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OSCILLATIONS OF VOLTERRA INTEGRAL EQUATIONS WITH DELAY

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Abstract. The oscillatory behavior of the solutions of a Volterra type equation with delay is investigated. Sufficient conditions on the kernel are given which guarantee that the oscillatory character of the forcing term is inherited by the solutions.

1. Introduction and preliminaries. In this paper we investigate when the oscillatory character of the forcing term f of the Volterra integral equation with delays of the form

$$x(t)=f(t)-\int_0^t K(t, s, x_s)ds\,, \qquad t\geq 0 \tag{1.1}$$

is inherited by the solutions.

As we can see from [2], [6], [27] and the references cited therein, the equations of the type (1.1) arise, for example, in certain applications to impulsive theory. It has also been a very interesting subject to study how the behavior (e.g., boundedness, convergence, periodicity, asymptotic periodicity, slow (or almost slow) varyingness) of the forcing term f produces the same property of the solutions of a Volterra integral equation under certain conditions on the kernel K (cf. [8], [9], [12], [13], [15], [16], [17], [18], [19]). Therefore it is natural to investigate how oscillation of the forcing term f can be inherited by the solutions of (1.1). Our aim here is to establish conditions on the nonlinear (in general) kernel K under which if the function f is oscillatory, strongly, quickly, moderately or slowly oscillatory, then every solution of (1.1) is oscillatory, strongly, quickly, moderately or slowly oscillatory, respectively.

Before giving the definitions of various types of oscillations mentioned above, we have to present some preliminaries needed in the sequel.

Let $C:=C([-r, 0], \boldsymbol{R})$ denote the Banach space of all continuous functions mapping the interval $[-r, 0]$ into $\boldsymbol{R}$ endowed with the sup-norm $\|\cdot\|$. For any function $x: \boldsymbol{R}\mapsto\boldsymbol{R}$ and $t\in\boldsymbol{R}^+$, we define x_t by $x_t(\theta):=x(t+\theta)$, $\theta\in[-r, 0]$.

The following assumptions will be used throughout this chapter without any further mention.

(A_1) $f: \boldsymbol{R}^+\mapsto\boldsymbol{R}$ is continuous.

(A_2) K maps the set $\{(t, s): 0\leq s\leq t,\ t\geq 0\}\times C$ into $\boldsymbol{R}$.

(A_3) The function $\varphi\mapsto K(t, s, \varphi)$ is continuous, and maps bounded sets into

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bounded sets. Moreover for each bounded subset $B \subseteq C$ the function $s \mapsto \sup_{\varphi \in B} K(t, s, \varphi)$ is measurable in s for fixed t.

(A_4) The function $t \mapsto K(t, s, \varphi)$ is continuous for each fixed (s, φ).

(A_5) For all (t, s) the following holds:

$$K(t, s, \varphi) \geq 0 \quad \text{if } \varphi \geq 0 \quad \text{and} \quad K(t, s, \varphi) \leq 0 \quad \text{if } \varphi \leq 0 . \tag{1.2}$$

By a solution of (1.1) with initial function $\varphi \in C$, we mean a function x which is continuous on $[0, +\infty)$, satisfies (1.1) and $\varphi(0) = f(0)$ and $x(s) = \varphi(s)$ hold, where $s \in [-r, 0]$.

For existence, uniqueness and continuous dependence of solutions of (1.1), we refer to the papers [3], [4], [6] and for some other related results we refer to [1], [2], [3], [4], [14], [20], [27], [28].

We use the following definitions of oscillations.

A function x is said to be *oscillatory* if for any $t_1 > 0$

$$\inf_{t \geq t_1} x(t) < 0 < \sup_{t \geq t_1} x(t) .$$

A function x is said to be *strongly oscillatory* if

$$\liminf_{t \to +\infty} x(t) < 0 < \limsup_{t \to +\infty} x(t) .$$

A function x is said to be *quickly oscillatory* if there exists a sequence of points $\{t_n\}$ such that

$$x(t_n) = 0 , \qquad n = 1, 2, 3, \ldots$$

and

$$t_{n+1} > t_n , \quad \lim_{n \to +\infty} t_n = +\infty \quad \text{and} \quad \lim_{n \to +\infty} (t_{n+1} - t_n) = 0 .$$

The phenomenon of quick oscillations can be simulated by a bouncing ball under the force of gravity. Each bounce will be progressively shorter until the ball comes to rest (cf. [5], [22], [25]).

The following two definitions can be found in [23], [24].

A function x is said to be *moderately oscillatory* if there exists a sequence of points $\{t_n\}$ such that

$$x(t_n) = 0 , \qquad n = 1, 2, 3, \ldots$$

and

$$t_{n+1} > t_n , \quad \lim_{n \to +\infty} t_n = +\infty \quad \text{and} \quad \sup\{t_{n+1} - t_n, n = 1, 2, 3, \ldots\} < +\infty .$$

A function x is said to be *slowly oscillatory* if for any large positive numbers T,

M, there exist two consecutive zeros t_1 and t_2 of $x(t)$, $t_2 > t_1 > T$, such that $t_2 - t_1 > M$.

The following notion is an example of slow oscillation. The trajectory of a point, moving with a constant velocity v along the path p: (x, y), where

$$x(t) = e^{\theta(t)} \cos\theta(t) , \quad y(t) = e^{\theta(t)} \sin\theta(t) , \quad \text{and} \quad \theta(t) = \ln\left(1 + \frac{vt}{\sqrt{2}}\right) ,$$

has a projection

$$(P(t), Q(t)) = \frac{1}{\sqrt{2}} (\cos\theta(t), \sin\theta(t)) ,$$

through $(0, 0)$ on the unit circle $x^2 + y^2 = 1$. Then the functions P and Q are slowly oscillatory.

From the above definitions it is clear that if x is quickly oscillatory, then it is moderately oscillatory. Also if x is slowly oscillatory, then it is not moderately oscillatory. For example, $\sin\sqrt{t}$ is slowly oscillatory, $\sin t$ is moderately oscillatory and $\sin t^2$ is quickly oscillatory.

New we define the following classes of functions.

- S_q is the class of all continuous functions $f: \boldsymbol{R}^+ \mapsto \boldsymbol{R}$ for which there exists an $\eta > 0$ such that for any $\varepsilon > 0$ there exists a $T > 0$ such that for every $t > T$ there exist two points s_1 and s_2 in the interval $[0, \varepsilon]$ with the property that

$$f(t + s_1) < -\eta \quad \text{and} \quad f(t + s_2) > \eta .$$

- S_m is the class of all continuous functions $f: \boldsymbol{R}^+ \mapsto \boldsymbol{R}$ for which there exist positive numbers η, ζ, T such that for every $t > T$ there exist two points s_1 and s_2 in the interval $[0, \zeta]$ such that

$$f(t + s_1) < -\eta \quad \text{and} \quad f(t + s_2) > \eta .$$

- S_s is the class of all continuous and strongly oscillatory functions $f: \boldsymbol{R}^+ \mapsto \boldsymbol{R}$ for which there exists an $\eta > 0$ and for any real numbers $p > 0$ and $T > 0$ there exists a $\tau > T$ such that

$$|f(t + \tau)| > \eta , \qquad \text{for all} \quad t \in [0, p] .$$

From the above definitions we observe that $f \in S_q$ (resp. S_m, resp. S_s) implies that f is continuous, strongly and quickly (resp. moderately, resp. slowly) oscillatory. However, the converse is not always true. For example, the function

$$f(t) := \begin{cases} e^t \sin t , & t \in [(2n)^2\pi, (2n+1)^2\pi) \\ e^{-t} \sin t , & t \in [(2n+1)^2\pi, (2n+2)^2\pi) \end{cases}$$

$n = 0, 1, 2, \ldots$, is continuous, strongly and moderately oscillatory. However $f \notin S_m$. Similarly, one can easily find functions which are continuous, strongly and slowly (resp.

quickly) oscillatory but they do not belong to S_s (resp. S_q).

LEMMA 1.1. *$S_q \subseteq S_m$ and $S_q \neq S_m$, $S_m \cap S_s = \emptyset$ (the empty set) and $S_m \cup S_s$ is the class of all strongly oscillatory continuous functions.*

The following result is also obvious by the definitions of S_q, S_m and S_s.

LEMMA 1.2. *If x belongs to the set S_q (resp. S_m, resp. S_s), then any shifting function x_c along $c \in \boldsymbol{R}$ of x belongs to the set S_q (resp. S_m, resp. S_s).*

A function x is said to be eventually positive (resp. nonnegative) if there is a large $T > 0$ such that $x(t) > 0$ (resp. ≥ 0) for $t > T$. A function x is said to be eventually negative (resp. nonpositive) if $x(t) < 0$ (resp. ≤ 0) for $t > T$.

2. Oscillation and asymptotic behavior. As we know, oscillation criteria for functional differential equations can be obtained by using asymptotic results on the solutions of functional differential inequalities (cf. [10], [11], [26]). We study the oscillatory behavior of the solutions of (1.1) through the asymptotic results on the solutions of the following two Volterra integral inequalities

$$x(t) \geq f(t) - \int_0^t K(t, s, x_s)ds, \qquad t \geq 0, \tag{2.1}$$

$$x(t) \leq f(t) - \int_0^t K(t, s, x_s)ds, \qquad t \geq 0. \tag{2.2}$$

For the sake of convenience, we define a quantity $L(k, T, b)$, which depends on two positive constants k, T and a function b, as follows:

$$L(k, T, b) := \limsup_{t \to +\infty} \int_0^T \sup_{\varphi \in B(0,k)} |b(t)|\,|K(t, s, \varphi)|\,ds, \tag{2.3}$$

where $b: [0, +\infty) \mapsto \boldsymbol{R}$ is a measurable and eventually positive function and $B(0, k) := \{\varphi \in C, \|\varphi\| \leq k\}$.

THEOREM 2.1. (i) *Assume that*

$$\limsup_{t \to +\infty} b(t)f(t) > L(k, T, b) \tag{2.4}$$

for every real number $k > 0$ and every large number $T > 0$. Then (2.1) *has no eventually nonpositive solutions.*

(ii) *Assume that*

$$\liminf_{t \to +\infty} b(t)f(t) < -L(k, T, b) \tag{2.5}$$

for every real number $k > 0$ and every large number $T > 0$. Then (2.2) *has no eventually* *nonnegative solutions.*

(iii) *Assume that both* (2.4) *and* (2.5) *are satisfied for every real number* $k>0$ *and every large number* $T>0$. *Then every solution of* (1.1) *is oscillatory.*

PROOF. First consider Case (i). Assume, for the sake of contradiction, that x is an eventually nonpositive solution of (2.1). Then there exists a sufficiently large number T such that $x(t)\leq 0,\ t>T$. Since the function b is eventually positive, there exists a number $T_1>T$ such that

$$b(t)>0\,, \qquad t>T_1\,.$$

So we have

$$b(t)x(t)\leq 0\,, \qquad t>T_1\,. \tag{2.6}$$

For this number T_1, there exists a positive real number k such that

$$|x_s(\theta)|\leq k\,, \qquad \text{for} \quad \theta\in[-r,0] \quad \text{and} \quad s\in[0,T_1+r]\,. \tag{2.7}$$

By (2.1), (1.2) and (2.6), we have

$$\begin{aligned} 0\geq b(t)x(t)&=b(t)f(t)-\int_0^t b(t)K(t,s,x_s)ds \\ &=b(t)f(t)-\int_0^{T_1+r} b(t)K(t,s,x_s)ds-\int_{T_1+r}^t b(t)K(t,s,x_s)ds \\ &\geq b(t)f(t)-\int_0^{T_1+r} b(t)K(t,s,x_s)ds\,, \qquad t>T_1+r\,. \end{aligned}$$

In view of (2.7), the last inequality yields

$$b(t)f(t)\leq\int_0^{T_1+r}\sup_{\varphi\in B(0,k)}|b(t)||K(t,s,\varphi)|ds\,, \qquad t>T_1+r\,.$$

Taking limit superior on both sides of this inequality, we have

$$\limsup_{t\to+\infty} b(t)f(t)\leq\limsup_{t\to+\infty}\int_0^{T_1+r}\sup_{\varphi\in B(0,k)}|b(t)||K(t,s,\varphi)|ds\,.$$

Also, in view of (2.3) and (2.6), we obtain

$$\limsup_{t\to+\infty} b(t)f(t)\leq L(k,T_1+r,b)\,,$$

which contradicts (2.4). The proof of (i) is complete.

Case (ii) can be proved similarly.

Case (iii) is a combination of Case (i) and Case (ii).

COROLLARY 2.1. *Assume that*

$$\lim_{t\to+\infty}\int_0^T \sup_{\varphi\in B(0,k)} |K(t,s,\varphi)|\,ds=0$$

for every $k>0$ *and every large number* $T>0$, *and that* f *is strongly oscillatory. Then every solution of* (1.1) *is oscillatory.*

PROOF. Here we have $L(k, T, b)\equiv 0$, where b is a positive constant, and f is strongly oscillatory. Therefore both conditions (2.4) and (2.5) are satisfied and so the result in Theorem 2.1 (iii) holds.

EXAMPLE 2.1. For the integral equation

$$x(t)=\frac{t}{t+\pi}\cos t-\int_0^t \frac{s^2}{t+\pi}x(s-\pi)\,ds\,,\qquad t\geq 0\,,$$

all conditions of Corollary 2.1 are satisfied. Therefore all solutions of this equation oscillate. For example

$$x(t)=\frac{\sin t}{t+\pi}\,,$$

is an oscillatory solution.

COROLLARY 2.2. *Assume that* (2.4) *and* (2.5) *hold for any large* $T>0$ *and some* $k>0$. *Then every solution* x *of* (1.1) *with* $|x(t)|<k$, $t>0$, *is oscillatory.*

PROOF. Take such a solution x of (1.1). If x is not oscillatory, then there exists a sufficiently large $T>0$ such that either $x(t)\geq 0$ or $x(t)\leq 0$ for all $t>T$. First let us consider the case $x(t)\geq 0$, $t>T$. Since the function b is eventually positive, there exists a $T_1>T$ such that $b(t)>0$ for $t>T_1$. So we have $b(t)x(t)\geq 0$, for $t>T_1$.

By (1.1), and taking (1.2) into account, we have

$$\begin{aligned}-b(t)f(t)&\leq -\int_0^{T_1+r} b(t)K(t,s,x_s)\,ds\leq \int_0^{T_1+r} |b(t)|\,|K(t,s,x_s)|\,ds\\ &\leq \int_0^{T_1+r} |b(t)| \sup_{\varphi\in B(0,k)} |K(t,s,\varphi)|\,ds\,.\end{aligned}$$

Taking limit superior of both sides of this inequality, we obtain

$$\limsup_{t\to+\infty}(-b(t)f(t))=-\liminf_{t\to+\infty} b(t)f(t)\leq L(k, T_1+r, b)\,.$$

In view of (2.5), we have a contradiction.

For the case where $x(t)\leq 0$ eventually, we can follow the same way.

EXAMPLE 2.2. Consider the following equation

$$x(t)=f(t)-\int_0^t e^{-s}x(s-\pi)ds\,, \qquad t\geq 0\,,$$

where

$$f(t):=\cos t+\frac{1}{2}+\frac{1}{\sqrt{2}}e^{-t}\sin\left(t-\frac{\pi}{4}\right).$$

It is easy to check that (2.4) and (2.5) hold for $b=1$, $k=1$ and any large $T>0$. Indeed,

$$L(k,\,T,\,1)=k(1-e^{-T})\,,\quad \liminf_{t\to+\infty} f(t)=-\frac{1}{2}\,,\quad \limsup_{t\to+\infty} f(t)=\frac{3}{2}\,.$$

Thus, any solution x, with the condition $|x(t)|\leq 1$, $t\geq 0$, is oscillatory. For example, $x(t)=\cos t$, $t\geq -\pi$, is an oscillatory solution of this equation.

As we can see from Corollary 2.2 and Example 2.2, the oscillation of solutions of (2.1) can be affected by k and T in the quantity $L(k, T, b)$. The following theorem gives us some results when the quantity $L(k, T, b)$ does not depend on k and T.

THEOREM 2.2. *Assume that the quantity $L(k, T, b)$, say $L(b)$, does not depend on k and T.*

(i) *If*

$$\limsup_{t\to+\infty} b(t)f(t)\geq L(b)\,, \tag{2.8}$$

then every solution x of (2.1) *satisfies* $\limsup_{t\to+\infty} b(t)x(t)\geq 0$;

(ii) *If*

$$\liminf_{t\to+\infty} b(t)f(t)\leq -L(b)\,,$$

then every solution of (2.2) *satisfies* $\liminf_{t\to+\infty} b(t)x(t)\leq 0$;

(iii) *If*

$$\min\left\{\limsup_{t\to+\infty} b(t)f(t),\ -\liminf_{t\to+\infty} b(t)f(t)\right\}>L(b)\,, \tag{2.9}$$

then every solution of (1.1) *is oscillatory;*

(iv) *If*

$$-\liminf_{t\to+\infty} b(t)f(t)>\limsup_{t\to+\infty} b(t)f(t)\geq L(b)\,, \tag{2.10}$$

then every nonoscillatory solution x of (1.1) *satisfies* $\limsup_{t\to+\infty} b(t)x(t)=0$;

(v) *If*

$$\limsup_{t\to+\infty} b(t)f(t) > -\liminf_{t\to+\infty} b(t)f(t) \geq L(b),$$

then every nonoscillatory solution x of (1.1) *satisfies* $\liminf_{t\to+\infty} b(t)x(t)=0$.

PROOF. (i) For the sake of contradiction, assume that x is a solution of (2.1) such that

$$\limsup_{t\to+\infty} b(t)x(t) < 0.$$

Then there exists a sufficiently large $T>0$ such that

$$b(t)>0, \qquad x(t)<0,$$

for $t>T$. From (2.1), in view of (1.2) and the last inequalities, we obtain

$$b(t)x(t) \geq b(t)f(t) - \int_0^{T+r} b(t)K(t, s, x_s)ds, \qquad t>T+r.$$

Suppose k is a bound of $x_s(\theta)$, for $\theta\in[-r, 0]$ and $s\in[0, T+r]$. Then the last inequality yields

$$b(t)x(t) \geq b(t)f(t) - \int_0^{T+r} \sup_{\varphi\in B(0,k)} |b(t)||K(t, s, \varphi)|ds.$$

Taking limit superior on both sides of this inequality, we have

$$0 > \limsup_{t\to+\infty} b(t)x(t) \geq \limsup_{t\to+\infty} b(t)f(t) - \limsup_{t\to+\infty} \int_0^{T+r} \sup_{\varphi\in B(0,k)} |b(t)||K(t, s, \varphi)|ds$$
$$= \limsup_{t\to+\infty} b(t)f(t) - L(b),$$

which contradicts (2.8). The proof of (i) is complete.

Case (ii) can be proved similarly.

Case (iii) is an immediate consequence of Case (iii) of Theorem 2.1.

For Case (iv), from (2.10) we see that (2.8) holds. Thus every solution x of (2.1) has the property that

$$\limsup_{t\to+\infty} b(t)x(t) \geq 0. \tag{2.11}$$

On the other hand, (2.10) implies (2.5), so every solution x of (2.2) satisfies

$$\limsup_{t\to+\infty} b(t)x(t) \leq 0. \tag{2.12}$$

Take a nonoscillatory solution x of (1.1); then x satisfies (2.1) and (2.2) and therefore from (2.11) and (2.12), we get $\limsup_{t\to+\infty} b(t)x(t)=0$.

Case (v) can be proved as Case (iv).
The proof of the theorem is complete.

Now assume that $K(t, s, \varphi)$ satisfies the following conditions.

(2.13) There exists a function $a(\cdot, \cdot): \boldsymbol{R}^+ \times \boldsymbol{R}^+ \mapsto \boldsymbol{R}^+$, such that $a(t, s)=0$ for $t<s$ and for every large number $T>0$, $a(t, \cdot) \in L^p[0, T]$ and for every positive number k there exists a function $m_k(\cdot) \in L^q[0, T]$ such that

$$|K(t, s, \varphi)| \leq a(t, s) \cdot m_k(s)$$

for all $\varphi \in B(0, k)$. Here $1/p+1/q=1$, $1 \leq p<+\infty$, $1<q \leq +\infty$.

THEOREM 2.3. *Assume that* (2.13) *holds and*

$$\limsup_{t \to +\infty} \int_0^T a^p(t, s) ds = 0 . \tag{2.14}$$

(i) *If*

$$\limsup_{t \to +\infty} f(t)>0 \quad (\textit{resp.} \liminf_{t \to +\infty} f(t)<0) ,$$

then (2.1) ((2.2)) *has no eventually nonpositive* (*resp. nonnegative*) *solutions.*

(ii) *If f is strongly oscillatory then every solution of* (1.1) *is oscillatory.*

(iii) *If*

$$\limsup_{t \to +\infty} f(t) \geq 0 > \liminf_{t \to +\infty} f(t) \ (\textit{resp.} \limsup_{t \to +\infty} f(t)>0 \geq \liminf_{t \to +\infty} f(t)) ,$$

then for every nonoscillatory solution x of (1.1), *we have*

$$\limsup_{t \to +\infty} x(t)=0 \quad (\textit{resp.} \liminf_{t \to +\infty} x(t)=0) .$$

PROOF. Let $b(t) \equiv 1$. Then from (2.3), using (2.13) and the Hölder inequality, we obtain

$$\begin{aligned} L(k, T, 1) &:= \limsup_{t \to +\infty} \int_0^T \sup_{\varphi \in B(0,k)} |K(t, s, \varphi)| ds \\ &\leq \limsup_{t \to +\infty} \int_0^T a(t, s) m_k(s) ds \\ &\leq \limsup_{t \to +\infty} \left(\int_0^T a^p(t, s) ds \right)^{1/p} \left(\int_0^T m_k^q(s) ds \right)^{1/q} . \end{aligned}$$

In view of (2.14), we obtain $L(k, T, 1) \equiv 0$ for every $k>0$ and $T>0$. Observe now that (i) follows from Case (i) (resp. Case (ii)) of Theorem 2.1; (ii) follows from Case (iii) of Theorem 2.2 and (iii) follows from Case (iv) (resp. Case (v)) of Theorem 2.2.

In order to investigate how the strong oscillation of the forcing term f affects the strong oscillation of the solutions of (1.1), we make the following assumptions.

(2.15) There exists a function a: $\boldsymbol{R}^+ \times \boldsymbol{R}^+ \mapsto \boldsymbol{R}^+$, such that $a(t, s) = 0$ for $t < s$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ with the property that

$$|K(t, s, \xi)| \leq \varepsilon \cdot a(t, s)$$

holds for all constant functions ξ with $|\xi| < \delta$, $t \geq s \geq T_0$ and for some $\mu > 0$,

$$p(t) := \int_{T_0}^{t} b(t)a(t, s)ds \leq \mu, \qquad t > T_0,$$

where T_0 is a sufficiently large real number.

THEOREM 2.4. *Assume that* (2.9) *and* (2.15) *hold. Assume further that* $K(\cdot, \cdot, \varphi)$ *depends only on* $\varphi(-r)$, *namely,*

$$K(t, s, \varphi) := K(t, s, \varphi(-r)) \tag{2.16}$$

and that

$$\liminf_{t \to +\infty} b(t) =: \beta > 0. \tag{2.17}$$

Then for every solution x *of* (1.1), *the function* $b \cdot x$ *is strongly oscillatory.*

PROOF. We have to prove that for every solution x of (1.1), it holds

$$\liminf_{t \to +\infty} b(t)x(t) < 0 < \limsup_{t \to +\infty} b(t)x(t).$$

From Case (iii) of Theorem 2.2, it follows that x is oscillatory. By (2.17), we see that the function $b \cdot x$ is also oscillatory. Therefore we have

$$\liminf_{t \to +\infty} b(t)x(t) \leq 0 \leq \limsup_{t \to +\infty} b(t)x(t).$$

For the sake of contradiction, suppose that x is a solution of (1.1) such that

$$\limsup_{t \to +\infty} b(t)x(t) = 0. \tag{2.18}$$

Since x is oscillatory, the following two sets

$$E^+(t, T) := \{s \in \boldsymbol{R}: T \leq s < t, x(s) > 0\},$$

$$E^-(t, T) := \{s \in \boldsymbol{R}: T \leq s < t, x(s) \leq 0\},$$

are nonempty for T and $t - T$ sufficiently large. It is clear that for T and $t - T$ sufficiently large it holds

$$E^+(t, T) \cup E^-(t, T) = [T, t), \qquad E^+(t, T) \cap E^-(t, T) = \varnothing .$$

From (2.9), there exists a number $N > 0$ such that

$$\limsup_{t \to +\infty} b(t)f(t) - L(b) \geq N .$$

Take an $\varepsilon > 0$ such that $\varepsilon < N/\mu$ (see (2.15)); then, by (2.15), there exists a $\delta > 0$ such that

$$(2.19) \qquad |K(t, s, \xi)| \leq \varepsilon \cdot a(t, s) \qquad \text{for} \quad |\xi| < \delta, \quad t \geq s \geq T_0 > T .$$

On the other hand, for such a $\delta > 0$, from (2.17) and (2.18), there exists a sufficiently large number $T_1 > T_0$ such that

$$b(t) > 0, \quad t \geq T_1, \quad 0 < x(s) < \delta, \qquad \text{on} \quad E^+(t, T_1) .$$

Therefore

$$|K(t, s, x(s-r)| \leq \varepsilon \cdot a(t, s), \qquad \text{for} \quad s \in E^+(t, T_1 + r) .$$

Now from (1.1), in view of (1.2) and (2.16), we obtain

$$\begin{aligned}
b(t)x(t) &= b(t)f(t) - \int_0^t b(t)K(t, s, x(s-r))ds \\
&= b(t)f(t) - \int_0^{T_1+r} b(t)K(t, s, x(s-r))ds - \int_{T_1+r}^t b(t)K(t, s, x(s-r))ds \\
&= b(t)f(t) - \int_0^{T_1+r} b(t)K(t, s, x(s-r))ds - \int_{E^+(t,T_1+r)} b(t)K(t, s, x(s-r))ds \\
&\quad - \int_{E^-(t,T_1+r)} b(t)K(t, s, x(s-r))ds \\
&\geq b(t)f(t) - \int_0^{T_1+r} b(t)K(t, s, x(s-r))ds \\
&\quad - \int_{E^+(t,T_1+r)} b(t)K(t, s, x(s-r))ds , \qquad \text{for} \quad t > T_1 + r .
\end{aligned}$$

Since $x(s-r)$ is bounded for $s \in [0, T_1 + r]$, there exists a number $k > 0$ such that

$$|x(s-r)| \leq k \qquad \text{for} \quad s \in [0, T_1 + r] .$$

By using (2.19) and (2.15), from the last inequality, we have

$$b(t)x(t) \geq b(t)f(t) - \int_0^{T_1+r} \sup_{|\xi| \leq k} |b(t)||K(t, s, \xi)|ds - \varepsilon \int_{T_1+r}^t b(t)a(t, s)ds$$

$$\geq b(t)f(t) - \int_0^{T_1+r} \sup_{|\xi|\leq k} |b(t)||K(t,s,\xi)|ds - \varepsilon\cdot\mu .$$

So we have, by (2.18), that

$$0 = \limsup_{t\to+\infty} b(t)x(t) \geq \limsup_{t\to+\infty} b(t)f(t) - \limsup_{t\to+\infty} \int_0^{T_1+r} \sup_{|\xi|\leq k} |b(t)||K(t,s,\xi)|ds - \varepsilon\cdot\mu$$

$$= \limsup_{t\to+\infty} b(t)f(t) - L(b) - \varepsilon\cdot\mu \geq N - \varepsilon\cdot\mu ,$$

which contradicts the choice of ε. Therefore $\limsup_{t\to+\infty} b(t)x(t) > 0$.

In a similar way, we can prove that $\liminf_{t\to+\infty} b(t)x(t) < 0$.

The proof is complete.

REMARK 2.1. In Theorem 2.4, assume that $L(1)=0$ and that the forcing function f is strongly oscillatory. Then every solution of (1.1) is strongly oscillatory. For example, all conditions of Theorem 2.4 are satisfied for the integral equation

$$x(t) = \sin t + \frac{\cos t - 1}{1+t} - \int_0^t \frac{1}{1+t} x(s-\pi)ds , \qquad t\geq 0 .$$

Thus all solutions of this equation are strongly oscillatory. For example, $x(t)=\sin t$ is a strongly oscillatory solution. Note, however, that in Example 2.1, there are solutions which are not strongly oscillatory, though the forcing term f is strongly oscillatory. This is due to the fact that the function $p(\cdot)$ in (2.15) is not bounded. Indeed.

$$p(t) = \int_{T_0}^t \frac{s^2}{\pi+t} ds = \frac{t^3 - (T_0)^3}{3(\pi+t)} \to +\infty , \qquad \text{as} \quad t\to+\infty .$$

THEOREM 2.5. *Suppose that* (2.13) *holds where we now assume that* $m_k(\cdot)\in L^q[0,+\infty)$. *If*

$$\lim_{t\to+\infty} \int_0^t a^p(t,s)ds = 0 , \tag{2.20}$$

then for any bounded solution x *of* (1.1), *we have*

$$\left[\liminf_{t\to+\infty} x(t), \limsup_{t\to+\infty} x(t)\right] \supseteq \left[\liminf_{t\to+\infty} f(t), \limsup_{t\to+\infty} f(t)\right].$$

PROOF. Take a bounded solution x of (1.1) and let δ be a number in the interval $[\liminf_{t\to+\infty} f(t), \limsup_{t\to+\infty} f(t)]$. For the sake of contradiction, assume that either

$$\delta < \liminf_{t\to+\infty} x(t) , \quad \text{or} \quad \delta > \limsup_{t\to+\infty} x(t) .$$

This is equivalent to saying that there exist positive numbers T and η such that

$$\text{either} \quad x(t) > \delta + \eta \quad \text{or} \quad x(t) < \delta - \eta \qquad \text{for} \quad t > T.$$

Let us consider the first case. In view of boundedness of x and (2.13), by (1.1), we have

$$\delta + \eta < x(t) \le f(t) + \left(\int_0^t a^p(t, s) ds \right)^{1/p} \left(\int_0^t m_k^q(s) ds \right)^{1/q},$$

for $t > T$, where k is a bound of x on $[0, +\infty)$. Taking (2.20) into account, we have

$$\delta + \eta \le \liminf_{t \to +\infty} x(t) \le \liminf_{t \to +\infty} f(t) \le \delta,$$

a contradiction. In a similar way, we can see that $x(t) < \delta - \eta$ does not hold for $t > T$ and the proof is complete.

Note that if all conditions in Theorem 2.5 hold and f is strongly oscillatory, then x is strongly oscillatory. Indeed,

$$\liminf_{t \to +\infty} x(t) \le \liminf_{t \to +\infty} f(t) < 0 \quad \text{and} \quad \limsup_{t \to +\infty} x(t) \ge \limsup_{t \to +\infty} f(t) > 0.$$

Observe that the results in Theorems 2.1, 2.2, 2.4 and 2.5 are valid when the delay r is zero. So it is interesting to look for conditions which involve the delay r.

Define a set $\tilde{C}$ as follows:

$$\tilde{C} := \left\{ \varphi \in C_r;\ \sigma < \liminf_{t \to +\infty} \int_{-r}^0 K(t, s+r, \varphi(s)) ds \le \limsup_{t \to +\infty} \int_{-r}^0 K(t, s+r, \varphi(s)) ds < \varrho \right\},$$

where $\sigma = L(1) + \liminf_{t \to +\infty} f(t)$, $\varrho = -L(1) + \limsup_{t \to +\infty} f(t)$.

It is obvious that $\varphi \in \tilde{C}$ implies

$$L(1) < \min \left\{ \limsup_{t \to +\infty} f(t) - \limsup_{t \to +\infty} \int_{-r}^0 K(t, s+r, \varphi(s)) ds, \right. \tag{2.21}$$
$$\left. -\liminf_{t \to +\infty} f(t) + \liminf_{t \to +\infty} \int_{-r}^0 K(t, s+r, \varphi(s)) ds \right\}.$$

THEOREM 2.6. *Assume that* (2.15) *and* (2.16) *hold. Then every solution of* (1.1) *with initial function* $\varphi \in \tilde{C}$ *is strongly oscillatory.*

PROOF. We observe that, in view of (2.16), the equation (1.1) can be written as a Volterra integral equation of the form

$$x(t) = F(t) - \int_0^t K_1(t, s+r, x(s)) ds, \qquad t \ge 0, \tag{1.1$_1$}$$

where

$$K_1(t, s+r, \xi) := \begin{cases} K(t, s+r, \xi), & \text{for } s \in [0, t-r], \\ 0, & \text{for } s \in [t-r, t], \end{cases}$$

and

$$F(t) := f(t) - \int_{-r}^{0} K(t, s+r, \varphi(s))ds .$$

Suppose that x is a solution of (1.1) with initial function $\varphi \in \tilde{C}$. It is clear that the function $K_1(t, s+r, \xi)$ satisfies (2.15) and (2.16). Also $F(t)$ satisfies (2.9) for $b(t) = 1$. Indeed,

$$\begin{aligned} \limsup_{t \to +\infty} F(t) - L(1) &= \limsup_{t \to +\infty} \left[f(t) - \int_{-r}^{0} K(t, s+r, \varphi(s))ds - L(1) \right] \\ &\geq \limsup_{t \to +\infty} f(t) - L(1) - \limsup_{t \to +\infty} \int_{-r}^{0} K(t, s+r, \varphi(s))ds \\ &= \varrho - \limsup_{t \to +\infty} \int_{-r}^{0} K(t, s+r, \varphi(s))ds > 0 , \end{aligned}$$

and

$$\liminf_{t \to +\infty} F(t) + L(1) \leq \sigma - \liminf_{t \to +\infty} \int_{-r}^{0} K(t, s+r, \varphi(s))ds < 0 .$$

So (2.9) is satisfied for $(1.1)_1$ and therefore, by Theorem 2.4, x is strongly oscillatory.

REMARK 2.2. Observe that for $b = 1$ and for $r = 0$ or

$$\lim_{t \to +\infty} \int_{-r}^{0} K(t, s+r, \varphi(s))ds = 0 ,$$

the condition (2.21) is equivalent to (2.9). However if

$$\lim_{t \to +\infty} \int_{-r}^{0} K(t, s+r, \varphi(s))ds = \zeta \neq 0 ,$$

then (2.21) is different from (2.9). Therefore, in some cases solutions of (1.1) may be strongly oscillatory though f is not strongly oscillatory.

Now the question is when the set $\tilde{C}$ is nonempty. In [29, p. 66] the existence of solutions of the following Fredholm integral equation

$$\int_{-r}^{0} A(t, s+r)\varphi(s)ds = g(t)$$

was studied under certain conditions on the functions A and g. Thus, it suffices to assume that

$$\sigma<\liminf_{t\to+\infty} g(t)\leq\limsup_{t\to+\infty} g(t)<\varrho .$$

We can also give an answer to this question as follows.

If $\sigma<0<\varrho$, set

$$\xi:=\min\left\{\frac{\varrho}{\limsup\limits_{t\to+\infty}\int_0^r |A(t,s)|ds},\ \frac{-\sigma}{\liminf\limits_{t\to+\infty}\int_0^r |A(t,s)|ds}\right\}.$$

Then $B(0,\xi)\subseteq\tilde{C}$. Indeed,

$$\limsup_{t\to+\infty}\int_{-r}^0 A(t,s+r)\varphi(s)ds\leq\|\varphi\|\limsup_{t\to+\infty}\int_0^r |A(t,s)|ds<\varrho ,$$

and

$$\liminf_{t\to+\infty}\int_{-r}^0 A(t,s+r)\varphi(s)ds\geq-\|\varphi\|\liminf_{t\to+\infty}\int_0^r |A(t,s)|ds>\sigma .$$

If $0<\sigma<\varrho$ or $\sigma<\varrho<0$ and $A(t,s)>0$ for all t, s, then we suppose that

$$\sigma\cdot\limsup_{t\to+\infty}\int_0^r A(t,s)ds<\varrho\cdot\liminf_{t\to+\infty}\int_0^r A(t,s)ds .$$

Define a subset $\hat{C}$ of C_r as follows

$$\hat{C}:=\left\{\varphi\in C_r;\ \frac{\sigma}{\liminf\limits_{t\to+\infty}\int_0^r A(t,s)ds}\leq\varphi(s)\leq\frac{\varrho}{\limsup\limits_{t\to+\infty}\int_0^r A(t,s)ds},\ s\in[-r,0]\right\}.$$

Then it is easy to see that $\hat{C}$ is a nonempty subset of $\tilde{C}$.

THEOREM 2.7. *Assume that* (2.13) *holds.*

(i) *If*

$$\limsup_{t\to+\infty}\frac{f(t)}{\left(\int_0^T a^p(t,s)ds\right)^{1/p}}=+\infty , \tag{2.22}$$

then (2.1) *has no eventually nonpositive solutions.*

(ii) *If*

$$\liminf_{t\to+\infty}\frac{f(t)}{\left(\int_0^T a^p(t,s)ds\right)^{1/p}}=-\infty , \tag{2.23}$$

then (2.2) *has no eventually nonnegative solutions.*

(iii) *If both* (2.22) *and* (2.23) *hold, then every solution of* (1.1) *is oscillatory.*

PROOF. First consider Case (i). Assume, for the sake of contradiction, that x is an eventually nonpositive solution of (2.1). Then, there exists a sufficiently large T such that

$$x(t) \le 0, \qquad t > T.$$

From (2.1), in view of (1.2), we obtain

$$0 \ge x(t) \ge f(t) - \int_0^{T+r} K(t, s, x_s)ds \ge f(t) - N\left(\int_0^{T+r} a^p(t, s)ds\right)^{1/p}, \quad t > T+r, \tag{2.24}$$

where

$$N := \left(\int_0^{T+r} m_k^q(t)ds\right)^{1/q}.$$

Now, from (2.22), it is clear that there exists a sequence $\{t_n\}$ such that $\lim_{t\to+\infty} t_n = +\infty$, and

$$\limsup_{t\to+\infty} \frac{f(t)}{\left(\int_0^T a^p(t, s)ds\right)^{1/p}} = \lim_{n\to+\infty} \frac{f(t_n)}{\left(\int_0^T a^p(t_n, s)ds\right)^{1/p}} = +\infty.$$

This is equivalent to saying that for any large number $M > 0$, there is an $n_0 > 0$ such that

$$f(t_n) \ge M\left(\int_0^T a^p(t_n, s)ds\right)^{1/p}, \qquad \text{for} \quad n > n_0. \tag{2.25}$$

Choose M so that $M > N$ and n_0 so that $t_n > T+r$ for $n > n_0$. Combining (2.24) and (2.25), we have

$$0 \ge x(t_n) \ge (M-N)\left(\int_0^T a^p(t_n, s)ds\right)^{1/p} > 0, \qquad \text{for} \quad n > n_0, \tag{2.26}$$

which is a contradiction. So we have completed the proof for (i).

Case (ii) can be proved similarly.

Case (iii) is an immediate consequence of Case (i) and Case (ii).

REMARK 2.3. Note that in Theorem 2.7, the function f has to be oscillatory. Otherwise (2.22) and (2.23) cannot be true. However in some cases, the function f may not be strongly oscillatory. An example is presented in the following remark.

REMARK 2.4. Observe that Theorem 1 in [20] is a special case of Theorem 2.6 (iii). Indeed, in [20], the stronger conditions

$$\limsup_{t \to +\infty} f(t) = \liminf_{t \to +\infty} f(t) = +\infty ,$$

are required. Our conditions (2.22) and (2.23) are satisfied by the assumptions in [20]. However the converse is not true. For example, for the integral equation

$$x(t) = f(t) - \int_0^t \frac{1+s}{(1+t)^2} x(s-\pi) ds , \qquad t > 0 ,$$

where

$$f(t) = \frac{\sin t}{1+t+\pi} + \frac{\cos t - 1}{(1+t)^2} ,$$

we have

$$\limsup_{t \to +\infty} f(t) = \liminf_{t \to +\infty} f(t) = 0 .$$

Namely, the conditions of Theorem 1 in [20] are not satisfied. However, the conditions (2.22) and (2.23) are satisfied. Therefore, all solutions of this equation oscillate. For example,

$$x(t) = \frac{\sin t}{1+t+\pi},$$

is an oscillatory solution.

The following result can be obtained from the proof of Theorem 2.7.

COROLLARY 2.3. *Assume that* (2.13) *holds and* $\|m_k(\cdot)\|_{L^q[0,+\infty)} \leq N$, *where* N *is a positive number which does not depend on* k. *If*

$$\limsup_{t \to +\infty} \frac{f(t)}{\left(\int_0^T a^p(t, s) ds\right)^{1/p}} > N , \qquad \liminf_{t \to +\infty} \frac{f(t)}{\left(\int_0^T a^p(t, s) ds\right)^{1/p}} < -N ,$$

then every solution of (1.1) *is oscillatory.*

2. 3. Quick, moderate and slow oscillations. In this section sufficient conditions are established under which the solutions of (1.1) belong to the set S_q, S_m or S_s, respectively, when the forcing term f belongs to the set S_q, S_m or S_s, respectively.

THEOREM 3.1. *Assume that all conditions in Theorem* 2.5 *are satisfied. Then the following statements hold.*

(i) *If* $f \in S_q$, *then every bounded solution of* (1.1) *belongs to* S_q.
(ii) *If* $f \in S_m$, *then every bounded solution of* (1.1) *belongs to* S_m.
(iii) *If* $f \in S_s$, *then every bounded solution of* (1.1) *belongs to* S_s.

PROOF. In Section 1, we have seen that if $f \in S_q \cup S_m \cup S_s$, then f is strongly oscillatory. Also by Theorem 2.5, we know that every bounded solution of (1.1) is strongly oscillatory.

Now consider Case (i). Take a bounded solution x of (1.1); by (1.1), we have

$$|x(t)-f(t)| \leq \left(\int_0^t a^p(t,s)ds\right)^{1/p}\left(\int_0^t m^q(t)ds\right)^{1/q},$$

and in view of (2.20), we obtain

$$\lim_{t\to+\infty} |x(t)-f(t)|=0. \tag{3.1}$$

On the other hand, since $f \in S_q$, there exists an $\eta>0$, and for any $\varepsilon: 0<\varepsilon<\eta/2$, there exists a $T>0$ such that for every $t>T$ there exist two points s_1 and s_2 in $[0, \varepsilon]$ satisfying

$$f(t+s_1)<-\eta \quad \text{and} \quad f(t+s_2)>\eta. \tag{3.2}$$

Also for this ε, by (3.1), there exists a sufficiently large number $T_1>T$ such that

$$|x(t)-f(t)|<\varepsilon, \qquad \text{for} \quad t>T_1. \tag{3.3}$$

From (3.2) and (3.3), we see that

$$x(t+s_1)<f(t+s_1)+\varepsilon<-\eta+\varepsilon<-\frac{\eta}{2},$$

and

$$x(t+s_2)>f(t+s_2)-\varepsilon>\eta-\varepsilon>\frac{\eta}{2}.$$

Therefore, $x \in S_q$.

Consider Case (ii). Since $f \in S_m$, there exist positive numbers η, ζ, T and for every $t>T$ there exist two points s_1, $s_2 \in [0, \zeta]$ such that

$$f(t+s_1)<-\eta, \qquad f(t+s_2)>\eta. \tag{3.4}$$

Take an ε such that $0<\varepsilon<\eta/2$, from (3.1), there exists a $T_1>T$ such that

$$|x(t)-f(t)|<\varepsilon, \qquad \text{for} \quad t>T_1. \tag{3.5}$$

From (3.4) and (3.5), we see that

$$x(t+s_1)<f(t+s_1)+\varepsilon<-\eta+\varepsilon<-\frac{\eta}{2}, \qquad \text{for} \quad t>T_1,$$

and

$$x(t+s_2)>f(t+s_2)-\varepsilon>\eta-\varepsilon>\frac{\eta}{2}, \qquad \text{for} \quad t>T_1.$$

Therefore $x\in S_m$.

For Case (iii), since $f\in S_m$, there exists a positive number η and for any large T and large M, there exist two points γ_1 and γ_2 such that

$$\gamma_1>\gamma_2>T, \qquad \gamma_1-\gamma_2>M$$

and

$$|f(t)|>\eta \qquad \text{for} \quad t\in(\gamma_1, \gamma_2). \tag{3.6}$$

Take an ε: $0<\varepsilon<\eta/2$. From (3.1), there exists a $T_1\geq T$ such that

$$|x(t)-f(t)|<\varepsilon, \qquad \text{for} \quad t>T_1.$$

Without loss of generality we may assume that $\gamma_1>\gamma_2>T_1$. Then we have

$$|x(t)|>|f(t)|-\varepsilon, \qquad \text{for} \quad t\in(\gamma_1, \gamma_2). \tag{3.7}$$

Then (3.6) and (3.7) yield

$$|x(t)|>\eta-\varepsilon>\frac{\eta}{2}, \qquad \text{for} \quad t\in(\gamma_1, \gamma_2).$$

This means that $x\in S_s$. The proof is complete.

EXAMPLE 3.1. For the Volterra integral equation

$$x(t)=f(t)-\int_0^t \frac{1+s}{(1+t)^3}x(s-\pi)ds, \qquad t\geq 0,$$

where

$$f(t):=\sin\sqrt{t+\pi}+\frac{1}{(1+t)^3}[(10\sqrt{t}-2t^{3/2})\cos\sqrt{t}+(6t-10)\sin\sqrt{t}],$$

all conditions of Theorem 3.1 are satisfied. Indeed, $p=1$, $q=+\infty$, $\|m_k(\cdot)\|_{L^\infty}=k$ and

$$\int_0^t a(t, s)ds=\int_0^t \frac{1+s}{(1+t)^3}\, ds=\frac{t+t^2/2}{(1+t)^3}\to 0, \qquad \text{as} \quad t\to+\infty.$$

We observe that $f\in S_s$ and therefore every bounded solution belongs to S_s. For example, $x(t)=\sin\sqrt{t+\pi}$ is a bounded solution and belongs to S_s.

EXAMPLE 3.2. Consider the Volterra integral equation

$$x(t)=\frac{t}{t+\pi}\cos t-\int_0^t \frac{s^2}{\pi+t}\, x(s-\pi)ds, \qquad t\geq 0.$$

We see that f is strongly oscillatory and moderately oscillatory. In fact, $f\in S_m$. It is easy to check that

$$x(t)=\frac{\sin t}{t+\pi},$$

is a moderately oscillatory solution but not strongly oscillatory. So $x\notin S_m$. As a matter of fact, the condition (2.20) in Theorem 3.1 is false, since

$$\lim_{t\to+\infty}\int_0^t a(t,s)ds=\lim_{t\to+\infty}\int_0^t \frac{s^2}{\pi+t}ds=+\infty .$$

Theorem 3.1 gives information about bounded solutions only. In order to have information for oscillation of all solutions we restrict ourselves to the following equation

$$x(t)=f(t)-\int_0^t K(t,s,x(s-r))ds, \qquad t\geq 0. \tag{3.8}$$

Here assume the following:

(3.9) $f:[0,+\infty)\to \boldsymbol{R}$ is bounded;

(3.10) $K(t,s,\xi)$ is measurable in s, continuous in t and $\xi\in\boldsymbol{R}$, $0\leq s\leq t$ and, $|K(t,s,\xi)|\leq B(t)L(s,|\xi|)$, where $B:\boldsymbol{R}^+\mapsto\boldsymbol{R}^+$, $L:\boldsymbol{R}^+\times\boldsymbol{R}^+\mapsto\boldsymbol{R}^+$ are continuous and B is bounded;

(3.11) $0\leq L(s,u)-L(s,v)\leq M(s,v)(u-v)$, $u\geq v\geq 0$, where $M:\boldsymbol{R}^+\times\boldsymbol{R}^+\mapsto\boldsymbol{R}^+$ is continuous;

(3.12) $\int_0^{+\infty} L(u+r,\xi)ds<+\infty$ and $\int_0^{+\infty} M(u+r,\xi)ds<+\infty$, hold for all $\xi:|\xi|\leq N$, where N is a positive constant.

The following lemma is borrowed from [7, p. 6].

LEMMA 3.1. *Let $A, B:[\alpha,\beta)\mapsto\boldsymbol{R}^+$, $L:[\alpha,\beta)\times\boldsymbol{R}^+\mapsto\boldsymbol{R}^+$ be continuous and suppose L satisfies* (3.11). *Then every nonnegative continuous solution x of the integral inequality*

$$x(t)\leq A(t)+B(t)\int_\alpha^t L(s,x(s))ds, \qquad t\in[\alpha,\beta)$$

satisfies

$$x(t)\leq A(t)+B(t)\int_\alpha^t L(u,A(u))\exp\left(\int_u^t M(s,A(s))B(s)ds\right)du, \qquad \text{for} \quad t\in[\alpha,\beta).$$

THEOREM 3.2. *Assume that* (3.9)–(3.12) *hold and* $\lim_{t\to+\infty}B(t)=0$. *Then every solution of* (3.8) *is bounded. Furthermore we have the following*:

(i) *If $f\in S_q$, then every solution of* (3.8) *belongs to S_q.*

(ii) *If $f\in S_m$, then every solution of* (3.8) *belongs to S_m.*

(iii) *If $f \in S_s$, then every solution of* (3.8) *belongs to* S_s.

PROOF. First we prove that every solution of (3.8) is bounded. From (3.8), we have

$$|x(t)| \leq |f(t)| + \int_0^t |K(t, s, x(s-r))| ds ,$$

and in view of (3.9)–(3.11), we have

$$|x(t)| \leq m + b \int_{-r}^{0} L(s+r, \|\varphi\|) ds + b \int_0^{t-r} L(s+r, |x(s)|) ds$$

$$\leq m + b \int_{-r}^{0} L(s+r, \|\varphi\|) ds + b \int_0^{t} L(s+r, |x(s)|) ds ,$$

for $t > 0$, where φ is the initial function of x. By Lemma 3.1, we have

$$|x(t)| \leq A + b \int_0^t L(u+r, A) \exp\left(b \int_u^t M(s+r, A) ds\right) du , \qquad \text{for} \quad t \geq 0 ,$$

where $A = m + b \int_{-r}^{0} L(s+r, \|\varphi\|) ds$.

In view of (3.12), we see that every solution is bounded.

On the other hand, by (3.8), we obtain

$$|x(t) - f(t)| \leq B(t) \int_0^t L(s, \xi) ds , \qquad t > 0 ,$$

where $\xi > 0$ is a bound of $x(t)$. Taking (3.12) into account and in view of the assumption

$$\lim_{t \to +\infty} B(t) = 0 ,$$

we have

$$\lim_{t \to +\infty} |x(t) - f(t)| = 0 .$$

Now we can follow a procedure analogous to that for the proof of Theorem 3.1, and prove the statements (i), (ii) and (iii).

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