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VALUES OF p-ADIC L-FUNCTIONS AT POSITIVE INTEGERS AND p-ADIC LOG MULTIPLE GAMMA FUNCTIONS

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Abstract. We consider p-adic analogues of multiple gamma functions, and express values of p-adic L-functions at positive integers in terms of these p-adic multiple gamma functions.

Introduction. For a prime number p and for a Dirichlet character defined modulo some integer, the p-adic L-function was constructed by interpolating the values of the complex analytic L-function at non-positive integers. Diamond [6] obtained formulas which express the values of p-adic L-function at positive integers in terms of the p-adic log gamma function. In this paper, we generalize his results to the case of the p-adic L-functions constructed by the author in [9], and obtain formulas which express their values at positive integers in terms of the p-adic log multiple gamma functions. Since the p-adic L-functions of a totally real algebraic number field can be expressed in terms of the p-adic L-functions we are considering, their values at positive integers can also be expressed in terms of the p-adic log multiple gamma functions.

1. Some p-adic integrals. Let p be a prime number. Let $\boldsymbol{Z}$, $\boldsymbol{Z}_p$, $\boldsymbol{Q}_p$, Ω_p, $\mathcal{O}_p$ and $\mathfrak{m}$ be the ring of rational integers, the ring of p-adic rational integers, the p-adic number field, the completion of an algebraic closure of $\boldsymbol{Q}_p$, the integer ring of Ω_p and the maximal ideal of $\mathcal{O}_p$, respectively.

We first define some twists of the Bernoulli numbers. Let c be a positive integer prime to p, and $\xi \in \Omega_p$ a c-th root of 1 different from 1. We define numbers $B_{k,\xi}$ and polynomials $B_{k,\xi}(x)$ for $k \geqq 0$ by the following formulas:

$$(\xi \exp(t) - 1)^{-1} = \sum_{k \geqq 0} B_{k,\xi} t^k / k!\,,$$

$$\exp(xt)(\xi \exp(t) - 1)^{-1} = \sum_{k \geqq 0} B_{k,\xi}(x) t^k / k!\,.$$

Then, by using the method which was used in [10, pp. 7–15], we can prove the following lemma.

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LEMMA 1.

$$B_{k,\xi}=\frac{1}{k+1}\lim_{N\to\infty}\frac{1}{cp^N}\sum_{0\leqq m<cp^N}\xi^m m^{k+1}.$$

For any $\xi\in\Omega_p$ which satisfies the above condition for some $c\in\boldsymbol{N}$, $(c,p)=1$, we denote by μ_ξ the p-adic measure on $\boldsymbol{Z}_p$ constructed in Koblitz [11, Proposition 2]:

$$\mu_\xi(a+p^N\boldsymbol{Z}_p)=\xi^a(1-\xi^{p^N})^{-1}\qquad\text{for}\quad 0\leqq a<p^N.$$

In what follows, we fix a positive integer r. For each $1\leqq j\leqq r$, let c_j be a positive integer prime to p, and let ξ_j be a nontrivial c_j-th root of 1. Let μ_{ξ_j} be Koblitz' p-adic measure on $\boldsymbol{Z}_p$, and let $\mu_\xi=\prod_{1\leqq j\leqq r}\mu_{\xi_j}$ be the product measure on the product space $\boldsymbol{Z}_p^r$. Let $y=(y_1,\dots,y_r)$ be a variable on $\boldsymbol{Z}_p^r$.

LEMMA 2. *For any* $b_1,\dots,b_r\in\boldsymbol{Z}$, $b_1,\dots,b_r\geqq 0$, $\int_{\boldsymbol{Z}_p^r}y_1^{b_1}\cdots y_r^{b_r}d\mu_\xi(y)$ *is the coefficient of* $t_1^{b_1}\cdots t_r^{b_r}/(b_1!\cdots b_r!)$ *in the Laurent expansion of the function* $\prod_{1\leqq j\leqq r}(1-\xi_j\exp(t_j))^{-1}$.

PROOF. Let $t_1,\dots,t_r$ be p-adic variables with sufficiently small absolute values so that $\exp(y_1t_1+\cdots+y_rt_r)$ converges for any $(y_1,\dots,y_r)\in\boldsymbol{Z}_p^r$. It is known that Koblitz' measure μ_{ξ_j} satisfies

$$\int_{\boldsymbol{Z}_p}\exp(y_jt_j)d\mu_{\xi_j}(y_j)=(1-\xi_j\exp(t_j))^{-1}.$$

Since μ_ξ is the product measure, we obtain

$$\int_{\boldsymbol{Z}_p^r}\exp(y_1t_1+\cdots+y_rt_r)d\mu_\xi(y)=\prod_{1\leqq j\leqq r}(1-\xi_j\exp(t_j))^{-1}.$$

Taking the coefficient of $t_1^{b_1}\cdots t_r^{b_r}/(b_1!\cdots b_r!)$ in the above formula, we obtain the lemma. ■

Let n be a positive integer. Let $D_1\subset\Omega_p^n$, $D_2\subset\Omega_p^r$ be balls in respective spaces such that $\mathcal{O}_p^r\subset D_2$. Let $f(x,y)=f(x_1,\dots,x_n,y_1,\dots,y_r)$ be a holomorphic function on $D_1\times D_2$. $f(x,y)$ is given by a convergent power series in $x_1-a_1,\dots,x_n-a_n,y_1,\dots,y_r$ for some $a_1,\dots,a_n$, which we write as $f(x,y)=\sum_{a_{IJ}}(x-a)^Iy^J$, where $a_{IJ}=a_{i_1\cdots i_nj_1\cdots j_r}\in\Omega_p$, $(x-a)^I=(x_1-a_1)^{i_1}\cdots(x_n-a_n)^{i_n}$, $y^J=y_1^{j_1}\cdots y_r^{j_r}$; for $x\in D_1$, $y\in D_2$, convergence of $f(x,y)$ implies

$$|a_{IJ}(x-a)^Iy^J|_p\to 0\qquad\text{for}\quad |I|+|J|=i_1+\cdots+i_n+j_1+\cdots+j_r\to\infty.$$

Let $g(x,y)$ be the power series of the form $\sum b_{IJ}(x-a)^Iy^J$ ($b_{IJ}\in\Omega_p$, $b_{IJ}=0$ if some component of J is zero) such that

$$\frac{\partial}{\partial y_1}\cdots\frac{\partial}{\partial y_r}g(x,y)=f(x,y).$$

Hence

$$g(x,y)=\sum a_{i_1\cdots i_n j_1\cdots j_r}(x_1-a_1)^{i_1}\cdots(x_n-a_n)^{i_n}\frac{y_1^{j_1+1}}{j_1+1}\cdots\frac{y_r^{j_r+1}}{j_r+1},$$

if

$$f(x,y)=\sum a_{i_1\cdots i_n j_1\cdots j_r}(x_1-a_1)^{i_1}\cdots(x_n-a_n)^{i_n}y_1^{j_1}\cdots y_r^{j_r}.$$

We *assume* that this power series $g(x,y)$ converges on $D_1\times\mathcal{O}_p^r$.

THEOREM 1. *Under the above assumptions,*

$$\int_{\mathbf{Z}_p^r}f(x,y)d\mu_\xi(y)=(-1)^r\lim_{N\to\infty}\frac{1}{c_1\cdots c_r p^{rN}}\sum_{1\leq j\leq r}\sum_{0\leq m_j<c_jp^N}\xi_1^{m_1}\cdots\xi_r^{m_r}g(x,m),$$

where $m=(m_1,\ldots,m_r)$.

PROOF. It suffices to prove this formula for each fixed $x=x_0\in D_1$. Then

$$\frac{\partial}{\partial y_1}\cdots\frac{\partial}{\partial y_r}g(x_0,y)=\left[\frac{\partial}{\partial y_1}\cdots\frac{\partial}{\partial y_r}g(x,y)\right]_{x=x_0}=[f(x,y)]_{x=x_0}=f(x_0,y).$$

Hence $g(x_0,y)$ is the power series of y which is obtained from $f(x_0,y)$ by integration. Thus the theorem for a holomorphic function $f(x,y)$ in x, y follows from that for the restricted holomorphic functions $f(x_0,y)$ $(x_0\in D_1)$ in y. Since $f(x_0,y)$ depends only on y, it suffices to consider the case where $f(x,y)=f(y)=\sum a_J y^J$ is a power series convergent on D_2. Since $\mathcal{O}_p^r\subset D_2$, we have $|a_J|_p\to 0$ when $|J|\to\infty$. Substituting this power series expression in the left hand side of the equation and using the above estimate for the coefficients, we see it suffices to consider the coefficient of $y_1^{b_1}\cdots y_r^{b_r}$. By Lemmas 1 and 2, we have

$$\int_{\mathbf{Z}_p^r}y_1^{b_1}\cdots y_r^{b_r}d\mu_\xi(y)=(-1)^r\lim_{N\to\infty}\frac{1}{c_1\cdots c_r p^{rN}}\sum_{1\leq j\leq r}\sum_{0\leq m_j<c_jp^N}\frac{\xi_1^{m_1}m_1^{b_1+1}}{b_1+1}\cdots\frac{\xi_r^{m_r}m_r^{b_r+1}}{b_r+1}.$$

This proves the theorem. ∎

2. p-adic log multiple gamma functions. For positive integers r and n, let $L_i(y)=L_i(y_1,\ldots,y_r)=\sum_{1\leq j\leq r}a_{ij}y_j$ $(1\leq i\leq n)$ be linear forms in r variables with all coefficients $a_{ij}\in\mathfrak{m}$. Let x_i $(1\leq i\leq n)$ be elements of Ω_p such that $x_i\equiv 1 \pmod{\mathfrak{m}}$. In [9], the p-adic L-function (in n variables) was constructed as the integral

$$Z_p(s)=Z_p(s_1,\ldots,s_n)=\int_{\mathbf{Z}_p^r}\prod_{1\leq i\leq n}(x_i+L_i(y))^{-s_i}d\mu_\xi(y).$$

(In this construction of $Z_p(s)$, the elements x_i are fixed parameters; later we regard x_i as variables.)

Let $\log x = \sum_{k \geq 1} (-1)^{k-1} (x-1)^k / k$ be the p-adic log function. This sum is convergent for $|x-1|_p < 1$ (cf., e.g., Iwasawa [10]). Let $\lambda(L, x, y) = \lambda(L_1, \ldots, L_n;\ x_1, \ldots, x_n;\ y_1, \ldots, y_r)$ be the power series which we obtain by formally integrating

$$\prod_{1 \leqq i \leqq n} \log(x_i + L_i(y)) = \prod_{1 \leqq i \leqq n} (\log x_i + \log(1 + L_i(y)/x_i))$$

with respect to $y_1, \ldots, y_r$. We denote this symbolically as follows:

$$\lambda(L, x, y) = \int_0^{y_r} dy_r \cdots \int_0^{y_1} \prod_{1 \leqq i \leqq n} \log(x_i + L_i(y)) dy_1 .$$

After we express $\log(1 + L_i(y)/x_i) = \sum_{k \geq 1} (-1)^{k-1} (L_i(y) x_i^{-1})^k / k$ as a power series in $y_1, \ldots, y_r$, it is easy to see that $\lambda(L, x, y)$ is holomorphic on $(1+\mathfrak{m})^n \times \mathcal{O}_p^r$.

Now we define a function $G_\xi(L, x)$ generalizing the p-adic log gamma function of Diamond [5], and call it the p-adic log multiple gamma function.

DEFINITION.

$$\begin{aligned} G_\xi(L, x) &= G_{(\xi_1, \ldots, \xi_r)}(L_1, \ldots, L_n;\ x_1, \ldots, x_n) \\ &= (-1)^r \lim_{N \to \infty} \frac{1}{c_1 \cdots c_r p^{rN}} \sum_{1 \leqq j \leqq r} \sum_{0 \leqq m_j < c_j p^N} \xi_1^{m_1} \cdots \xi_r^{m_r} \lambda(L, x, m) , \end{aligned}$$

where $m = (m_1, \ldots, m_r)$.

By [5, Theorem 2], $G_\xi(L, x)$ is a holomorphic function defined for $x \in (1+\mathfrak{m})^n$. By Theorem 1, we have:

PROPOSITION 1.

$$G_\xi(L, x) = \int_{\mathbf{Z}_p^r} \prod_{1 \leqq i \leqq n} \log(x_i + L_i(y)) d\mu_\xi(y) .$$

By the definition of $Z_p(s)$ and by Proposition 1, we obtain the following theorem, which is the main result of this paper:

THEOREM 2. *Let $Z_p(s_1, \ldots, s_n)$ be the p-adic L-function in n variables constructed in* [9], *and $G_\xi(L, x)$ the p-adic log multiple gamma function constructed above. Then we have*

$$\frac{\partial}{\partial s_1} \cdots \frac{\partial}{\partial s_n} Z_p(0, \ldots, 0) = (-1)^n G_\xi(L, x) ,$$

$$Z_p(a_1, \ldots, a_n) = \prod_{1 \leqq i \leqq n} \frac{(-1)^{a_i - 1}}{(a_i - 1)!} \cdot \frac{\partial^{a_1}}{\partial x_1^{a_1}} \cdots \frac{\partial^{a_n}}{\partial x_n^{a_n}} G_\xi(L, x)$$

for any positive integers $a_1, \ldots, a_n$.

REMARK. As explained in the introduction, the p-adic L-functions of a totally real algebraic number field can be expressed in terms of the p-adic L-functions of our type, in fact in terms of $Z_p(s, \ldots, s)$, cf. [2, Théorèmes 22 and 26]. In particular their values at positive integers can also be expressed in terms of the p-adic log multiple gamma functions. To write down these values explicitly, it suffices to quote a formula in the proof of [2, Théorème 26]. Also note that the derivative at 0 of the Kubota-Leopoldt p-adic L-function was expressed in terms of the p-adic log gamma function (cf. [6, Theorem 8]), but the derivative at 0 of the p-adic L-function of a totally real algebraic number field is related to $dZ_p(s, \ldots, s)/ds$ so it cannot be expressed in terms of the p-adic log multiple gamma functions.

Next we prove some properties of the p-adic log multiple gamma functions. Let $L_i(y)=\sum_{1\leqq j\leqq r} a_{ij}y_j$ be as before. We fix a suffix j, and put $\delta_j=(a_{1j}, \ldots, a_{nj})$. For any r-dimensional vector $y=(y_1, \ldots, y_r)$, let $y^{(j)}=(y_1, \ldots, \hat{y}_j, \ldots, y_r)$ be the $(r-1)$-dimensional vector in which the component y_j is omitted. Let $L_i^{(j)}(y^{(j)})=\sum_{k\neq j} a_{ik}y_k$ $(1\leqq i\leqq n)$ be linear forms in $y^{(j)}$. In the above construction of $G_\xi(L, x)$, we replace r by $r-1$, $\xi=(\xi_1, \ldots, \xi_r)$ by $\xi^{(j)}=(\xi_1, \ldots, \hat{\xi}_j, \ldots, \xi_r)$, L_i by $L_i^{(j)}$, and write the resulting function as $G_{\xi^{(j)}}(L^{(j)}, x)$. Note that $G_{\xi^{(j)}}(L^{(j)}, x)$ is defined only for $r\geqq 2$. We omit the suffix j if $r=1$. Then after some calculations, we obtain the following proposition (cf. [11, Proposition 4]).

PROPOSITION 2.

(i) $$\xi_j G_\xi(L, x+\delta_j)-G_\xi(L, x)=-G_{\xi^{(j)}}(L^{(j)}, x), \qquad \textit{if} \quad r\geqq 2,$$

$$\xi G_\xi(L, x+\delta)-G_\xi(L, x)=-\prod_{1\leqq i\leqq n}\log x_i, \qquad \textit{if} \quad r=1.$$

(ii) $$G_\xi(L, x)=\sum_{a_1, \ldots, a_r=0}^{p-1} \xi_1^{a_1}\cdots\xi_r^{a_r} G_{\xi^p}(L, (x+L(a))/p),$$

where $\xi^p=(\xi_1^p, \ldots, \xi_r^p)$ *and* $(x+L(a))/p=((x_1+L_1(a))/p, \ldots, (x_n+L_n(a))/p)$ *with* $a=(a_1, \ldots, a_r)$.

In some cases, logarithms of complex analytic multiple gamma functions are constructed and they are related to some special value of complex analytic L-functions (cf. [16], Theorem 1).

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