in S} |\gamma'(x)|^\delta$ by Lemma 5. Therefore we get $\Theta(x, \delta) = \sum_{\gamma \in S} |\gamma'(x)|^\delta < \infty$ and this is the required result. q.e.d.

5. Radii of isometric spheres. In this last section we describe a property of radii of the isometric spheres of discrete groups.

Let Γ be a discrete subgroup of $M(H^n)$ and Γ' a subgroup of Γ. We say that elements $\gamma_i, \gamma_j \in \Gamma$ are equivalent with respect to Γ' if there exist $h_1, h_2 \in \Gamma'$ such that $\gamma_i = h_1 \gamma_j h_2$.

For a non-elementary discrete subgroup Γ of $M(H^n)$, let $\{\gamma_m\}$ be an enumeration of $\Gamma - \Gamma_\infty$. For each m we denote by R_m the radius of the isometric sphere $I(\gamma_m)$ of γ_m. It is well-known that $\liminf_{m \to \infty} R_m = 0$. We show the following:

THEOREM 8. *Let Γ be a non-elementary discrete subgroup of $M(H^n)$. Then the following holds*:

(1) *If* $\infty \in H(\Gamma)$ then $\limsup_{m\to\infty} R_m = \infty$.

(2) *If* $\infty \in Q(\Gamma)$ *then* $\limsup_{m\to\infty} R_m > 0$.

(3) *If* $\infty \in \Omega(\Gamma)$ *then* $\lim_{m\to\infty} R_m = 0$.

(4) *If* $\infty \in C(\Gamma)$ *then for any subsequence* $\{\gamma_{m_j}\}$, *which consists of inequivalent elements of* $\{\gamma_m\}$ *with respect to* Γ_∞, *it follows that* $\lim_{j\to\infty} R_{m_j} = 0$.

PROOF. For each m let $\gamma_m \in \Gamma - \Gamma_\infty$ be of the form $\gamma_m(x) = \lambda_m T_m(x - a_m)^* + b_m$. Since $|\gamma'_m(x)| = \lambda_m / |x - a_m|^2$, we see $R_m = \lambda_m^{1/2} = |\gamma'_m(x)|^{1/2} |x - a_m|$ for all $x \in H^n$. Note that a_m belongs to R^{n-1} $(\subset \partial H^n)$ for every m. Then there exists a positive number c which depend on x and Γ such that $|x - a_m| > c$ for all m. Hence the statements (1), (2) are immediate consequences of definitions of horocyclic limit points and Garnett points.

Now we show (3). Since ∞ is an ordinary point, there exist a compact set $W \subset R^{n-1}$ and a positive integer N such that $\gamma_m^{\pm 1}(\infty) \in W$ for all $m \geqq N$. Assume that there exists a subsequence $\{\gamma_{m_j}\}$ of $\{\gamma_m\}$ such that $\lim_{j\to\infty} R_{m_j} = \alpha$. If $\alpha = \infty$, then there exists a point $x \in H^n$ such that $\gamma_{m_j}(x) \to \infty$ as $j \to \infty$. This means that ∞ is a limit point, a contradiction. Next we consider the case where α is positive finite. By taking a subsequence $\{\gamma_{m_j}\}$, if necessary, we have $(h_{m_j}\gamma_{m_j}g_{m_j})^{-1}(\infty) \to \zeta \in W$, $(h_{m_j}\gamma_{m_j}g_{m_j})(\infty) \to \eta \in W$ and $T_{m_j} \to T \in O(n)$ as $j \to \infty$. It follows that $(h_{m_j}\gamma_{m_j}g_{m_j})(x) \to \alpha^2 T(x - \zeta)^* + \xi$ as $j \to \infty$ for all $x \in H^n$. Then Γ is not discrete, a contradiction. Hence we have $\alpha = 0$.

Finally we deal with (4). Assume that there exists a subsequence $\{\gamma_{m_j}\}$ of $\{\gamma_m\}$ so that $\lim_{j\to\infty} R_{m_j} = R > 0$. Since $\infty \in C(\Gamma)$, there exists $d > 0$ such that the subspace H_d is precisely invariant under Γ_∞ in Γ by Lemma 2. So R is positive finite. Then by Lemma 1 there exist a compact set K in R^{n-1} and $h_{m_j}, g_{m_j} \in \Gamma_\infty$ such that $(h_{m_j}\gamma_{m_j}g_{m_j})^{\pm 1}(\infty) \in K$ for every j. By an argument similar to that in (3), we have a contradiction. Hence the statement (4) is proved. q.e.d.

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