CONFORMAL DEFORMATION TO PRESCRIBED SCALAR CURVATURE ON COMPLETE NONCOMPACT Riemannian manifolds with nonpositive curvature

Dedicated to Professor Hideki Ozeki on his 60th birthday

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Abstract. We consider the problem of deforming the metric on a complete negatively curved manifold conformally to another complete metric whose scalar curvature is positive in an unbounded domain. We also consider the case of the Euclidean space.

1. Introduction. Let \( (M, g) \) be a Riemannian manifold with or without boundary \( (n = \dim M \geq 2) \), and \( f \) a smooth function on \( M \). In this paper, we consider the problem of deforming the given metric \( g \) conformally to another metric
\[
\tilde{g} = \begin{cases} 
e^ug & \text{if } n = 2 \\ u^{4/(n-2)}g & \text{if } n \geq 3 \end{cases}
\]
with the prescribed scalar curvature \( f \). It is well-known that this problem is equivalent to solving the following elliptic differential equation:
\[
\begin{align*}
(\ast2) & \quad -\Delta g u + S_g u = fe^u \quad \text{if } n = 2, \\
(\ast n) & \quad \begin{cases}
-4^{n-1} \Delta g u + S_g u = fu^{(n+2)/(n-2)} & \text{if } n \geq 3, \\
u > 0
\end{cases}
\end{align*}
\]
where \( \Delta_g \) is the Laplacian with respect to \( g \), namely, \( \Delta_g = \text{trace} \nabla^2_g \), and \( S_g \) is the scalar curvature of \( g \). This problem and related ones have been extensively investigated, mainly in the case \( (M, g) \) is a compact manifold. As for the case \( (M, g) \) is the Euclidean space \( (\mathbb{R}^n, g_0) \), since Ni [13] was published, many authors have refined and generalized his results, and applied the method to other equations (see, for example, [7], [11], and their references).

We study first the case of \( (\mathbb{R}^n, g_0) \), and show the following sufficient conditions for the existence of infinitely many metrics each of which is pointwise conformal and uniformly equivalent to \( g_0 \), and each of whose scalar curvature is the prescribed
function $f$.

**THEOREM I.** Let $\Sigma$ be a submanifold of $\mathbb{R}^n$ ($n \geq 3$) with $m = \dim \Sigma \leq n-3$, and $f$ a bounded smooth function on $\mathbb{R}^n$. Suppose $\Sigma$ and $f$ satisfy the following conditions:

(R.1) $\Sigma$ is the graph of some $C^1$-map from $\mathbb{R}^m$ to $\mathbb{R}^{n-m}$ whose gradient is bounded;

(R.2) $|f| \leq C/r^2_\Sigma$ on $\mathbb{R}^n \setminus B_R(\Sigma)$

for positive constants $C$, $l > 2$ and $R$, where $r_\Sigma(x) = \inf_{y \in \Sigma} |x-y|$, and $B_R(\Sigma)$ is the $R$-neighborhood of $\Sigma$.

Then, for any small enough positive number $\beta$, the equation

\[
\left\{ \begin{array}{l}
-4\frac{n-1}{n-2} \Delta u = fu^{\frac{(n+2)}{(n-2)}} \\
u > 0
\end{array} \right.
\]

possesses a bounded smooth solution $u$ which is also bounded away from zero, and which has the following property:

(R.3) $|u-\beta| \leq \left\{ \begin{array}{ll}
C'/r^{l-2}_\Sigma & \text{if } l < n-m \\
C'/r^{n-m-2-\epsilon}_\Sigma & \text{if } n-m \leq l < n-m+2 \\
C'/r^{n-m}_\Sigma & \text{if } n-m+2 \leq l
\end{array} \right.$

for a positive constant $C'$, where a positive number $\epsilon$ can be chosen arbitrarily small.

The same assertion holds when $\Sigma$ is the union of a finite family of submanifolds of $\mathbb{R}^n$ each of which satisfies the condition (R.1) (cf. Remark 2.2). Ni [13] proved the same assertion as above in the case where $\Sigma$ is an affine subspace of $\mathbb{R}^n$ (see [ibid., Theorem 1.4], and also [12] and [11]). He constructed a supersolution and a subsolution of the equation ($\ast n'$) which are symmetric with respect to $\Sigma$ by solving certain ordinary differential equations. However, it seems to be difficult to apply the method to our case. Actually, we try to construct a supersolution and a subsolution directly, based on the condition on $\Sigma$.

Moreover, our method is applicable to other situations. In fact, it yields the following results.

**THEOREM II.** Let $\Sigma$ be a subset of $\mathbb{R}^n$ ($n \geq 3$) and $f$ a bounded smooth function on $\mathbb{R}^n$. Suppose $\Sigma$ and $f$ satisfy the following conditions:

(R.1') \[
\sup_{x \in \mathbb{R}^n} \int_{B_\delta(\Sigma)} \frac{dy}{(|x-y|^2 + 1)^{n/2}} < +\infty
\]

for positive numbers $\delta$ and $\alpha \leq n-2$;

(R.2) as in Theorem I with
Then the equation \((n)\) possesses infinitely many bounded smooth solutions each of which is also bounded away from zero.

**Theorem III.** Let \((M, g)\) be a complete, noncompact, simply connected Riemannian manifold \((n = \dim M \geq 2)\) satisfying
\[
- A^2 \leq \text{the sectional curvatures} \leq -B^2
\]
for some positive constants \(A\) and \(B\) such that
\[
\left(\frac{A}{B}\right)^2 \leq \frac{(n-1)^2}{n(n-2)}.
\]
Let \(\Sigma\) be a subset of \(M\), and \(f\) a bounded smooth function on \(M\). Suppose \(\Sigma\) and \(f\) satisfy the following conditions:

\((H.1)\)
\[
\sup_{x \in M} \int_{B_{\delta}(x)} \frac{dy}{\cosh \{Bd(y, x)\}^a} < +\infty
\]
for a positive number \(\delta\), where
\[
\alpha = \frac{n - 1 + \{(n - 1)^2 - n(n - 2)(A/B)^2\}^{1/2}}{2} \quad (= 1 \text{ if } n = 2);
\]
\[(H.2)\]
\[
-a^2 \leq f \leq \left\{ \begin{array}{ll}
b^2 & \text{on } M \setminus B_{R}(\Sigma) \\ 0 & \text{on } M
\end{array} \right.
\]
for positive constants \(a, b, R\) and a certain positive constant \(c\) depending only on \(A, B, a, b, R\) and \(\Sigma\), where \(\rho(x) = \inf_{y \in \Sigma} d_{g}(x, y)\).

Then the equation \((n)\) possesses a bounded smooth solution (which is also bounded away from zero if \(n \geq 3\)).

Aviles and McOwen [2] proved the same assertion as above in the case \(\Sigma\) is a point or \(B_{R}(\Sigma)\) is compact (see [ibid., Theorems 1 and 4]). Indeed, when \(\Sigma\) is compact, the condition \((H.1)\) is obviously satisfied. On the other hand, even if \(\Sigma\) is noncompact, we can construct examples of \(\Sigma\) satisfying the condition \((H.1)\). For instance, when \(\Sigma\) is the union of a certain family of totally geodesic submanifolds of \(M\), the condition \((H.1)\) is satisfied (cf. Section 5). Furthermore, if \(\Sigma\) is invariant under the action of a certain nontrivial subgroup \(\Gamma\) of \(\text{Isom}(M)\), then our supersolution and subsolution are also \(\Gamma\)-invariant. Hence, we can regard them as those on \(M/\Gamma\) which is not simply connected (cf. Section 7).

Sections 2–3 (resp. 4–7) are devoted to the case where \((M, g)\) is the Euclidean space \((\mathbb{R}^n, g_0)\) (resp. the case where \((M, g)\) has negative curvature).
In Sections 2 and 3, we give the proofs of Theorems I and II, examples of \( \Sigma \), and some remarks on generalization to other equations (cf. [14] etc.). A proof of Theorem III and examples of \( \Sigma \) are given in Sections 4 and 5. We observe in Section 6 that the condition (H.1) is sharp in a sense, when \( M \) is the hyperbolic plane \( H^2 = H^2(-1) \) of constant curvature \(-1\). In Section 7, we discuss the case where \( M = H^2/G \) is not simply connected.

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2. Proofs of Theorems I and II.  We recall first the following:

THE METHOD OF SUPERSOLUTIONS AND SUBSOLUTIONS.  Let \((M, g)\) be a complete Riemannian manifold, and \( f \) a smooth function on \( M \). If there exist a weak supersolution \( u^+ \) and a weak subsolution \( u^- \) of the equation (\( *n \)) such that \( u^+, u^- \in H^p_{\text{loc}}(M) \) \((p > n)\) and \( u^- \leq u^+ \), then the equation (\( *n \)) possesses a smooth solution \( u \) satisfying \( u^- \leq u \leq u^+ \).

This is well-known and we omit the proof (see, for instance, [8], [10], [13] and [3]). First, we give a proof of Theorem II for convenience.

PROOF OF THEOREM II.  For any \( y \in \mathbb{R}^n \), let \( r_y := r(y) \) and \( u_y := 1/(r_y^2 + 1)^{\alpha/2} \). By direct computation, we see that

\[
-\Delta u_y = \alpha(r_y^2 + 1)^{-\alpha/2 - 2}(n - \alpha)r_y^2 + n.
\]

Set

\[
u_x := \int_{B_\delta(x)} u_y dy - \inf \int_{B_\delta(x)} u_y dy
\]

and \( u_x := \sup_{\mathbb{R}^n} u_x \) which is finite by the assumption (R.1'). Moreover, it follows easily that \( u_x \) is a smooth function on \( \mathbb{R}^n \), and \( -\Delta u_x \) satisfies

\[
-\Delta u_x \geq \begin{cases} 
\alpha(n - 2 - \alpha) \int_{B_\delta(x)} (r_y^2 + 1)^{-\alpha/2 - 1} dy & \text{if } \alpha < n - 2 \\
\alpha n \int_{B_\delta(x)} (r_y^2 + 1)^{-\alpha/2 - 2} dy & \text{if } \alpha = n - 2 .
\end{cases}
\]

Now, for any \( x \in \mathbb{R}^n \) and every \( r_0 > r_x(x) \), there is an \( x_0 \in \Sigma \) such that \( r_x(x) \leq |x - x_0| < r_0 \). Since

\[
r_x(x) = |x - y| \leq |x - x_0| + |x_0 - y| < r_0 + \delta
\]

for any \( y \in B_\delta(x_0) \subset B_\delta(\Sigma) \), it follows that
$-\Delta u_\pm \geq (n-2-\alpha) \int_{B_\rho(x_0)} (r_\rho^2 + 1)^{-\alpha/2 - 1} \, dy$

$\geq (n-2-\alpha) \text{vol}(B_\rho) [(r_\rho + \delta)^2 + 1]^{-\alpha/2 - 1}$

$= (n-2-\alpha) \text{vol}(B_\rho) [(r_\rho + \delta)^2 + 1]^{-1/2}$ \quad \text{if } \alpha < n-2,

and

$-\Delta u_\pm \geq \alpha \int_{B_\rho(x_0)} (r_\rho^2 + 1)^{-\alpha/2 - 2} \, dy$

$\geq \alpha \cdot \text{vol}(B_\rho) [(r_\rho + \delta)^2 + 1]^{-\alpha/2 - 2}$

$= \alpha \cdot \text{vol}(B_\rho) [(r_\rho + \delta)^2 + 1]^{-1/2}$ \quad \text{if } \alpha = n-2.$

Since we can take $r_\rho$ arbitrarily near $r_\Sigma$, we get

$-\Delta u_\pm \geq C_1/[(r_\Sigma + \delta)^2 + 1]^{1/2},$

where

$C_1 := \begin{cases} 
(n-2-\alpha) \text{vol}(B_\rho) & \text{if } \alpha < n-2 \\
\alpha \cdot \text{vol}(B_\rho) & \text{if } \alpha = n-2.
\end{cases}$

Now, we may assume $R \geq 1$. It follows from simple computations that

$(r_\Sigma + \delta)^2 + 1 \leq [2 + (2\delta^2 + 1)/R^2] r_\Sigma^2$ \quad \text{on } \mathbb{R}^n \setminus B_R(\Sigma).

Hence we get

$-\Delta u_\pm \geq \begin{cases} 
C_2/r_\Sigma^2 & \text{on } \mathbb{R}^n \setminus B_R(\Sigma) \\
C_3 & \text{in } B_R(\Sigma),
\end{cases}$

where

$C_2 := C_1/[2 + (2\delta^2 + 1)/R^2]^{1/2}, \quad C_3 := C_1/[(R + \delta)^2 + 1]^{1/2}.$

Let us now set $u_\pm := \beta(1 \pm \eta_\pm u_\Sigma)$ with positive numbers $\beta$, $\eta_+$ and $\eta_-$, where $\eta_+$ is chosen arbitrarily, $\eta_-$ is chosen so as to satisfy $\eta_- < 1/|u_\Sigma|$, while $\beta$ is chosen so as to satisfy

$\beta \leq \beta_0 := \left[ \frac{n-2}{4(n-1)} \max \left\{ \left( \frac{1 + \eta_+ u_\Sigma}{\eta_+ C_2} \right)^{(n-2)/(n+2)} C \sup f, \left( \frac{1 + \eta_+ u_\Sigma}{\eta_+ C_3} \right)^{(n-2)/(n+2)} \inf f \right\} \right]^{-\frac{(n-2)/4}{n-2}}.$

Then we get
\( u_+^{-(n+2)/(n-2)} \left( -4 \frac{n-1}{n-2} \Delta u_+ \right) \)

\[ = \beta^{-(n+2)/(n-2)}(1 + \eta_+ u_+^{-2})^{-(n+2)/(n-2)} \left( -4 \frac{n-1}{n-2} \beta \eta_+ \Delta u_+ \right) \]

\[ \geq \beta^{-4/(n-2)}(1 + \eta_+ u_+^{-2})^{-(n+2)/(n-2)} \cdot 4 \frac{n-1}{n-2} \eta_+ (-\Delta u_+) \]

\[ \leq \beta^{-4/(n-2)}(1 + \eta_+ u_+^{-2})^{-(n+2)/(n-2)} \cdot 4 \frac{n-1}{n-2} \eta_+ C_2 / r_2^2 \geq C / r_2^2 \geq f \quad \text{on} \quad \mathbb{R}^n \backslash B_R(\Sigma) \]

\[ \leq \beta^{-4/(n-2)}(1 + \eta_+ u_+^{-2})^{-(n+2)/(n-2)} \cdot 4 \frac{n-1}{n-2} \eta_+ C_3 \geq \sup f \geq f \quad \text{in} \quad B_R(\Sigma) , \]

that is

\[ -4 \frac{n-1}{n-2} \Delta u_+ \geq f u_+^{-(n+2)/(n-2)} \quad \text{on} \quad \mathbb{R}^n . \]

On the other hand, we get

\( -u_-^{-(n+2)/(n-2)} \left( -4 \frac{n-1}{n-2} \Delta u_- \right) \)

\[ = \beta^{-(n+2)/(n-2)}(1 - \eta_- u_-^{-2})^{-(n+2)/(n-2)} \left( -4 \frac{n-1}{n-2} \beta \eta_- \Delta u_- \right) \]

\[ \geq \beta^{-4/(n-2)} \cdot 4 \frac{n-1}{n-2} \eta_- (-\Delta u_-) \]

\[ \leq \beta^{-4/(n-2)} \cdot 4 \frac{n-1}{n-2} \eta_- C_2 / r_2^2 \geq C / r_2^2 \geq -f \quad \text{on} \quad \mathbb{R}^n \backslash B_R(\Sigma) \]

\[ \leq \beta^{-4/(n-2)} \cdot 4 \frac{n-1}{n-2} \eta_- C_3 \geq \inf f \geq -f \quad \text{in} \quad B_R(\Sigma) , \]

that is

\[ -4 \frac{n-1}{n-2} \Delta u_- \leq f u_-^{-(n+2)/(n-2)} \quad \text{on} \quad \mathbb{R}^n . \]

Hence \( u_+ \) and \( u_- \) are respectively a supersolution and a subsolution of the equation \((\ast n')\). Since \( u_- \leq u_+ \), by the method of supersolutions and subsolutions, the equation \((\ast n')\) possesses a bounded smooth solution \( u \) satisfying \( u_- \leq u \leq u_+ \). It is clear that

\[ |u - \beta| \leq \beta \max \{ \eta_+, \eta_- \} u_\Sigma \quad \text{on} \quad \mathbb{R}^n , \]
from which, for any positive number $\beta \leq \beta_0$, the equation (*n') possesses a solution satisfying $\lim_{i \to \infty} u(x_i) = \beta$ for any minimizing sequence $\{x_i\}_{i=1}^{\infty}$ of $u_\Sigma$. Namely the equation (*n') possesses infinitely many solutions. q.e.d.

**PROOF OF THEOREM I.** Since the assertion in the case $m=0$ is known as we referred to in Section 1, we assume $m \geq 1$ in what follows.

By the condition (R.1), we may assume that the submanifold $\Sigma$ is given as

$$\Sigma = \{(x_1, h(x_1)) \in \mathbb{R}^m \times \mathbb{R}^{n-m}\},$$

where $h: \mathbb{R}^m \to \mathbb{R}^{n-m}$ is a $C^1$-map with $|\partial h/\partial x_1| \leq C_\Sigma$ for a positive constant $C_\Sigma$.

Let $r_y$ and $u_y$ be as in the proof of Theorem II. Set

$$u_\Sigma := \int_\Sigma u_y ds(y) = \int_\Sigma (r^2 + 1)^{-\alpha/2} ds(y),$$

where $ds(y)$ is the volume element of $\Sigma$, and

$$\alpha := \begin{cases} l + m - 2 & \text{if } l < n - m \\ n - 2 - \varepsilon & \text{for some } \varepsilon > 0 \text{ if } n - m \leq l < n - m + 2 \\ n - 2 & \text{if } n - m + 2 \leq l. \end{cases}$$

For any $x = (x_1, x_2) \in \mathbb{R}^m \times \mathbb{R}^{n-m} = \mathbb{R}^n$, let $s_\Sigma(x) := |x_2 - h(x_1)|$. By the assumption (R.1),

$$\Sigma \subset \{y = (y_1, y_2) \mid |y_2 - h(x_1)| \leq C_\Sigma |y_1 - x_1|\}$$

for any $x_1 \in \mathbb{R}^m$, from which it follows that

$$\frac{s_\Sigma}{\sqrt{C_\Sigma^2 + 1}} \leq r_\Sigma \leq s_\Sigma \quad \text{on } \mathbb{R}^n.$$

On the other hand, since $ds(x) = \{\det (g_{ij})\}^{1/2} dx_1$ and

$$g_{ij} = \delta_{ij} + \sum_{k=1}^{n-m} \frac{\partial h^k}{\partial x_1} \cdot \frac{\partial h^k}{\partial x_1} \quad \text{for } 1 \leq i, j \leq m$$

and $|\partial h/\partial x_1| \leq C_\Sigma$, we get

$$dx_1 \leq ds(x) \leq V dx_1$$

for some positive constant $V \geq 1$ depending only on $C_\Sigma$.

Now, if we denote $s_1 := s_\Sigma/(C_\Sigma^2 + 1)^{1/2}$ and $r := |x_1 - y_1|$ for convenience, then obviously, for any $y \in \Sigma$,

$$u_y = (r^2 + 1)^{-\alpha/2} \leq \min \{(s_1^2 + 1)^{-\alpha/2}, (r^2 + 1)^{-\alpha/2}\} \quad \text{on } \mathbb{R}^n,$$

from which it follows that
\[
\int_{\Sigma} u_y ds(y) \leq \int_{\Sigma} u_y V dy_1 \\
\leq V \left\{ \int_{r \leq s_1} (s_1^2 + 1)^{-a/2} dy_1 + \int_{r \geq s_1} (r^2 + 1)^{-a/2} dy_1 \right\} \\
\leq V \omega \left( s_1^{-a} \int_0^{s_1} r^{m-1} dr + \int_{s_1}^{\infty} r^{-a+m-1} dr \right) \\
= V \omega \left( \frac{1}{m} + \frac{1}{\alpha-m} \right) s_1^{m-a},
\]
where \( \omega := \text{vol}\{S^{m-1}(1)\} \). On the other hand, since \( u_y \leq 1 \),
\[
\int_{\Sigma} u_y ds(y) \leq V \left\{ \int_{r \leq 1} dy_1 + \int_{r \geq 1} (r^2 + 1)^{-a/2} dy_1 \right\} \\
\leq V \omega \left( \frac{1}{m} + \frac{1}{\alpha-m} \right).
\]
Hence we get
\[
u \leq C_0 \min\{1, s_1^{m-a}\},
\]
where
\[
C_0 := V \omega \left( \frac{1}{m} + \frac{1}{\alpha-m} \right).
\]
From this estimate and the calculation in the proof of Theorem II, it follows easily that \( u_\Sigma \) is a smooth function on \( \mathbb{R}^n \), and \( \Delta u_\Sigma \) satisfies
\[
- \Delta u_\Sigma \geq \left\{ \begin{array}{ll}
\frac{\alpha(n-2-\alpha)}{2} \int_{\Sigma} (r^2 + 1)^{-a/2-1} ds(y) & \text{if } \alpha < n-2 \\
\alpha n \int_{\Sigma} (r^2 + 1)^{-a/2-2} ds(y) & \text{if } \alpha = n-2.
\end{array} \right.
\]
Now, for any \( x \in \mathbb{R}^n \) and every \( y \in \Sigma \), we have
\[
(r_y(x))^2 = |x-y|^2 = |x_1 - y_1|^2 + |x_2 - y_2|^2 \leq |x_1 - y_1|^2 + (s_2(x) + C_2 |x_1 - y_1|)^2 \\
\leq |x_1 - y_1|^2 + 2s_2(x)^2 + 2C_2^2 |x_1 - y_1|^2 = (2C_2^2 + 1) |x_1 - y_1|^2 + 2s_2(x)^2.
\]
Hence
\[
\int_{\Sigma} (r^2 + 1)^{-a/2-1} ds(y) \geq \omega \int_0^{\infty} ((2C_2^2 + 1)r^2 + 2s_2^2 + 1)^{-a/2-1} r^{m-1} dr
\]
If we denote \( s_2 := (2 + s_x^{-2})^{1/2} \) for convenience, then obviously

\[
(2C_2^2 + 1) t^2 + s_2^2 \leq \begin{cases} 
2(C_2^2 + 1) s_2^2 & \text{for any } t \leq s_2 \\
2(C_2^2 + 1) t^2 & \text{for any } t \geq s_2 
\end{cases}
\]

from which it follows that

\[
\int_\Sigma (r_x^2 + 1)^{-a/2 - 1} d\gamma(y) 
\geq \omega s_x^{m-a-2} \left[ \int_0^{s_2} \{2(C_2^2 + 1) s_2^2\}^{-a/2 - 1} t^{m-1} dt + \int_{s_2}^{\infty} \{2(C_2^2 + 1) t^2\}^{-a/2 - 1} t^{m-1} dt \right] 
\]

\[
= \omega s_x^{m-a-2} \left[ (2(C_2^2 + 1))^{-a/2 - 1} s_2^{-a/2 - 2} \frac{1}{m} s_x^m + (2(C_2^2 + 1))^{-a/2 - 1} \frac{1}{\alpha - m + 2} s_x^{m-4} \right] 
\]

\[
= \omega (2(C_2^2 + 1))^{-a/2 - 1} \left( \frac{1}{m} + \frac{1}{\alpha - m + 2} \right) (s_x s_2)^{m-2} 
\]

\[
\geq \omega (2(C_2^2 + 1))^{-a/2 - 1} \left( \frac{1}{m} + \frac{1}{\alpha - m + 2} \right) (2(C_2^2 + 1) r_x^2 + 1)^{(m-2) a/2} 
\]

Similarly

\[
\int_\Sigma (r_x^2 + 1)^{-a/2 - 2} d\gamma(y) 
\geq \omega (2(C_2^2 + 1))^{-a/2 - 2} \left( \frac{1}{m} + \frac{1}{\alpha - m + 4} \right) (2(C_2^2 + 1) r_x^2 + 1)^{(m-4) a/2} 
\]

Hence we get

\[- \Delta u_x \geq C_1/(2(C_2^2 + 1) r_x^2 + 1)^{l/2},\]

where

\[
C_1 := \begin{cases} 
\alpha(n-2-\alpha)\omega (2(C_2^2 + 1))^{-a/2 - 1} \left( \frac{1}{m} + \frac{1}{\alpha - m + 2} \right) & \text{if } l < n-m+2 \\
\alpha n \omega (2(C_2^2 + 1))^{-a/2 - 2} \left( \frac{1}{m} + \frac{1}{\alpha - m + 4} \right) & \text{if } n-m+2 \leq l.
\end{cases}
\]

Now, we may assume \( R \geq 1 \). It follows easily that

\[
2(C_2^2 + 1) r_x^2 + 1 \leq \{2(C_2^2 + 1) + 1/R^2\} r_x^2 \quad \text{on } \mathbb{R}^n \setminus B_R(\Sigma).
\]
Hence we get
\[-\Delta u_\Sigma \geq \begin{cases}
C_2/r_\Sigma^l & \text{on } \mathbb{R}^n \setminus B_R(\Sigma) \\
C_3 & \text{in } B_R(\Sigma),
\end{cases}\]
where
\[C_2 := C_1/[2(C_0^2 + 1) + 1/R^2]^{l/2}, \quad C_3 := C_1/[2(C_0^2 + 1)R^2 + 1]^{l/2}.\]

Using this estimate, we can prove, by the same method as in the proof of Theorem II, that there exist positive numbers \(\eta_+, \eta_-\) and \(\beta_0\) such that, for any positive number \(\beta \leq \beta_0\), the equation \((*)'\) possesses a smooth bounded solution \(u\) satisfying
\[|u - \beta| \leq \beta \cdot \max \{\eta_+, \eta_-\} u_\Sigma \text{ on } \mathbb{R}^n.\]

Moreover, since
\[0 < u_\Sigma \leq C_0 e^{-\alpha_\Sigma m - \alpha} = C_0 \left(\frac{s_\Sigma}{\sqrt{C_\Sigma^2 + 1}}\right)^m \leq C_0 \left(\frac{r_\Sigma}{\sqrt{C_\Sigma^2 + 1}}\right)^m = C_0(C_0^2 + 1)^{(m-m)/2}/r_\Sigma^{m-m} \text{ on } \mathbb{R}^n,
\]
we get the estimate (R.3) with
\[C' := \beta \cdot \max \{\eta_+, \eta_-\} C_0(C_0^2 + 1)^{(m-m)/2}.\]

**Remark 2.1.** From our proof, it is not hard to see that we can replace the condition (R.1) by the following condition:

(R.1') \[\langle \langle G(x), q \rangle \rangle := \min \{|\pi_q(v)| | v \in G(x), |v| = 1\} > \varepsilon_\Sigma \text{ on } \Sigma \setminus \Sigma_0,
\]
for some \(q \in G(m, n - m)\), a positive number \(\varepsilon_\Sigma\), and a compact subset \(\Sigma_0\) of \(\Sigma\), where \(G : \Sigma \to G(m, n - m)\) is the Gauss map of \(\Sigma\).

**Remark 2.2.** We can replace \(\Sigma\) in Theorem I by the union of a finite family \(\{\Sigma_i\}_{i=1}^k\) of submanifolds of \(\mathbb{R}^n\) with \(m_i = \dim \Sigma_i \leq n - 3\) such that each \(\Sigma_i\) satisfies the condition (R.1) with \(h = h_i\). Indeed, for any \(1 \leq i \leq k\), set
\[u_{\Sigma_i} := \int_{\Sigma_i} u_y ds(y) = \int_{\Sigma_i} (r^2 + 1)^{-\alpha/2} ds(y),
\]
where
\[\alpha_i := \begin{cases}
l + m_i - 2 & \text{if } l < n - m_i \\
n - 2 - \varepsilon & \text{for some } \varepsilon > 0 & \text{if } n - m_i \leq l < n - m_i + 2 \\
n - 2 & \text{if } n - m_i + 2 \leq l.
\end{cases}\]
Then, by the proof of Theorem I,
\[-\Delta u_{\Sigma_i} \geq \begin{cases} C_{2i}/r_{\Sigma_i}^l & \text{on } \mathbb{R}^n \setminus B_R(\Sigma_i) \\ C_{3i} & \text{in } B_R(\Sigma_i) \end{cases},\]
\[u_{\Sigma_i} \leq C_{4i}/r_{\Sigma_i}^{a_i-m_i} \text{ on } \mathbb{R}^n,\]
where
\[C_{4i} := C_0(C_2^2 + 1)^{(a_i-m_i)/2}.\]

Set
\[u_\Sigma := \frac{1}{k} \sum_{i=1}^k u_{\Sigma_i}.
\]

Now, since \(r_{\Sigma} = \min_i r_{\Sigma_i}\), it is clear that
\[r_{\Sigma} \leq r_{\Sigma_i} \text{ for any } 1 \leq i \leq k,\]
\[r_{\Sigma}(x) = r_{\Sigma_i}(x) \text{ for some } i \text{ depending on } x \in \mathbb{R}^n,\]
from which it follows that
\[-\Delta u_{\Sigma} = \sum_{i=1}^k (-\Delta u_{\Sigma_i}) \geq \begin{cases} \min_{i} C_{2i}/r_{\Sigma_i}^l & \text{on } \mathbb{R}^n \setminus B_R(\Sigma) = \bigcap_i \{\mathbb{R}^n \setminus B_R(\Sigma_i)\} \\ \min_i C_{3i} & \text{in } B_R(\Sigma) = \bigcup_i B_R(\Sigma_i) \end{cases},\]
\[u_{\Sigma} \leq \sum_{i=1}^k C_{4i}/r_{\Sigma_i}^{a_i-m_i} \leq \left( \sum_{i=1}^k C_{4i} \right) / r_{\Sigma}^{\min\{a_i-m_i\}} \text{ on } \mathbb{R}^n \setminus B_R(\Sigma),\]
where
\[\min_i \{a_i-m_i\} = \begin{cases} l-2 & \text{if } l < n-m \\ n-m-2-\varepsilon & \text{if } n-m \leq l < n-m+2 \\ n-m-2 & \text{if } n-m+2 \leq l \end{cases}\]
and \(m := \max_i m_i\). Using these estimates, we can prove our assertion by the same method as in the proof of Theorem I.

3. Examples and generalization. In this section, we first give examples for submanifolds \(\Sigma\) of \(\mathbb{R}^n\) such that the assertion of Theorem I holds. Secondly, we discuss certain equations in more general forms.

Example 3.1. Let \(\vec{h} := \mathbb{R} \to \mathbb{R}\) be a \(C^1\)-function with \(d\vec{h}/dt\) bounded above or below, and
\[\Sigma := \{(t, \vec{h}(t), 0, \ldots, 0) \in \mathbb{R}^n | t \in \mathbb{R}\} \quad (n \geq 4).\]
Even if \( |\frac{d\bar{h}}{dt}| \) is unbounded, e.g., \( \bar{h}(t) = e^t \) or \( \bar{h}(t) \) is a polynomial of odd degree, we see in this case that \( \Sigma \) satisfies the condition (R.1) by a suitable coordinate change (c.f. Remark 2.1).

**Example 3.2.** Let \( \Sigma \) be as in Example 3.1. When \( \bar{h}(t) \) is a polynomial of even degree, it is clear that \( \frac{d\bar{h}}{dt} \) is unbounded above and below. However, we can easily show that \( \Sigma \) satisfies the condition (R.1') in Theorem II with \( \alpha > m \). Moreover, if we set

\[
\Sigma_i := \{(t, \bar{h}_i(t), 0, \ldots, 0)\} \quad \text{for} \quad i = 1, 2,
\]

\[
\bar{h}_1(t) := \begin{cases} \bar{h}(t) & \text{for} \ t \geq 0 \\ \bar{h}'(0)t + \bar{h}(0) & \text{for} \ t \leq 0 \end{cases},
\]

\[
\bar{h}_2(t) := \begin{cases} \bar{h}'(0)t + \bar{h}(0) & \text{for} \ t \geq 0 \\ \bar{h}(t) & \text{for} \ t \leq 0 \end{cases},
\]

then obviously both \( \Sigma_1 \) and \( \Sigma_2 \) satisfy the assumption in Example 3.1. Hence they satisfy the condition (R.1). Since \( \Sigma \subset \Sigma_1 \cup \Sigma_2 \), we see that the assertion of Theorem I holds for \( \Sigma \) with the property which is somewhat weaker than the property (R.3) (cf. Remark 2.2).

In the remainder of this section, we mention some generalization of the method used so far to the following equation:

\[
-\Delta u(x) = f(x)F(x, u(x)) \quad \text{on} \quad \mathbb{R}^n,
\]

where \( f(x) \) is a bounded locally Hölder continuous function on \( \mathbb{R}^n \), and \( F(x, t) \) is a nonnegative locally Hölder continuous function on \( \mathbb{R}^n \times (a, b) \) (\( -\infty \leq a < b \leq +\infty \)) with one of the following properties:

(F.1) \( a > -\infty \) and \( F(x, t) \to 0 \) as \( t \to a + 0 \) uniformly in \( x \);

(F.2) \( a = -\infty \) and \( F(x, t) \to 0 \) as \( t \to -\infty \) uniformly in \( x \);

(F.3) \( b < +\infty \) and \( F(x, t) \to 0 \) as \( t \to b - 0 \) uniformly in \( x \);

(F.4) \( b = +\infty \) and \( F(x, t) \to 0 \) as \( t \to +\infty \) uniformly in \( x \);

(F.5) \( a = -\infty \), \( f(x) \leq 0 \) and \( F(x, t) \) is bounded on \( (a, c] \) for any \( c \in (a, b) \);

(F.6) \( b = +\infty \), \( f(x) \geq 0 \) and \( F(x, t) \) is bounded on \( [c, b) \) for any \( c \in (a, b) \);

(F.7) \( a = -\infty \), \( b = +\infty \) and \( F(x, t) \) is bounded.

In this situation, we can apply the same method as in the proofs of Theorems I and II to showing the following existence results which respectively include the asser-
TENSIONS OF THEOREMS I AND II.

**Theorem 3.3.** Let $\Sigma$ be a submanifold of $\mathbb{R}^n$ ($n \geq 3$) with $m = \dim \Sigma \leq n - 3$, and $f$ and $F$ as above. Suppose $\Sigma$ and $f$ satisfy the conditions (R.1) and (R.2). Then, for any $\beta \in I$, the equation $(**)$ possesses a $C^2$-solution $u$ which is bounded away from both $a$ and $b$, and which has the property (R.3), where

$$I = \begin{cases} (a, b_0) & \text{for some } b_0 \in (a, b) \text{ when (F.1) or (F.2) holds} \\ (a_0, b) & \text{for some } a_0 \in [a, b) \text{ when (F.3) or (F.4) holds} \\ (a, b) & \text{when (F.5), (F.6) or (F.7) holds}. \end{cases}$$

**Theorem 3.4.** Let $\Sigma$ be a subset of $\mathbb{R}^n$ ($n \geq 3$), and $f$ and $F$ as above. Suppose $\Sigma$ and $f$ satisfy the conditions (R.1') with $r \leq n - 2$ and (R.2) with the same $l$ as in Theorem II. Then the equation $(**)$ possesses infinitely many $C^2$-solutions each of which is bounded away from both $a$ and $b$.

**Remarks 3.5.** (1) The equation $-\Delta u = fu^p$ ($p > 1$) satisfies (F.1) with $a = 0$ and $b = +\infty$.

(2) The equation $-\Delta u = fu^p$ ($p < 1$) satisfies (F.4) with $a = 0$.

(3) The equation $-\Delta u = fe^u$ satisfies (F.2) with $b = +\infty$. In addition, if $f \leq 0$, then this equation satisfies (F.5).

4. Proof of Theorem III. We recall first the following standard theorem:

**Comparison Theorem.** Let $(M, g)$ be a complete, noncompact, simply connected Riemannian manifold ($n = \dim M \geq 2$) satisfying

$$-A^2 \leq \text{the sectional curvatures} \leq -B^2$$

for some positive constants $A$ and $B$, and let $\Sigma$ be a totally geodesic submanifold of $M$ with $m = \dim \Sigma \leq n - 1$. Then the distance $\rho_\Sigma := d_g(\cdot, \Sigma)$ to $\Sigma$ satisfies the following estimates on $M \setminus \Sigma$:

$$\begin{align*}
|\nabla_{\xi} \rho_\Sigma| &\equiv 1, \\
|\nabla^2_{\xi} \rho_\Sigma| &\leq nA \coth A \rho_\Sigma, \\
\Delta_\xi \rho_\Sigma &\leq B((n-m-1)(\coth B \rho_\Sigma) + m(\tanh B \rho_\Sigma)).
\end{align*}$$

The equality holds in (4.3) if and only if $(M, g)$ is the hyperbolic space $\mathbb{H}^n(-B^2)$ of constant curvature $-B^2$.

This is well-known and we omit the proof (see, for instance, [5] and [6]).

**Proof of Theorem III.** For any $y \in M$, let $\rho_y := \rho_{(y)}$, and $u_y := 1/(\cosh B \rho_y)^p$. By direct computation, we see that
\[-\Delta g u_x = B^2 (\cosh B \rho_y)^{-\alpha - 2} \left\{ -\frac{\alpha^2}{2} (\cosh B \rho_y)^2 + \frac{\alpha}{B} (\cosh B \rho_y)(\sinh B \rho_y) \Delta g \rho_y + \alpha (\alpha + 1) \right\}.\]

Now by (4.3) with \( \Sigma = \{y\} \) (hence \( m = \dim \{y\} = 0 \)), we have

\[ \Delta g \rho_y \geq B(n-1) \coth B \rho_y , \]

from which

\[-\Delta g u_x \geq B^2 (\cosh B \rho_y)^{-\alpha - 2} \left\{ -\alpha (\alpha - n + 1)(\cosh B \rho_y)^2 + \alpha (\alpha + 1) \right\} \cdot \]

Set

\[ u_\Sigma := \int_{B_\delta(\Sigma)} u_y dy , \]

and \( \bar{u}_\Sigma := \sup_M u_\Sigma \) which is finite by the assumption (H.1). By (4.1) and (4.2), we can easily get \( u_\Sigma \in C^2(M) \), and \( \Delta g u_\Sigma \) satisfies

\[ -\Delta g u_\Sigma \geq \int_{B_\delta(\Sigma)} B^2 (\cosh B \rho_y)^{-\alpha - 2} \left\{ -\alpha (\alpha - n + 1)(\cosh B \rho_y)^2 + \alpha (\alpha + 1) \right\} dy . \]

**The case \( n = 2 \).** In this case, since \( \alpha = 1 \),

\[ -\Delta g u_\Sigma \geq \int_{B_\delta(\Sigma)} 2B^2 (\cosh B \rho_y)^{-3} dy > 0 . \]

Now, for any \( x \in M \) and every \( \rho_0 > \rho_\delta(x) \), there is an \( x_0 \in \Sigma \) such that \( \rho_\delta(x) \leq d_\delta(x, x_0) < \rho_0 \). Since

\[ \rho_\delta(x) = d_\delta(x, y) \leq d_\delta(x, x_0) + d_\delta(x_0, y) < \rho_0 + \delta \]

for any \( y \in B_\delta(x_0) \subset B_\delta(\Sigma) \), it follows that

\[ -\Delta g u_\Sigma \geq \int_{B_\delta(x_0)} 2B^2 (\cosh B \rho_y)^{-3} dy \geq 2B^2 \text{vol}_B(B_\delta) [\cosh \{B(\rho_0 + \delta)\}]^{-3} , \]

where \( \text{vol}_B \) is the volume with respect to the metric of \( H^n(-B^2) \). Since we can take \( \rho_0 \) arbitrarily near \( \rho_\delta(x) \), we get

\[ -\Delta g u_\Sigma \geq 2B^2 \text{vol}_B(B_\delta) [\cosh \{B(\rho_\delta + \delta)\}]^{-3} . \]

It is clear that

\[ -\Delta g u_\Sigma \geq 2B^2 \text{vol}_B(B_\delta) [\cosh \{B(R + \delta)\}]^{-3} =: C_1 > 0 \quad \text{in} \quad B_R(\Sigma) . \]

Set \( u_+ := \beta u_\Sigma + \log(2A^2/b^2) \), where \( \beta \) is chosen so as to satisfy \( \beta > 2A^2/C_1 \). If we take \( \varepsilon := (\beta C_1 - 2A^2)b^2/2A^2 \exp(\beta u_\Sigma) > 0 \), then we get
\[-\Delta_g u_+ + S_g \geq -\beta \Delta_g u - 2A^2 \geq \beta C_1 - 2A^2 = \varepsilon \exp\left(\beta \mu_x + \log \frac{2A^2}{b^2}\right) \geq \varepsilon e^{\mu_+} \equiv f e^{\mu_+} \quad \text{in } B_R(\Sigma),\]

and

\[-\Delta_g u_+ + S_g \geq -2A^2 = -b^2 \exp\left(\log \frac{2A^2}{b^2}\right) > -b^2 e^{\mu_+} \equiv f e^{\mu_+} \quad \text{on } M.\]

On the other hand, if we set \(u_- := \log(2B^2/a^2)\), then we have

\[-\Delta_g u_- + S_g \leq -2B^2 = -a^2 \exp\left(\log \frac{2B^2}{a^2}\right) = -a^2 e^{\mu_-} \leq f e^{\mu_-} \quad \text{on } M.\]

Hence \(u_+\) and \(u_-\) are respectively a supersolution and a subsolution of the equation (\(\ast 2\)). Since \(u_- \leq u_+\), by the method of supersolutions and subsolutions, the equation (\(\ast 2\)) possesses a bounded solution.

**The Case \(n \geq 3\).** We have first by (4.4),

\[-\frac{4}{n-2} \Delta_g \mu_x + S_g \mu_x \geq -\frac{4}{n-2} \Delta_g \mu_x + A^2 n(n-1) u_2 \]

\[\geq \int_{B_d(\Sigma)} \left[ \frac{4}{n-2} B^2 (\cosh B \rho_y)^{-a-2} \left\{ -\alpha (\alpha + n+1)(\cosh B \rho_y)^2 + \alpha (\alpha + 1) \right\} - A^2 n(n-1)(\cosh B \rho_y)^{-a} \right] dy \]

\[= \int_{B_d(\Sigma)} \frac{4}{n-2} B^2 (\cosh B \rho_y)^{-a-2} \left\{ -\left\{ \alpha^2 - (n-1)\alpha + \frac{n(n-2)}{4} \left( \frac{A}{B} \right)^2 \right\} \right\} \]

\[- (\cosh B \rho_y)^2 + \alpha (\alpha + 1) \right\} dy \]

\[= \int_{B_d(\Sigma)} \frac{4}{n-2} B^2 \alpha (\alpha + 1)(\cosh B \rho_y)^{-a-2} dy > 0.\]

Now, by the same observation as in the case \(n = 2\), we get

\[-\frac{4}{n-2} \Delta_g \mu_x + S_g \mu_x \geq \frac{4}{n-2} B^2 \alpha (\alpha + 1) \text{vol}_n(B_d)[\cosh\{B(R + \delta)\}]^{-a-2}.\]

It is clear that

\[-\frac{4}{n-2} \Delta_g \mu_x + S_g \mu_x \geq \frac{4}{n-2} B^2 \alpha (\alpha + 1) \text{vol}_n(B_d)[\cosh\{B(R + \delta)\}]^{-a-2} \]

\[=: C_2 > 0 \quad \text{in } B_R(\Sigma).\]
Set \( u_+ := \beta u_\Sigma + \{ A^2 n(n-1)/b^2 \}^{(n-2)/4} > 0 \), where \( \beta \) is chosen so as to satisfy \( \beta > \{ A^2 n(n-1)/b^2 \}^{(n-2)/4} \cdot A^2 n(n-1)/C_2 \). If we take
\[
\epsilon := \left[ \beta u_\Sigma + \left\{ \frac{A^2 n(n-1)}{b^2} \right\}^{(n-2)/4} \right] - (n+2)/(n-2)
\]
then we get
\[
-4 \frac{n-1}{n-2} \Delta_g u_+ + S_g u_+ = \beta \left[ -4 \frac{n-1}{n-2} \Delta_g u_\Sigma + S_g u_\Sigma \right] + S_g \left\{ \frac{A^2 n(n-1)}{b^2} \right\}^{(n-2)/4}
\]
\[
\geq \beta C_2 - A^2 n(n-1) \left\{ \frac{A^2 n(n-1)}{b^2} \right\}^{(n-2)/4}
\]
\[
= \epsilon \left[ \beta u_\Sigma + \left\{ \frac{A^2 n(n-1)}{b^2} \right\}^{(n-2)/4} \right]^{(n+2)/(n-2)}
\]
\[
\geq \epsilon u_+^{(n+2)/(n-2)} \geq f u_+^{(n+2)/(n-2)} \quad \text{in } B_R(\Sigma),
\]
and
\[
-4 \frac{n-1}{n-2} \Delta_g u_+ + S_g u_+ > - A^2 n(n-1) \left\{ \frac{A^2 n(n-1)}{b^2} \right\}^{(n-2)/4}
\]
\[
= - b^2 \left[ \left\{ \frac{A^2 n(n-1)}{b^2} \right\}^{(n-2)/4} \right]^{(n+2)/(n-2)}
\]
\[
> - b^2 u_+^{(n+2)/(n-2)} \geq u_+^{(n+2)/(n-2)} \quad \text{on } M.
\]
On the other hand, if we set \( u_- := \{ B^2 n(n-1)/a^2 \}^{(n-2)/4} > 0 \), then we have
\[
-4 \frac{n-1}{n-2} \Delta_g u_- + S_g u_- \leq - B^2 n(n-1) \left\{ \frac{B^2 n(n-1)}{a^2} \right\}^{(n-2)/4}
\]
\[
= - a^2 \left[ \left\{ \frac{B^2 n(n-1)}{a^2} \right\}^{(n-2)/4} \right]^{(n+2)/(n-2)}
\]
\[
= - a^2 u_-^{(n+2)/(n-2)} \leq u_-^{(n+2)/(n-2)} \quad \text{on } M.
\]
Hence \( u_+ \) and \( u_- \) are respectively a supersolution and a subsolution of the equation \((\ast n)\). Since \( 0 < u_- \leq u_+ \), by the method of supersolutions and subsolutions, the equation \((\ast n)\) possesses a solution \( u \) which is bounded between two positive constants. q.e.d.

5. Examples for the negative case. In this section, we give examples for subsets \( \Sigma \) of \( M \) such that the assertion of Theorem III holds. To begin with, we shall prove
the following:

**Theorem 5.1.** Let \((M, g), \Sigma\) and \(f\) be as in Theorem III. Suppose \(
\Sigma\) and \(f\) satisfy the following conditions:

\((H.1')\) \(\Sigma\) is the union of a family \(\{\Sigma_i\}_{i \in I}\) of totally geodesic submanifolds of \(M\) with

\[
m_i = \dim \Sigma_i \leq \begin{cases} 
1 & \text{if } n = 2 \\
\frac{n+1}{2} & \text{if } n \geq 3 \quad \text{and } \left(\frac{A}{B}\right)^2 < \frac{(n-1)^2}{n(n-2)} \\
\frac{n}{2} & \text{if } n \geq 3 \quad \text{and } \left(\frac{A}{B}\right)^2 = \frac{(n-1)^2}{n(n-2)},
\end{cases}
\]

such that the condition

\[
(5.1) \quad \sup_{x \in M} \sum_{i \in I} \frac{1}{[\cosh(Bd_{\Sigma}(x, \Sigma_i))]^e} < +\infty
\]

holds with the same \(\alpha\) as in Theorem III;

\((H.2)\) as in Theorem III.

Then the equation \((\ast n)\) possesses a bounded smooth solution (which is also bounded away from zero if \(n \geq 3\)).

Although we can derive this theorem (except when \(m_i = \dim \Sigma_i = (n+1)/2\) if \(n \geq 3\) and \((A/B)^2 \geq \{(n-1)^2/n(n-2)\}: \left\{1 - 1/(2n^2 - 4n + 1)^2\right\}\) as a corollary of Theorem III, we will prove it directly.

**Proof of Theorem 5.1.** For any \(i \in I\), let \(\rho_i := \rho_{\Sigma_i}\) and \(u_i := 1/(\cosh(B\rho_i))^\alpha\). By direct computation, we see that

\[-\Delta_g u_i = B^2(\cosh(B\rho_i))^{-\alpha - 2} \cdot \left\{-\alpha^2(\cosh(B\rho_i))^2 + \frac{\alpha}{B}(\cosh(B\rho_i))(\sinh(B\rho_i))\Delta_g \rho_i + \alpha(\alpha + 1)\right\}.\]

Now by (4.3) with \(\Sigma = \Sigma_i\), we have

\[-\Delta_g \rho_i \geq B\{(n - m_i - 1)(\coth(B\rho_i)) + m_i(\tanh(B\rho_i))\},\]

from which

\[-\Delta_g u_i \geq B^2(\cosh(B\rho_i))^{-\alpha - 2}\left\{-\alpha(\alpha - n + 1)(\cosh(B\rho_i))^2 + \alpha(\alpha - m_i + 1)\right\}.
\]

Set \(u_\Sigma := \sum_{i \in I} u_i\) and \(\bar{u}_\Sigma := \sup_{M} u_\Sigma\) which is finite by the assumption (5.1). By using (4.1) and (4.2), we can easily get \(u_\Sigma \in C^2(M)\), and \(\Delta_g u_\Sigma\) satisfies

\[-\Delta_g u_\Sigma \geq \sum_{i \in I} B^2(\cosh(B\rho_i))^{-\alpha - 2}\left\{-\alpha(\alpha - n + 1)(\cosh(B\rho_i))^2 + \alpha(\alpha - m_i + 1)\right\}.
\]
In the case $n = 2$,

$$-\Delta \mu^2 \geq \sum_{i \in I} (2 - m_i) B^2 (cosh B \rho_i)^{-3} > 0.$$  

Since, for any $x \in B_\rho(\Sigma)$, there exists an $i \in I$ such that $x \in B_\rho(\Sigma_i)$, it follows that

$$-\Delta \mu^2 \geq (2 - m_i) B^2 (cosh BR)^{-3} \geq B^2 (cosh BR)^{-3} > 0 \quad \text{in} \quad B_\rho(\Sigma).$$

In the case $n \geq 3$,

$$-4 \frac{n-1}{n-2} \Delta \mu^2 + S \mu^2 \geq \sum_{i \in I} 4 \frac{n-1}{n-2} B^2 (cosh B \rho_i)^{-2}$$

$$\cdot \left[ -\left\{ \alpha^2 - (n-1)\alpha + \frac{n(n-2)}{4} \left( \frac{A}{B} \right)^2 \right\} (cosh B \rho_i)^{2} + \alpha (\alpha - m_i + 1) \right]$$

$$= \sum_{i \in I} 4 \frac{n-1}{n-2} B^2 \alpha (\alpha - m_i + 1) (cosh B \rho_i)^{-2} > 0.$$  

Since, for any $x \in B_\rho(\Sigma)$, there exists an $i \in I$ such that $x \in B_\rho(\Sigma_i)$, it follows that

$$-4 \frac{n-1}{n-2} \Delta \mu^2 + S \mu^2 \geq 4 \frac{n-1}{n-2} B^2 \alpha (\alpha - m_i + 1) (cosh BR)^{-2}$$

$$\geq 4 \frac{n-1}{n-2} B^2 \alpha C_3 (cosh BR)^{-2} > 0 \quad \text{in} \quad B_\rho(\Sigma),$$

where

$$C_3 := \begin{cases} 
\frac{(n-1)^2 - n(n-2)(A/B)^2}{2} & \text{if} \quad \left( \frac{A}{B} \right)^2 < \frac{(n-1)^2}{n(n-2)} \\
1 & \text{if} \quad \left( \frac{A}{B} \right)^2 = \frac{(n-1)^2}{n(n-2)}. 
\end{cases}$$

Now we can prove our assertion by the same method as in the proof of Theorem III.

q.e.d.

In Theorem 5.1, if $\# I$ is finite, then the condition (5.1) is obviously satisfied, and hence the assertion holds. Even if $\# I$ is infinite, we can construct examples satisfying the condition (5.1), which is illustrated in the following:

**Proposition 5.2.** In Theorem 5.1, the condition (5.1) is satisfied provided that $I = N$, and that there exists a sequence $\{D_i\}_{i \in N}$ of domains of $M$ with the following properties:

1. $D_i \subset D_j$ for all $i < j$,
2. $d := \inf_{i \in N} d_\rho(\partial D_i, \partial D_{i+1}) > 0$,
3. $\Sigma_i$ is contained in $\overline{D_i \setminus D_{i-1}}$ for any $i \in N$, where $D_0 := \emptyset$.  


PROOF. For any \( i, j \in \mathbb{N} \) and \( x \in \overline{D}_j \setminus D_{j-1} \),

\[
\rho_i(x) = d_g(x, \Sigma_i) \geq \begin{cases} 
(j-i-1)d & \text{if } i<j \\
0 & \text{if } i=j \\
(i-j-1)d & \text{if } i>j.
\end{cases}
\]

Hence

\[
\sum_{i \in \mathbb{N}} \frac{1}{(\cosh B\rho_i)^2} \leq \sum_{i<j} \frac{1}{(\cosh B\rho_i)^2} + \frac{1}{(\cosh B\rho_j)^2} + \sum_{i>j} \frac{1}{(\cosh B\rho_i)^2}
\]

\[
\leq \sum_{i<j} \frac{1}{[\cosh((j-i-1)Bd)]^2} + \frac{1}{[\cosh 0]^2} + \sum_{i>j} \frac{1}{[\cosh((i-j-1)Bd)]^2}
\]

\[
< 1 + 2 \sum_{i \in \mathbb{N}} \frac{1}{[\cosh((i-1)Bd)]^2}
\]

\[
< 1 + \frac{2^{\alpha+1}}{1 - e^{-B\alpha}} < +\infty.
\]

Now the condition (5.1) is satisfied. q.e.d.

When \( M = H^2 \) or \( H^3 \), we can construct examples having the properties above with \( \partial D_i = \Sigma_i \) for any \( i \in \mathbb{N} \).

In Section 7, the idea of Proposition 5.2 will be applied to the case where \( M \) is not simply connected.

Now we replace the assumption on \( m_i = \dim \Sigma_i \) in Theorem 5.1 by another.

**Theorem 5.3.** Let \((M, g), \Sigma \) and \( f \) be as in Theorem III (without the assumption on \( A/B \)). Suppose \( \Sigma \) is a totally geodesic submanifold of \( M \) with \( m = \dim \Sigma \leq n-1 \), and \( f \) satisfies the condition (H.2) with \( R < R_0/B \), where \( R_0 \) is a positive constant (or +\( \infty \)) depending only on \( A/B, m, \) and \( n \). Then the equation (\( \ast n \)) possesses a bounded smooth solution (which is also bounded away from zero if \( n \geq 3 \)).

**Proof.** It is enough to prove the case \( n \geq 3 \). Set \( u_\Sigma := 1/(\cosh B\rho_2)^2 \), where a positive number \( \alpha \) will be chosen later. By direct computation, we see that

\[
-4 \frac{n-1}{n-2} \Delta_g u_\Sigma + S_g u_\Sigma \geq 4 \frac{n-1}{n-2} B^2 (\cosh B\rho_2)^{-2} \left\{ -F(\alpha)(\cosh B\rho_2)^2 + \alpha(\alpha - m + 1) \right\},
\]

where

\[
F(\alpha) := \alpha^2 - (n-1)\alpha + \frac{n(n-2)}{4} \left( \frac{A}{B} \right)^2.
\]

Set
Now since \( R < R_0 / B \), there exists some \( \alpha \) such that

\[ BR < R_1 := \text{Cosh}^{-1} \sqrt{\frac{\alpha(\alpha - m + 1)}{F(\alpha)}} \leq R_0. \]

For any \( x \in B_R(\Sigma) \),

\[ [\cosh \left( B\rho_2(x) \right)]^2 \leq (\cosh BR)^2 = \left( \frac{\cosh BR}{\text{cosh} R_1} \right)^2 \cdot \frac{\alpha(\alpha - m + 1)}{F(\alpha)}, \]

and hence

\[ -4 \frac{n-1}{n-2} \Delta g u_x + S_g u_x \geq 4 \frac{n-1}{n-2} B^2 (\cosh BR)^{-\alpha - 2} \left( 1 - \left( \frac{\cosh BR}{\text{cosh} R_1} \right)^2 \right) \alpha(\alpha - m + 1) > 0 \quad \text{in} \quad B_R(\Sigma). \]

On the other hand, we have

\[ -4 \frac{n-1}{n-2} \Delta g u_x + S_g u_x > -4 \frac{n-1}{n-2} B^2 F(\alpha) \quad \text{on} \quad M. \]

Hence we can prove our assertion by the same method as in the proof of Theorem III.

q.e.d.

When \( m \leq (n + 1)/2 \), we can easily see

\[ R_0 = +\infty \quad \text{if} \quad \left( \frac{A}{B} \right)^2 \leq \frac{(n-1)^2}{n(n-2)}, \]

and

\[ R_0 \to +\infty \quad \text{as} \quad \left( \frac{A}{B} \right)^2 \to \frac{(n-1)^2}{n(n-2)} + 0. \]

6. The case \( M = H^2 \). In this section, we consider the case \( n = 2 \). Under the condition (1.1), if \( n = 2 \), then by the Ahlfors-Schwarz Lemma (cf. [1]), \((M, g)\) is conformally and uniformly equivalent to the hyperbolic plane \( H^2 = H^2(-1) \). Hence we restrict our attention to the case \( M = H^2 \).

Now we provide a certain necessary condition for the same assertion as in Theorem III to hold. We begin with the following:

**Lemma 6.1.** If there exists a bounded solution of the equation (82) on \( H^2 \), then for
any \( 1 < \alpha \leq 2\sqrt{2} \) and \( x \in H^2 \),

(6.1) \[
\int_{H^2} \frac{f(y)}{\cosh\{d_g(x, y)\} + 1}^\alpha \; dy < 0.
\]

**Proof.** Under the assumption above, [2, Theorem 2] showed that if we regard \( H^2 \) as the Poincaré disk \( D \), then \( \int_D f dv < 0 \), where \( dv \) is the volume element with respect to the flat metric. By the same method, we can show that \( \int_D f(1 - r^2) \alpha^2 dv < 0 \) for any \( 1 < \alpha \leq 2\sqrt{2} \), where \( r \) is the distance to the origin with respect to the flat metric. Now if we regard \( x \in H^2 \) as the origin of \( D \), and use the hyperbolic distance \( d_\alpha \) then we get the condition (6.1). q.e.d.

**Theorem 6.2.** Let \( \Sigma \) be a subset of \( H^2 \), and \( f \) a bounded function \( H^2 \). Suppose \( \Sigma \) satisfies \( B_\delta(\Sigma') \subset \Sigma \) for a positive number \( \delta \) and a measurable subset \( \Sigma' \) of \( H^2 \),

(6.2) \[
\sup_{x \in H^2} \int_{\Sigma} \frac{dy}{\cosh\{d_g(x, y)\} + 1} = \frac{4\pi}{2\alpha(\alpha - 1)}
\]

with some \( 1 < \alpha \leq 2\sqrt{2} \), and \( f \geq \varepsilon \) on \( \Sigma' \) for a positive number \( \varepsilon \). Then the equation (6.2) possesses no bounded smooth solution.

**Proof.** Clearly, there is a positive number \( a \) such that \( f \geq -a^2 \) on \( H^2 \). If the equation (6.2) possesses a bounded solution \( u \), then, from Lemma 6.1,

\[
0 > \int_{H^2} \frac{f(y)}{\cosh\{d_g(x, y)\} + 1}^\alpha \; dy
\geq \int_{\Sigma} \frac{\varepsilon}{\cosh\{d_g(x, y)\} + 1}^\alpha \; dy - \int_{H^2 \setminus \Sigma} \frac{a^2}{\cosh\{d_g(x, y)\} + 1}^\alpha \; dy
\]

\[
= (\varepsilon + a^2) \int_{\Sigma} \frac{dy}{\cosh\{d_g(x, y)\} + 1}^\alpha - a^2 \int_{H^2} \frac{dy}{\cosh\{d_g(x, y)\} + 1}^\alpha
\]

for any \( x \in H^2 \). Hence

\[
\int_{\Sigma} \frac{dy}{\cosh\{d_g(x, y)\} + 1}^\alpha < \frac{a^2}{\varepsilon + a^2} \cdot \frac{4\pi}{2\alpha(\alpha - 1)}.
\]

Finally we have

\[
\sup_{x \in H^2} \int_{\Sigma} \frac{dy}{\cosh\{d_g(x, y)\} + 1}^\alpha \leq \frac{a^2}{\varepsilon + a^2} \cdot \frac{4\pi}{2\alpha(\alpha - 1)} < \frac{4\pi}{2\alpha(\alpha - 1)}.
\]

This contradicts the assumption (6.2). q.e.d.
Observe $4\pi/2^\delta(\alpha - 1) \to +\infty$ as $\alpha \to 1 + 0$. Moreover,

$$\int_{\Sigma} \frac{dy}{\cosh \{d_{\rho}(x, y)\} + 1} < +\infty,$$

if and only if

$$\int_{\Sigma} \frac{dy}{\cosh \{d_{\rho}(x, y)\}} < +\infty.$$

In this sense, the sufficient condition (H.1) in Theorem III is sharp.

**Example 6.3.** Let $\Sigma$ be a subset of $H^2$. Suppose there exists a horocyclic region $U \subset \Sigma$. Then, by replacing $U$, we may assume $B_{\delta}(U) \subset \Sigma$ with a positive number $\delta$. Now, for any positive number $R$, there exists an $x_R \in U$ satisfying $B_R(x_R) \subset U$. Hence

$$\int_{B_R(x_R)} \frac{dy}{\cosh \{d_{\rho}(x, y)\} + 1} > \int_{B_R(x_R)} \frac{dy}{\cosh \{d_{\rho}(x, y)\} + 1}^2,$$

from which

$$\sup_{x \in H^2} \int_{U} \frac{dy}{\cosh \{d_{\rho}(x, y)\} + 1}^2 \geq \lim_{R \to +\infty} \int_{B_R(x_R)} \frac{dy}{\cosh \{d_{\rho}(x, y)\} + 1}^2$$

$$= \int_{H^2} \frac{dy}{\cosh \{d_{\rho}(x, y)\} + 1}^2 = \frac{4\pi}{2^\delta(\alpha - 1)}.$$

Namely, the condition (6.2) is satisfied for $\Sigma' = U$, hence the same assertion as in Theorem III does not hold for $\Sigma$.

On the other hand, in the following case, the same conclusion as in Theorem III holds.

**Example 6.4.** Suppose $\Sigma$ is a horocycle of $H^2$, and $f$ satisfies the condition (H.2). Then the equation (*) possesses a bounded smooth solution. Indeed, let $\Sigma'$ be a horocycle which is the component of $\partial B_R(\Sigma)$ contained in the smaller component of $H^2 \setminus \Sigma$. Denote the Busemann function with respect to a point on $\Sigma'$ and the end point of $\Sigma'$ by $\rho$. Then we get $B_R(\Sigma) = \{x \in H^2 \mid 0 < \rho(x) < 2R\}$, $|\nabla_{\rho} \rho| = 1$, and $\Delta_{\rho} \rho = 1$. Let $u_{\Sigma} = 1/\cosh \rho$. By direct computation, we see that

$$-\Delta_{\rho} u_{\Sigma} = (\cosh \rho)^{-3} \{ - (\cosh \rho)^2 + (\cosh \rho)(\sinh \rho) \rho + 2 \}$$

$$= (\cosh \rho)^{-3} \{ - (\cosh \rho)^2 + (\cosh \rho)(\sinh \rho) + 2 \}$$

$$= (\cosh \rho)^{-3} (3 - e^{-2\rho})/2.$$

Since $0 < \rho(x) < 2R$ for any $x \in B_R(\Sigma)$,

$$-\Delta_{\rho} u_{\Sigma} > (\cosh \rho)^{-3} > (\cosh 2R)^{-3} > 0 \quad \text{in} \quad B_R(\Sigma).$$
On the other hand, we have,

\[-\Delta_g \mu \geq \inf \{ (\cosh \rho)^{-3}(3-e^{-2\rho})/2 \mid 3-e^{-2\rho} < 0 \} \]

\[-4 \sup \left\{ e^\rho \left| \rho < -\frac{1}{2} \log 3 \right. \right\} = -\frac{4}{\sqrt{3}} \quad \text{on } H^2.\]

Hence we can prove our assertion by the same method as in the proof of Theorem III.

**Remark 6.5.** We can give an example similar to that above when $M$ is the hyperbolic space $H^3$ and $\Sigma$ is a horosphere.

Next, as a generalization of [2, Theorem 3] on the behavior of a solution of the equation (*2), we get the following:

**Theorem 6.6.** Let $\Sigma$ be a subset of $H^2$, and $f$ a bounded smooth function on $H^2$. Suppose $\Sigma$ is the union of a finite family $\{ \Sigma_i \}_{i \in I}$ of complete geodesics of $H^2$, and $f$ satisfies the following condition:

\[(H.2') \quad f \leq 0, \quad \text{and} \quad \sum_{i \in I} e^{-\alpha \rho_i} \]

for positive constants $b$, $C$ and $\alpha < 1$.

Then the equation (*2) possesses a bounded smooth solution $u$ which has the following property:

\[(H.3) \quad |u + 2 \log b| \leq C \sum_{i \in I} e^{-\alpha \rho_i} \]

for a positive constant $C'$.

**Proof.** Let $\rho_i$, $u_i$ and $u_\Sigma$ be as in the proof of Theorem 5.1. Then

\[-\Delta_g u_\Sigma = \sum_{i \in I} (\cosh \rho_i)^{-\alpha} \{ \alpha(1-\alpha)(\cosh \rho_i)^2 + \alpha^2 \} \]

\[> \alpha(1-\alpha) \sum_{i \in I} (\cosh \rho_i)^{-\alpha} > \alpha(1-\alpha) \sum_{i \in I} e^{-\alpha \rho_i}. \]

Set $u_\pm := \pm \beta u_\Sigma - 2 \log b$, where $\beta := b^{-2} C / \alpha(1-\alpha)$. Now we get

\[-\Delta_g u_\pm + S_g - f e^{u_\pm} = -\beta \Delta_g u_\Sigma - 2 - f b^{-2} e^{u_\Sigma} \geq -\beta \Delta_g u_\Sigma - 2 - f b^{-2} \]

\[= -\beta \Delta_g u_\Sigma - b^{-2} (2b^2 + f) > \{ \beta \alpha(1-\alpha) - b^{-2} C \} \sum_{i \in I} e^{-\alpha \rho_i} = 0. \]

On the other hand, we get
Hence $u_+$ and $u_-$ are respectively a supersolution and a subsolution of the equation \((\ast 2)\). Since $u_- \leq u_+$, by the method of supersolutions and subsolutions, the equation \((\ast 2)\) possesses a bounded solution $u$ satisfying $u_- \leq u \leq u_+$. Namely, $u$ satisfies the estimate \((H.3)\). q.e.d.

Under the assumption of Theorem 6.6, we do not have much information on $u$ at $\Sigma(\infty)$, but we see

$$
u \rightharpoonup -2 \log b \quad \text{as} \quad x \to H^2(\infty) \setminus \Sigma(\infty).$$

7. The case $M = H^2/\Gamma$. In this section, we consider the case where $M = H^2/\Gamma$ is not simply connected. First, from Theorem 5.1 and Example 6.4, we immediately get the following:

**Corollary 7.1.** Let $\Sigma$ be a subset of $M = H^2/\Gamma$, and $f$ a bounded smooth function on $M$. Suppose $\Gamma \cong \mathbb{Z}$, $\Sigma$ is compact, and $f$ satisfies the condition \((H.2)\). Then the equation \((\ast 2)\) possesses a bounded smooth solution.

**Proof.** There are two cases. When $\Gamma$ is a hyperbolic subgroup of Isom($H^2$), let $\Sigma$ be the lift of the minimal closed geodesic of $M$, and when $\Gamma$ is a parabolic subgroup of Isom($H^2$), let $\Sigma$ be a suitable horocycle on $M$. In both cases, since $u_\Sigma = 1/\cosh(d_g(\cdot, \Sigma))$ on $\tilde{M} = H^2$ is $\Gamma$-invariant, we can regard $u_\Sigma$ as a function on $M$. Hence, by the method of supersolutions and subsolutions, we get a bounded solution of the equation \((\ast 2)\) also on $M$. q.e.d.

Now we consider the case where $\Gamma$ is purely hyperbolic.

**Definition 7.2.** Let $(M = H^2/\Gamma, g)$ be a complete, noncompact, oriented surface which is finitely connected with $h$ handles and $e$ ends. Set

$$d_\Gamma := \sup \left[ \min_{i \neq j} \{d_g(T_i, T_j)\} \right],$$

where $\{T_i\}_{i=1}^N$ runs through families of complete geodesics of $\tilde{M} = H^2$ which bounds a fundamental domain of $\Gamma$, $N := 2(2h + e - 1)$, and $\tilde{g}$ is the standard metric on $H^2$.

It is easy to verify that $d_\Gamma > 0$ if and only if $\Gamma$ is purely hyperbolic.

**Theorem 7.3.** Let $(M, g)$ be as in Definition 7.2, $\Sigma$ a subset of $M$, and $f$ a bounded smooth function on $M$. Suppose $d_\Gamma > \log(N - 1)$, $\Sigma$ is the union of a finite family $\{\Sigma_i\}_{i \in I}$ of complete geodesics of $M$, and $f$ satisfies the condition \((H.2)\). Then the equation \((\ast 2)\) possesses a bounded smooth solution.
PROOF. Let \( \{T_i\}_{i=1}^{N} \) be a family of complete geodesics of \( \tilde{M} = \mathbb{H}^2 \) which bounds a fundamental domain \( \Omega \) of \( \Gamma \), and \( d := \min_{i \neq j} d_{\tilde{g}}(T_i, T_j) > \log(N-1) \), that is \( (N-1)e^{-d} < 1 \). Without loss of generality, we may assume that some lift \( \tilde{\Sigma}_i \) of \( \Sigma_i \) is contained in \( \tilde{\Omega} \) for any \( i \in I \). Now, since

\[
\#\{\gamma \in \Gamma \mid d_{\tilde{g}}(\gamma \tilde{x}, \tilde{\Omega}) \leq ld\} \leq 1 + \sum_{j=1}^{l} N(N-1)^{j-1}
\]

for any \( \tilde{x} \in \mathbb{H}^2 \) and every \( l \in N \), it is clear that

\[
\sum_{\gamma \in \Gamma} \frac{1}{\cosh\{d_{\tilde{g}}(\gamma \tilde{x}, \tilde{\Sigma}_i)\}} \leq 1 + \sum_{j=1}^{\infty} \frac{N(N-1)^{j-1}}{\cosh((j-1)d)} \leq 1 + 2N \sum_{j=1}^{\infty} \left((N-1)e^{-d}\right)^{j-1}
\]

for any \( i \in I \). Hence we can define

\[
u_{\tilde{g}}(\tilde{x}) := \sum_{i \in I} \sum_{\gamma \in \Gamma} \frac{1}{\cosh\{d_{\tilde{g}}(\gamma \tilde{x}, \tilde{\Sigma}_i)\}}
\]

on \( \mathbb{H}^2 \). Since \( \nu_{\tilde{g}} \) is \( \Gamma \)-invariant, we can regard \( \nu_{\tilde{g}} \) as a function on \( M = \mathbb{H}^2 / \Gamma \). Hence by the method of supersolutions and subsolutions, we get a bounded solution of the equation \((\ast 2)\) on \( M \). q.e.d.

The hyperbolic case of Corollary 7.1 is obtained also as a corollary to Theorem 7.3. Indeed, in this case, we have \( N = 2 \) since \( h = 0 \) and \( e = 2 \). Hence \( \log(N-1) = 0 \).

**Example 7.4.** Let \( D (= \mathbb{H}^2) \) be the Poincaré disk, and \( \{T_i\}_{i=1}^{8} \) a family of circular arcs in \( D \) which are orthogonal to \( \partial D (= \mathbb{H}^2(\infty)) \) (those are geodesics of \( (\mathbb{H}^2, \tilde{g}) \)) as in [Diagram].
Figure. We can easily take \( \{ T_i \}_{i=1}^8 \) satisfying \( d = \min_{i \neq j} d_\mathcal{g}(T_i, T_j) > \log 7 \). (In fact, we can take it for arbitrarily large \( d \).) Let \( \gamma_1 \) be the hyperbolic isometry (with respect to \( \mathcal{g} \)) such that \( \gamma_1(T_1) = T_8 \), and that the axis is the geodesic orthogonal to \( T_1 \) and \( T_8 \). Define \( \gamma_2 \) similarly by \( T_2 \) and \( T_7 \), \( \gamma_3 \) by \( T_3 \) and \( T_5 \), and \( \gamma_4 \) by \( T_4 \) and \( T_6 \). Suppose that \( \Gamma \) is the purely hyperbolic subgroup of \( \text{Isom}(\mathcal{H}^2, \mathcal{g}) \) generated by \( \gamma_1, \gamma_2, \gamma_3 \) and \( \gamma_4 \) (see, for instance, [4]). Then clearly \( M = \mathcal{H}^2/\Gamma \) has one handle and three ends. Hence \( N - 1 = 7 \), and \( M \) satisfies the assumption of Theorem 7.3.

REFERENCES