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AN EXPLICIT INTEGRAL REPRESENTATION OF WHITTAKER FUNCTIONS ON $Sp(2; \boldsymbol{R})$ FOR THE LARGE DISCRETE SERIES REPRESENTATIONS

TAKAYUKI ODA

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Abstract. We consider Whittaker model of the discrete series representations of the real symplectic group of degree 2. We obtain an integral formula for the radial part of the extreme vector of the minimal K-type of the Whittaker model.

Introduction. We shall prove an explicit integral formula for the Whittaker function associated to the highest weight vector in the representation space of the minimal K-type of a discrete series representation with the maximal Gelfand-Kirillov dimension for the real symplectic groups $Sp(2; \boldsymbol{R})$ of rank 2.

Let us explain the basic idea of this paper. Consider the case $G=SL_2(\boldsymbol{R})$. Put

$$N=\left\{\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\middle|\, x\in\boldsymbol{R}\right\},$$

and let

$$\eta:\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\mapsto\exp(2\pi icx)\quad(c\in\boldsymbol{R})$$

be a non-trivial unitary character of N. Let $C_\eta^\infty(N\backslash G)$ be the space of C^∞-functions φ satisfying $\varphi(ng)=\eta(n)\varphi(g)$ for all $(n, g)\in N\times G$.

For an irreducible unitary representation (π, H_π) of G, we denote by H_π^∞ the space of smooth vectors in G. When (π, H_π) is a principal series representation of $SL_2(\boldsymbol{R})$, the image of a vector in H_π^∞ with respect to the unique continuous intertwining operator from H_π^∞ to $C_\eta^\infty(N\backslash G)$ is represented by the modified Bessel function, i.e. the Whittaker function, if it is restricted to the split torus

$$A=\left\{\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}\middle|\, a\in\boldsymbol{R},\ a>0\right\}.$$

However when (π, H_π) is a discrete series representation of formal degree $k-1$ of $SL_2(\boldsymbol{R})$, then the image of minimal K-type vector of H_π with respect to the intertwining operator from H_π^∞ to $C_\eta^\infty(N\backslash G)$ (if it exists), is written by a constant times $a^k e^{-2\pi|c|\cdot a^2}$

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on A (cf. Jacquet-Langlands [J-L]).

Thus as special functions on A, the functions realizing the Whittaker model of the discrete series representations of $SL_2(\boldsymbol{R})$ are "degenerate" elementary functions, much simpler than those of the principal series representations.

We hope similar phenomena occur in higher rank groups. The purpose of this paper is to confirm this for the case $G=Sp(2; \boldsymbol{R})$.

Let $\eta: N \to \boldsymbol{C}^*$ be a non-degenerate character of the standard maximal unipotent subgroup N of G. In general for generic principal series representations π of G, the dimension of the intertwing space $\mathrm{Hom}_{(\mathfrak{g},K)}(\pi, C_\eta^\infty(N\backslash G))$ between $(\mathfrak{g}, K)$-modules (i.e. the space of algebraic Whittaker functionals) equals 8, the order of the Weyl group. However when π belongs to the discrete series representations which are large in the sense of Kostant-Vogan, the dimension of algebraic Whittaker functionals is reduced to 4. We consider the restriction map

$$\mathrm{res}: \mathrm{Hom}_{(\mathfrak{g},K)}(\pi_\Lambda^*, C_\eta^\infty(N\backslash G)) \to \mathrm{Hom}_K(\tau_\lambda^*, C_\eta^\infty(N\backslash G)) \cong C_{\eta,\tau_\lambda}^\infty(N\backslash G/K)$$

to the minimal K-type τ_λ^* of π_Λ^*, and find differential equations which characterize the image of the above restriction map (Lemma (6.2), Proposition (7.2), Lemma (8.1)). These formulae constitute a holonomic system of rank 4. Because the obtained formulae happen to be very simple, we can find an integral expression for one solution, which gives a solution rapidly decreasing at infinity (§9).

Let us explain the contents of this paper. We recall basic notation for the structure of $Sp(2; \boldsymbol{R})$ and associated Lie algebras in §1, and we review the Harish-Chandra parametrization of the representations of discrete series for $Sp(2; \boldsymbol{R})$ in §2. We recall some basic results on the representations of $U(2)$ ($\cong K$) in §3, and the definition of non-degenerate characters of the maximal unipotent subgroup of $Sp(2; \boldsymbol{R})$ in §4. In §5–§8, we write down explicitly the system of partial differential equations characterizing the radial part of the Whittaker functions of the minimal K-type of a discrete series representation, using the Schmid operator. In this step we follow the method of Yamashita [Y-I], [Y-II] who discussed the case $G=SU(2,2)$. Actually the author noticed the fact that it is possible to obtain a simple integral formula for Whittaker functions of the discrete series of $Sp(2; \boldsymbol{R})$ by reading these papers.

New parts different from [Y-I], [Y-II] are Proposition (8.1) and §9. §9 contains the main result of this paper: an explicit integral expression for the Whittaker function of the highest weight vector of the minimal K-type of a discrete series representation of $Sp(2; \boldsymbol{R})$.

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to clarify the meaning of the multiplicity one theorem.

1. Basic notation, and the structure of Lie groups and algebras. In this section, we determine basic notation on the symplectic group of degree 2, its maximal compact subgroup and associated Lie algebras.

⟨*Lie groups.*⟩ Let $M_4(\boldsymbol{R})$ be the space of real 4×4 matrices. Put

$$J=\begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}\in M_4(\boldsymbol{R}),$$

where 1_2 is a unit matrix of size 2. The symplectic group $Sp(2;\boldsymbol{R})$ of degree 2 is given by

$$Sp(2:\boldsymbol{R})=\{g\in M_4(\boldsymbol{R})\,|\,{}^tgJg=J,\ \det(g)=1\}.$$

Here ${}^tg$ denotes the transpose of g, and $\det(g)$ the determinant of g. A maximal compact subgroup K of $G=Sp(2;\boldsymbol{R})$ is given by

$$K=\left\{\begin{pmatrix} A & B \\ -B & A \end{pmatrix}\in Sp(2;\boldsymbol{R})\,\middle|\,A,B\in M_2(\boldsymbol{R})\right\},$$

which is isomorphic to the unitary group

$$U(2)=\{g\in GL(2;\boldsymbol{C})\,|\,{}^t\bar{g}\cdot g=1_2\}$$

of size 2 via a homomorphism

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}\in K\mapsto A+\sqrt{-1}B\in U(2).$$

⟨*Lie algebras.*⟩ The Lie algebra of G is given by

$$\mathfrak{g}=\mathfrak{sp}(2;\boldsymbol{R})=\{X\in M_4(\boldsymbol{R})\,|\,JX+{}^tXJ=0\},$$

and that of K is given by

$$\mathfrak{k}=\left\{X=\begin{pmatrix} A & B \\ -B & A \end{pmatrix}\,\middle|\,A,B\in M_2(\boldsymbol{R});\ {}^tA=-A,\ {}^tB=B\right\}.$$

The Cartan involution for $\mathfrak{k}$ is given by

$$\theta(X)=-{}^tX \qquad \text{for} \quad X\in\mathfrak{g}.$$

Hence the subspace

$$\mathfrak{p}=\{X\in\mathfrak{g}\,|\,\theta(X)=X\}=\left\{\begin{pmatrix} A & B \\ B & -A \end{pmatrix}\,\middle|\,{}^tA=A,\ {}^tB=B;\ A,B\in M_2(\boldsymbol{R})\right\}$$

given a Cartan decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$. The linear map

$$\mathfrak{k} \ni \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto A + \sqrt{-1}B \in \mathfrak{u}(2)$$

defines an isomorphism of Lie algebras from $\mathfrak{k}$ to the unitary Lie algebra

$$\mathfrak{u}(2) = \{C \in M_2(\boldsymbol{C}) \mid {}^t\bar{C} + C = 0\}$$

of degree 2.

An $\boldsymbol{R}$-basis of $\mathfrak{u}(2)$ is given by

$$\sqrt{-1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sqrt{-1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Y' = \sqrt{-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $\mathfrak{u}(2)_{\boldsymbol{C}} = \mathfrak{u}(2) \otimes_{\boldsymbol{R}} \boldsymbol{C}$ be the complexification of $\mathfrak{u}(2)$. Then a basis of $\mathfrak{u}(2)_{\boldsymbol{C}}$ is given by

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad H' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$X = \frac{1}{2}(Y - \sqrt{-1}Y') = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \bar{X} = \frac{1}{2}(-Y - \sqrt{-1}Y') = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then $\{H', X, \bar{X}\}$ is an $\mathfrak{sl}_2$-triple, i.e.

$$[H', X] = 2X; \quad [H', \bar{X}] = -2\bar{X}; \quad [X, \bar{X}] = H'.$$

Via the isomorphism $\mathfrak{k}_{\boldsymbol{C}} \xrightarrow{\sim} \mathfrak{u}_{\boldsymbol{C}}$, the preimage of the above basis of $\mathfrak{u}_{\boldsymbol{C}}$ is given by

$$Z = (-\sqrt{-1})\left(\begin{array}{cc|cc} & & 1 & \\ & & & 1 \\ \hline -1 & & & \\ & -1 & & \end{array}\right); \quad H' = (-\sqrt{-1})\left(\begin{array}{cc|cc} & & 1 & \\ & & & -1 \\ \hline -1 & & & \\ & 1 & & \end{array}\right);$$

$$Y = \left(\begin{array}{cc|cc} 0 & 1 & & \\ -1 & 0 & & \\ \hline & & 0 & 1 \\ & & -1 & 0 \end{array}\right); \quad Y' = \left(\begin{array}{cc|cc} & & & 1 \\ & & 1 & \\ \hline & -1 & & \\ -1 & & & \end{array}\right).$$

From now on we use the convention that unwritten components of a matrix are zero. Now we fix a compact Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ by

$$\mathfrak{h} = \boldsymbol{R}(\sqrt{-1}Z) + \boldsymbol{R}(\sqrt{-1}H').$$

Write $T_+ = \sqrt{-1}Z$ and $T_- = \sqrt{-1}H'$, and set

$$T_1 = \frac{1}{2}(T_+ + T_-) \quad \text{and} \quad T_2 = \frac{1}{2}(T_+ - T_-).$$

Put

$$H'_1=\frac{1}{2}(Z+H')\,,\qquad H'_2=\frac{1}{2}(Z-H')\,.$$

Then $T_1=\sqrt{-1}H'_1$, $T_2=\sqrt{-1}H'_2$, and

$$T_1=\left(\begin{array}{cc|cc} & & 1 & \\ & & & 0 \\ \hline -1 & & & \\ & 0 & & \end{array}\right),\quad T_2=\left(\begin{array}{cc|cc} & & 0 & \\ & & & 1 \\ \hline 0 & & & \\ & -1 & & \end{array}\right)\in\mathfrak{h}\,.$$

⟨*Root system.*⟩ We consider a root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$. For a linear form $\beta\colon\mathfrak{h}\to\boldsymbol{C}$, we write $\beta(T_i)=\beta_i\in\boldsymbol{C}$. For each $\beta\in\mathfrak{h}^*=\mathrm{Hom}(\mathfrak{h},\boldsymbol{C})$, set

$$\mathfrak{g}_\beta=\{X\in\mathfrak{g}_{\boldsymbol{C}}=\mathfrak{g}\otimes_{\boldsymbol{R}}\boldsymbol{C}\,|\,[H,X]=\beta(H)X,\ \forall H\in\mathfrak{h}\}\,.$$

Then the roots of $(g,\mathfrak{h})$ is given by

$$\begin{aligned}\textstyle\sum&=\{\beta=(\beta_1,\beta_2)\,|\,\mathfrak{g}_\beta\neq 0,\ \beta\neq 0\}\\ &=\sqrt{-1}\{\pm(2,0),\ \pm(0,2),\ \pm(1,1),\ \pm(1,-1)\}\,.\end{aligned}$$

We determine a root vector X_β in $\mathfrak{g}_\beta$, i.e. a generator of $\mathfrak{g}_\beta$ as in Table 1.

TABLE 1.

$-\sqrt{-1}\beta$	$(2,0)$	$(1,1)$	$(0,2)$	$(1,-1)$
X_β	$\left(\begin{array}{cc\|cc} 1 & & i & \\ & 0 & & 0 \\ \hline i & & -1 & \\ & 0 & & 0 \end{array}\right)$	$\left(\begin{array}{cc\|cc} & 1 & & i \\ 1 & & i & \\ \hline & i & & -1 \\ i & & -1 & \end{array}\right)$	$\left(\begin{array}{cc\|cc} 0 & & 0 & \\ & 1 & & i \\ \hline 0 & & 0 & \\ & i & & -1 \end{array}\right)$	$\left(\begin{array}{cc\|cc} & 1 & & -i \\ -1 & & i & \\ \hline & i & & 1 \\ -i & & -1 & \end{array}\right)$
$X_{-\beta}$	$\left(\begin{array}{cc\|cc} 1 & & -i & \\ & 0 & & 0 \\ \hline -i & & -1 & \\ & 0 & & 0 \end{array}\right)$	$\left(\begin{array}{cc\|cc} & 1 & & -i \\ 1 & & -i & \\ \hline & -i & & -1 \\ -i & & -1 & \end{array}\right)$	$\left(\begin{array}{cc\|cc} 0 & & 0 & \\ & 1 & & -i \\ \hline 0 & & 0 & \\ & -i & & -1 \end{array}\right)$	$\left(\begin{array}{cc\|cc} & 1 & & i \\ -1 & & -i & \\ \hline & -i & & 1 \\ i & & -1 & \end{array}\right)$

Then

$$\mathfrak{k}_{\boldsymbol{C}}=\mathfrak{h}_{\boldsymbol{C}}+\boldsymbol{C}X_{(1,-1)}+\boldsymbol{C}X_{(-1,1)}\,,$$

and set

$$\mathfrak{p}_+=\boldsymbol{C}X_{(2,0)}+\boldsymbol{C}X_{(1,1)}+\boldsymbol{C}X_{(0,2)}=\left\{X=\begin{pmatrix}X_1 & iX_1\\ iX_1 & -X_1\end{pmatrix}\,\middle|\,X_1\in M_2(\boldsymbol{C})\right\},$$

and

$$\mathfrak{p}_- = \boldsymbol{C}X_{-(2,0)} + \boldsymbol{C}X_{-(1,1)} + \boldsymbol{C}X_{-(0,2)} = \left\{ X = \begin{pmatrix} X_1 & -iX_1 \\ -iX_1 & -X_1 \end{pmatrix} \middle| X_1 \in M_2(\boldsymbol{C}) \right\}.$$

Then $\mathfrak{g}_{\boldsymbol{C}} = \mathfrak{k}_{\boldsymbol{C}} \oplus \mathfrak{p}_+ \oplus \mathfrak{p}_-$. For each root $\beta = (\beta_1, \beta_2)$, we put $\|\beta\| = \sqrt{|\beta_1|^2 + |\beta_2|^2}$. Then $\|\beta\|^2 = 4$ or $= 2$.

Then set

$$\{c \cdot \|\beta\|(X_\beta + X_{-\beta}),\ c \cdot \sqrt{-1}\,\|\beta\|(X_\beta - X_{-\beta}) \quad (\beta \in \Sigma_n^+)\}$$

forms an orthonormal basis of $\mathfrak{p} = \mathfrak{p}_{\boldsymbol{R}}$ with respect to the Killing form for some constant c. Here $\Sigma_n^+ = \{(2, 0),\ (1, 1),\ (0, 2)\}$ is the set of non-compact positive roots. $\Sigma_c^+ = \{(1, -1)\}$ is the set of compact positive roots. $\Sigma_c = \Sigma_c^+ \cup (-\Sigma_c^+)$ and $\Sigma_n = \Sigma_n^+ \cup (-\Sigma_n^+)$ are the set of compact roots and the set of non-compact roots, respectively.

⟨*Root system of* $(\mathfrak{g}, \mathfrak{a})$ *and Iwasawa decomposition.*⟩ We choose a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ given by

$$\mathfrak{a} = \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & -A \end{array} \right) \middle| A = \mathrm{diag}(t_1, t_2) \quad (t_1, t_2 \in \boldsymbol{R}) \right\}.$$

Here $\mathrm{diag}(t_1, t_2)$ is a diagonal matrix with (1, 1)-entry t_1 and (2, 2)-entry t_2. Set

$$H_1 = \left(\begin{array}{cc|cc} 1 & & & \\ & 0 & & \\ \hline & & -1 & \\ & & & 0 \end{array} \right) \quad \text{and} \quad H_2 = \left(\begin{array}{cc|cc} 0 & & & \\ & 1 & & \\ \hline & & 0 & \\ & & & -1 \end{array} \right).$$

Then $\{H_1, H_2\}$ forms a basis of $\mathfrak{a}$.

Let $\{e_1 = (1, 0),\ e_2 = (0, 1)\}$ be a standard basis of the 2-dimensional Euclidean plane $\boldsymbol{R}^2$. Then the root system Ψ of $(\mathfrak{g}, \mathfrak{a})$ is given by

$$\Psi = \{\pm 2e_1,\ \pm 2e_2,\ \pm e_1 \pm e_2\}.$$

A positive root system Ψ_+ is fixed by

$$\Psi_+ = \{2e_1, 2e_2, e_1 + e_2, e_1 - e_2\}.$$

Then $\mathfrak{n} = \sum_{\alpha \in \Psi_+} \mathfrak{g}_\alpha$ is a nilradical of a minimal parabolic subalgebra. We choose generators E_α of $\mathfrak{g}_\alpha$ $(\alpha \in \Psi_+)$ as follows:

$$E_{2e_1} = \left(\begin{array}{cc|cc} & & 1 & 0 \\ & & 0 & 0 \\ \hline & & & \\ & & & \end{array} \right); \quad E_{e_1 + e_2} = \left(\begin{array}{cc|cc} & & 0 & 1 \\ & & 1 & 0 \\ \hline & & & \\ & & & \end{array} \right);$$

$$E_{2e_2}=\left(\begin{array}{cc|cc} & & 0 & 0 \\ & & 0 & 1 \\ \hline & & & \\ & & & \end{array}\right); \quad E_{e_1-e_2}=\left(\begin{array}{cc|cc} 0 & 1 & & \\ 0 & 0 & & \\ \hline & & 0 & 0 \\ & & -1 & 0 \end{array}\right).$$

The Iwasawa decomposition associated to $(\mathfrak{a}, \mathfrak{n})$ is given by $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{n}$. In $\mathfrak{g}_C$, the Iwasawa decomposition of the root vectors $\{X_\beta;\ \beta\in\Sigma\}$ are given in the following Lemma, which we obtain by direct computation.

LEMMA (1.1).

$$X_{(2,0)}=H'_1+H_1+2\sqrt{-1}E_{2e_1};\qquad X_{(-2,0)}=-H'_1+H_1-2\sqrt{-1}E_{2e_1};$$

$$X_{(1,1)}=2\cdot\bar{X}+2\cdot E_{e_1-e_2}+2\sqrt{-1}E_{e_1+e_2};$$

$$X_{(-1,-1)}=-2\cdot X+2\cdot E_{e_1-e_2}-2\sqrt{-1}E_{e_1+e_2};$$

$$X_{(0,2)}=H'_2+H_2+2\sqrt{-1}E_{2e_2};\qquad X_{(0,-2)}=-H'_2+H_2-2\sqrt{-1}E_{2e_2}.$$

2. Parametrization of the representation of the discrete series. Consider a compact Cartan subgroup of G

$$\exp(\mathfrak{h})=\left\{\left(\begin{array}{cc|cc} \cos\theta_1 & & \sin\theta_1 & \\ & \cos\theta_2 & & \sin\theta_2 \\ \hline -\sin\theta_1 & & \cos\theta_1 & \\ & -\sin\theta_2 & & \cos\theta_2 \end{array}\right) \,\middle|\, \theta_1, \theta_2\in\boldsymbol{R}\right\}$$

corresponding to $\mathfrak{h}$. Then the characters are given by

$$\left(\begin{array}{cc|cc} \cos\theta_1 & & \sin\theta_1 & \\ & \cos\theta_2 & & \sin\theta_2 \\ \hline -\sin\theta_1 & & \cos\theta_1 & \\ & -\sin\theta_2 & & \cos\theta_2 \end{array}\right) \mapsto \exp\{\sqrt{-1}(m_1\theta_1+m_2\theta_2)\}\in\boldsymbol{C}^*.$$

Here m_1, m_2 are some integers. The derivation of these characters determines an integral structure of $\mathfrak{h}^*=\mathrm{Hom}(\mathfrak{h}, \boldsymbol{C})$, the weight lattice.

The set of compact positive roots is given by $\Sigma_c^+=\{(1, -1)\}$. Hence the set of dominants weight is given by $\{(\lambda_1, \lambda_2)\in\boldsymbol{Z}^{\oplus 2}\,|\,\lambda_1\geqq\lambda_2\}$. In order to parametrize the representation of the discrete series of $Sp(2; \boldsymbol{R})$, we first enumerate all the positive root systems compatible to Σ_c^+. There are four such positive root systems:

(I) $\Sigma_{\mathrm{I}}^+=\{(1, -1), (2, 0), (1, 1), (0, 2)\}$;

(II) $\Sigma_{\mathrm{II}}^+=\{(1, -1), (1, 1), (2, 0), (0, -2)\}$;

(III) $\Sigma^+_{\mathrm{III}}=\{(1,\ -1),(2,0),(0,\ -2),(-1,\ -1)\}$;

(IV) $\Sigma^+_{\mathrm{IV}}=\{(1,\ -1),(-2,0),(-1,\ -1),(0,\ -2)\}$.

Let J be a variable running over the set of indices {I, II, III, IV}. Then we write $\Sigma^+_{J,n}=\Sigma^+_J-\Sigma^+_c$ for the set of non-compact positive roots for each index J.

Define a subset Ξ_J of dominant weights by

$$\Xi_J=\{\Lambda=(\Lambda_1,\Lambda_2) \text{ dominant with respect to } \Sigma^+_c \mid \langle\Lambda,\beta\rangle>0,\ \forall\beta\in\Sigma^+_{J,n}\}.$$

Then the set $\bigcup_{J=\mathrm{I}}^{\mathrm{IV}}\Xi_J$ gives the Harish-Chandra parametrization of the representation of the discrete series for $Sp(2;\boldsymbol{R})$. Let π_Λ be the associated representation of G for $\Lambda\in\bigcup_{J=\mathrm{I}}^{\mathrm{IV}}\Xi_J$. The K-types of $\pi_\Lambda|_K$ are described by the formula of Blattner proved finally by Hecht-Schmid [HS]. Among others the minimal K-type of π_Λ is given by $\lambda_{\min}=\Lambda-\rho_c+\rho_n$. Hence ρ_c or ρ_n is a half of the sum of compact positive roots or non-compact positive roots, respectively. The Blattner parameter $\lambda_{\min}$ is listed in Table 2.

TABLE 2.

type J	I	II	III	IV
$\lambda_{\min}$	$(\Lambda_1+1,\Lambda_2+2)$	(Λ_1+1,Λ_2)	(Λ_1,Λ_2-1)	$(\Lambda_1-2,\Lambda_2-1)$

3. Representations of the maximal compact subgroup. For our later computation, we recall some basic facts about the representation of the maximal compact subgroup K or its complexification $K_{\boldsymbol{C}}$. Since K is identified with the unitary group $U(2)$ of degree 2, $K_{\boldsymbol{C}}$ is isomorphic to $GL(2,\boldsymbol{C})$. Recall a basis of $\mathfrak{u}(2)_{\boldsymbol{C}}$ given in Section 1:

$$Z=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad H'=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad X=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad \bar{X}=\begin{pmatrix}0&0\\1&0\end{pmatrix}.$$

The irreducible finite-dimensional representations of the Lie algebra $\mathfrak{gl}(2,\boldsymbol{C})$ are parametrized by a set

$$\{\lambda=(\lambda_1,\lambda_2)\in\boldsymbol{Z}^{\oplus 2}\mid\lambda_1\geqq\lambda_2,\ \text{i.e. } \lambda \text{ is dominant}\}.$$

For each dominant weight λ, we set $d=\lambda_1-\lambda_2\geqq 0$. Then the dimension of the representation space V_λ associated to λ is $d+1$. We can choose a basis $\{v_k\mid 0\leqq k\leqq d\}$ in V_λ so that the associated representation τ_λ is given by

$$\begin{cases}\tau_\lambda(Z)v_k=(\lambda_1+\lambda_2)v_k\,;\\ \tau_\lambda(H')v_k=(2k-d)v_k\,;\\ \tau_\lambda(X)v_k=(k+1)v_{k+1}\,;\\ \tau_\lambda(\bar{X})v_k=(d+1-k)v_{k-1}=\{d-(k-1)\}v_{k-1}\,.\end{cases}$$

Since $H'_1=(Z+H')/2$ and $H'_2=(Z-H')/2$, we have

$$\tau_\lambda(H'_1)v_k=(k+\lambda_2)v_k \quad \text{and} \quad \tau_\lambda(H'_2)v_k=(-k+\lambda_1)v_k\,.$$

If it is necessary to refer explicitly to the dominant weight λ, we denote v_k by $v_{\lambda,k}$.

For the adjoint representation of K on $\mathfrak{p}_+$, we have an isomorphism $\mathfrak{p}_+\cong V_{(2,0)}$, and the correspondence of the basis is given by

$$(X_{(0,2)}, X_{(1,1)}, X_{(2,0)})\mapsto(v_0, v_1, v_2)\,.$$

Similarly for $\mathfrak{p}_-$, we have $\mathfrak{p}_-\cong V_{(0,-2)}$, and the identification of the basis is

$$(X_{(-2,0)}, X_{(-1,-1)}, X_{(0,-2)})\mapsto(v_0, -v_1, v_2)\,.$$

Let us consider the tensor product $V_\lambda\otimes\mathfrak{p}_+$. Then it has a decomposition into irreducible factors:

$$V_\lambda\otimes\mathfrak{p}_+\cong V_\lambda\otimes V_{(2,0)}=V_{(\lambda_1+2,\lambda_2)}\oplus V_{(\lambda_1+1,\lambda_2+1)}\oplus V_{(\lambda_1,\lambda_2+2)}\,.$$

Let $P^{(2,0)}$, $P^{(1,1)}$, and $P^{(0,2)}$ be the projectors from $V_\lambda\otimes\mathfrak{p}_+$ to the factors $V_{(\lambda_1+2,\lambda_2)}$, $V_{(\lambda_1+1,\lambda_2+1)}$, and $V_{(\lambda_1,\lambda_2+2)}$, respectively. We denote $v_{(2,0),k}$ $(k=0, 1, 2)$ by w_k $(k=0, 1, 2)$.

LEMMA (3.1). *Set $\mu=(\lambda_1+2, \lambda_2)$. Then up to scalars, the projector $P^{(2,0)}$ is given by*

(i) $$P^{(2,0)}(v_{\lambda,k}\otimes w_2)=\frac{(k+1)\cdot(k+2)}{2}\,v_{\mu,k+2}\,;$$

(ii) $$P^{(2,0)}(v_{\lambda,k}\otimes w_1)=(k+1)(d+1-k)v_{\mu,k+1}\,;$$

(iii) $$P^{(2,0)}(v_{\lambda,k}\otimes w_0)=\frac{(d+1-k)(d+2-k)}{2}\,v_{\mu,k}\,.$$

LEMMA (3.2). *Set $\nu=(\lambda_1+1, \lambda_2+1)$. Then up to scalars, the projector $P^{(1,1)}$ is given by*

(0) $P^{(1,1)}(v_{\lambda,d}\otimes w_2)=0$

(i) $P^{(1,1)}(v_{\lambda,k}\otimes w_2)=(k+1)v_{\nu,k+1}$ $\quad(0\leqq k\leqq d-1)\,;$

(ii) $P^{(1,1)}(v_{\lambda,k}\otimes w_1)=(d-2k)v_{\nu,k}$ $\quad(0\leqq k\leqq d)\,;$

(iii) $P^{(1,1)}(v_{\lambda,k}\otimes w_0)=-(d+1-k)v_{\nu,k-1}$ $\quad(1\leqq k\leqq d)\,.$

LEMMA (3.3). *Set $\pi=(\lambda_1, \lambda_2+2)$. Then up to scalars, the projector $P^{(0,2)}$ is given by*

(i) $P^{(0,2)}(v_{\lambda,k}\otimes w_2)=v_{\pi,k}$ $\quad(0\leqq k\leqq d-2)\,;$

(ii) $P^{(0,2)}(v_{\lambda,k}\otimes w_1)=-2\cdot v_{\pi,k-1}$ $\quad(1\leqq k\leqq d-1)\,;$

(iii) $P^{(0,2)}(v_{\lambda,k}\otimes w_0)=v_{\pi,k-2}$ $\quad(2\leqq k\leqq d)\,;$

(iv) $P^{(0,2)}(v_d\otimes w_2)=P^{(0,2)}(v_d\otimes w_1)=P^{(0,2)}(v_{d-1}\otimes w_2)=0\,.$

The proofs of the above lemmas are easy. It is enough to find the highest weight vectors in $V_\lambda \otimes \mathfrak{p}_+$ corresponding to the factors V_μ, V_ν, and V_π, respectively. The other steps of the proofs are settled by induction.

4. Characters of the unipotent radical. Put $N = \exp(\mathfrak{n})$. Then N is written as

$$N = \left\{ \left(\begin{array}{cc|cc} 1 & n_0 & & \\ 0 & 1 & & \\ \hline & & 1 & 0 \\ & & -n_0 & 1 \end{array} \right) \cdot \left(\begin{array}{c|c} 1_2 & \begin{matrix} n_1 & n_2 \\ n_2 & n_3 \end{matrix} \\ \hline & 1_2 \end{array} \right) \middle| n_0, n_1, n_2, n_3 \in \boldsymbol{R} \right\}.$$

The commutator group $[N, N]$ of N is given by

$$[N, N] = \left\{ \left(\begin{array}{c|c} 1_2 & \begin{matrix} n_1 & n_2 \\ n_2 & 0 \end{matrix} \\ \hline & 1_2 \end{array} \right) \middle| n_1, n_2 \in \boldsymbol{R} \right\}.$$

Hence a unitary character η of N is written as

$$\left(\begin{array}{cc|cc} 1 & n_0 & & \\ 0 & 1 & & \\ \hline & & 1 & 0 \\ & & -n_0 & 1 \end{array} \right) \cdot \left(\begin{array}{c|c} 1_2 & \begin{matrix} n_1 & n_2 \\ n_2 & n_3 \end{matrix} \\ \hline & 1_2 \end{array} \right) \mapsto \exp\{2\pi i(c_0 n_0 + c_3 n_3)\}$$

for some real numbers c_0, $c_3 \in \boldsymbol{R}$.

We denote by the same letter η, the derivative of η

$$\eta : \mathfrak{n} \to \boldsymbol{C}.$$

Since $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}] = \boldsymbol{R}E_{e_1 - e_2} \oplus \boldsymbol{R}E_{2e_2}$, η is determined by the purely imaginary numbers

$$\eta_0 = \eta(E_{e_1 - e_2}) \quad \text{and} \quad \eta_3 = \eta(E_{2e_2}).$$

ASSUMPTION (4.1). Throughout this paper, we assume that η is non-degenerate, i.e.

$$\eta_0 \neq 0 \quad \text{and} \quad \eta_3 \neq 0.$$

5. Characterization of the minimal K-type. Let $\eta : N = \exp(\mathfrak{n}) \to \boldsymbol{C}^*$ be a unitary character. Then we denote by $C_\eta^\infty(N\backslash G)$ the space

$$C_\eta^\infty(N\backslash G) = \{\phi : G \to \boldsymbol{C} \mid C^\infty\text{-function}, \ \phi(ng) = \eta(n)\phi(g), \ \forall (n, g) \in N \times G\}.$$

By the right regular action of G, $C_\eta^\infty(N\backslash G)$ has structures of a smooth G-module, and a $(\mathfrak{g}_{\boldsymbol{C}}, K)$-module.

For any finite-dimensional K-module (τ, V), we put

$$C^{\infty}_{\eta,\tau}(N\backslash G/K)$$
$$=\{F\colon G\to V \mid C^{\infty}\text{-function, } F(ngk^{-1})=\eta(n)\tau(k)F(g),\ \forall(n,g,k)\in N\times G\times K\}\,.$$

Let $(\pi_{\Lambda}, E_{\Lambda})$ be the representation of the discrete series with Harish-Chandra parameter Λ, and let $(\pi^*_{\Lambda}, E^*_{\Lambda})$ be its contragredient representation.

Assume that there exists a continuous homomorphism $W\colon (\pi^*_{\Lambda}, E^*_{\Lambda})\to C^{\infty}_{\eta}(N\backslash G)$ of smooth G-modules. Then the restriction of W to the minimal K-type τ^*_{λ} of π^*_{Λ} gives an element $F_W\in C^{\infty}_{\eta,\tau_{\lambda}}(N\backslash G/K)$ such that

$$W(v^*)=\langle v^*, F_W(\cdot)\rangle \qquad \text{for all} \quad v^*\in V^*_{\lambda}\,.$$

Here $\langle *, *\rangle$ is the canonical pairing on $V^*_{\lambda}\times V_{\lambda}$.

There is a characterization of the minimal K-type function F by means of a differential operator acting on $C^{\infty}_{\eta,\tau_{\lambda}}(N\backslash G/K)$.

Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$, and $\mathrm{Ad}=\mathrm{Ad}_{\mathfrak{p}_C}$ the adjoint representation of K on $\mathfrak{p}_{C}$. Then we have a canonical covariant differential operator $\nabla_{\lambda,\eta}$ from $C^{\infty}_{\eta,\tau_{\lambda}}(N\backslash G/K)$ to $C^{\infty}_{\eta,\tau_{\lambda}\otimes\mathrm{Ad}}(N\backslash G/K)$:

$$\nabla_{\eta,\lambda}F=\sum_i R_{X_i}F(\cdot)\otimes X_i\,,\quad F\in C^{\infty}_{\eta,\tau_{\lambda}}(N\backslash G/K)\,,$$

where $(X_i)_i$ is any fixed orthonormal basis of $\mathfrak{p}$ with respect to the Killing form of $\mathfrak{g}$, and

$$R_{X_i}F(g)=\left(\frac{d}{dt}F(g\cdot\exp(tX_i))\right)\bigg|_{t=0} \qquad (g\in G)\,.$$

Let $(\tau^-_{\lambda}, V^-_{\lambda})$ be the sum of irreducible K-submodules of $V_{\lambda}\otimes\mathfrak{p}_{C}$ with highest weights of the form $\lambda-\beta$, β being a non-compact root in Σ^+. Denote by P_{λ} a surjective K-homomorphism from $V_{\lambda}\otimes\mathfrak{p}_{C}$ to V^-_{λ}. We define $\mathscr{D}_{\eta,\lambda}$ as the composite of $\nabla_{\eta,\lambda}$ with P_{λ}:

$$\mathscr{D}_{\eta,\lambda}\colon C^{\infty}_{\eta,\tau_{\lambda}}(N\backslash G/K)\to C^{\infty}_{\eta,\tau^-_{\lambda}}(N\backslash G/K)\,,$$
$$\mathscr{D}_{\eta,\lambda}F=P_{\lambda}(\nabla_{\eta,\lambda}F(\cdot))\quad (F\in C^{\infty}_{\eta,\tau_{\lambda}}(N\backslash G/K))\,.$$

We have the following:

Proposition (5.1). Yamashita [Y-I, Proposition (2.1)].

Let π_{Λ} be a representation of discrete series with Harish-Chandra parameter Λ of $Sp(2, \boldsymbol{R})$. Set $\lambda=\Lambda-\rho_c+\rho_n$. Then the linear map

$$W\in\mathrm{Hom}_{(\mathfrak{g}_C,K)}(\pi^*_{\Lambda}, C^{\infty}_{\eta}(N\backslash G))\to F_W\in\mathrm{Ker}(\mathscr{D}_{\eta,\lambda})$$

is injective, and if Λ is far from the walls of the Weyl chambers, it is bijective.

By the results of Kostant [K, §6], we have

$$\dim_{C}\mathrm{Hom}_{(\mathfrak{g}_C,K)}(\pi^*_{\Lambda}, C^{\infty}_{\eta}(N\backslash G))+\dim_{C}\mathrm{Hom}_{(\mathfrak{g}_C,K)}(\pi_{\Lambda}, C^{\infty}_{\eta}(N\backslash G))=0 \quad \text{or} \quad =|W|\,.$$

Here $|W|=8$ is the order of the Weyl group of $Sp(2, \boldsymbol{R})$.

Since holomorphic discrete series and antiholomorphic discrete series are not large in the sense of Vogan [V], if $\pi_\Lambda \in \Xi_{\mathrm{I}} \cup \Xi_{\mathrm{IV}}$, we have

$$\mathrm{Hom}_{(\mathfrak{g}_C, K)}(\pi_\Lambda^*, C_\eta^\infty(N\backslash G)) = \{0\} .$$

In subsequent sections, we show that if $\Lambda \in \Xi_{\mathrm{II}} \cup \Xi_{\mathrm{III}}$, then

$$\dim \mathrm{Hom}_{(\mathfrak{g}_C, K)}(\pi_\Lambda^*, C_\eta^\infty(N\backslash G)) = \dim_{\boldsymbol{C}} \mathrm{Ker}(\mathscr{D}_{\eta,\lambda}) = \frac{1}{2} \, |W| = 4 ,$$

using the above proposition (cf. Proposition (8.2)).

6. Radial part of differential operators.

Put $A = \exp(\mathfrak{a})$, i.e.

$$A = \left\{ \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & a_1^{-1} & \\ & & & a_2^{-1} \end{pmatrix} \middle| \, a_1, a_2 \in \boldsymbol{R}, a_1 > 0, a_2 > 0 \right\} .$$

Then we have the Iwasawa decomposition $G = NAK$ of $Sp(2; \boldsymbol{R})$. The value of $F \in C_{\eta,\tau_\lambda}^\infty(N\backslash G/K)$ is determined by its restriction $\phi = F|_A$ to A.

We compute the radial parts $R(\nabla_{\eta,\lambda})$ and $R(\mathscr{D}_{\eta,\lambda})$ of $\nabla_{\eta,\lambda}$ and $\mathscr{D}_{\eta,\lambda}$, respectively.

As an orthogonal basis of $\mathfrak{p}$, we take

$$C\|\beta\|(X_\beta + X_{-\beta}) , \qquad \frac{C\|\beta\|}{\sqrt{-1}}(X_\beta - X_{-\beta}) \qquad (\beta \in \Sigma_n^+)$$

with some $C > 0$ depending on the Killing form. Then

$$2\nabla_{\eta,\lambda} F = C \sum_{\beta \in \Sigma_n^+} \|\beta\|^2 R_{X_{-\beta}} F \otimes X_\beta + C \sum_{\beta \in \Sigma_n^+} \|\beta\|^2 R_{X_\beta} F \otimes X_{-\beta} .$$

We write

$$\nabla_{\eta,\lambda}^+ F = \frac{1}{4} \Sigma \|\beta\|^2 \cdot R_{X_{-\beta}} F \otimes X_\beta ; \qquad \nabla_{\eta,\lambda}^- F = \frac{1}{4} \Sigma \|\beta\|^2 \cdot R_{X_\beta} F \otimes X_{-\beta} .$$

In order to write $R(\nabla_{\eta,\lambda}^\pm)$, it is better to introduce some "macro" symbols. We set $\partial_i = R_{H_i}$ restricted to A ($i = 1, 2$), and define linear differential operators $\mathscr{L}_i^\pm$ and $\mathscr{S}^\pm$ on $C^\infty(A, V_\lambda)$ by

$$\begin{cases} \mathscr{L}_i^\pm \phi = (\partial_i \pm 2\sqrt{-1}\, a_i^2 \eta(E_{2e_i}))\phi & (i = 1, 2) ; \\ \mathscr{S}^\pm \phi = \{a_1 a_2^{-1} \eta(E_{e_1 - e_2}) \pm \sqrt{-1}\, a_1 a_2 \eta(E_{e_1 + e_2})\}\phi . \end{cases}$$

PROPOSITION (6.1). *The operators* $R(\nabla_{\eta,\lambda}^\pm) = C^\infty(A, V_\lambda) \to C^\infty(A, V_\lambda \otimes \mathfrak{p}_\pm)$ *are expressed as*

(i) $$R(\nabla^{+}_{\eta,\lambda})\phi=(\mathscr{L}^{-}_{1}+\tau_{\lambda}\otimes \mathrm{Ad}_{\mathfrak{p}_{+}}(H'_{1})-4)(\phi\otimes X_{(2,0)}) \\ +(\mathscr{S}^{-}+\tau_{\lambda}\otimes \mathrm{Ad}_{\mathfrak{p}_{+}}(X))(\phi\otimes X_{(1,1)}) \\ +(\mathscr{L}^{-}_{2}+\tau_{\lambda}\otimes \mathrm{Ad}_{\mathfrak{p}_{+}}(H'_{2})-2)(\phi\otimes X_{(0,2)})$$

(ii) $$R(\nabla^{-}_{\eta,\lambda})\phi=(\mathscr{L}^{+}_{1}-\tau_{\lambda}\otimes \mathrm{Ad}_{\mathfrak{p}_{-}}(H'_{1})-4)(\phi\otimes X_{(-2,0)}) \\ +(\mathscr{S}^{+}-\tau_{\lambda}\otimes \mathrm{Ad}_{\mathfrak{p}_{-}}(\bar{X}))(\phi\otimes X_{(-1,-1)}) \\ +(\mathscr{L}^{+}_{2}-\tau_{\lambda}\otimes \mathrm{Ad}_{\mathfrak{p}_{-}}(H'_{2})-2)(\phi\otimes X_{(0,-2)})\,.$$

PROOF. In order to prove (i), we note that

$$(R_{X_{-(2,0)}}F)_{|A}\otimes X_{(2,0)}=\{-H'_{1}+H_{1}-2\sqrt{-1}\,E_{2e_{1}}\}F_{|A}\otimes X_{(2,0)} \\ =\{\mathscr{L}^{-}_{1}\phi+(\tau_{\lambda}(H'_{1})\cdot F)_{|A}\}\otimes X_{(2,0)}\,,$$

and

$$(\tau_{\lambda}(H'_{1})\cdot F)_{|A}\otimes X_{(2,0)}=\tau_{\lambda}\otimes \mathrm{Ad}_{\mathfrak{p}_{+}}(H'_{1})(\phi\otimes X_{(2,0)})-\phi\otimes[H'_{1},X_{(2,0)}] \\ =\tau_{\lambda}\otimes \mathrm{Ad}_{\mathfrak{p}_{+}}(H'_{1})(\phi\otimes X_{(2,0)})-2(\phi\otimes X_{(2,0)})\,.$$

The case of (ii) is similar. q.e.d.

For a non-compact positive root $\beta=(\beta_1,\beta_2)$ in Σ^{+}, let P^{β} be the projector from $V_{\lambda}\otimes\mathfrak{p}_{+}$ to $V_{\lambda+\beta}$, and $P^{-\beta}$ the projector from $V_{\lambda}\otimes\mathfrak{p}_{-}$ to $V_{\lambda-\beta}$.

Then, similarly as in Yamashita [Y-I, Lemma (5.2)] we can show the following:

LEMMA (6.2). *Let λ be the minimal K-type of the discrete series representation π_{Λ} with Harish-Chandra parameter Λ.*

(i) *When $\Lambda\in\Xi_{\mathrm{II}}$, $R(\mathscr{D}_{\eta,\lambda})\phi=0$ if and only if*

$$\begin{cases} P^{(0,2)}(R(\nabla^{+}_{\eta,\lambda})\phi)=0\,; \\ P^{(-1,-1)}(R(\nabla^{-}_{\eta,\lambda})\phi)=0\,; \\ P^{(-2,0)}(R(\nabla^{-}_{\eta,\lambda})\phi)=0\,. \end{cases}$$

(ii) *When $\Lambda\in\Xi_{\mathrm{III}}$, $R(\mathscr{D}_{\eta,\lambda})\phi=0$ if and only if*

$$\begin{cases} P^{(1,1)}(R(\nabla^{+}_{\eta,\lambda})\phi)=0\,; \\ P^{(0,2)}(R(\nabla^{+}_{\eta,\lambda})\phi)=0\,; \\ P^{(-2,0)}(R(\nabla^{-}_{\eta,\lambda})\phi)=0\,. \end{cases}$$

7. Difference-differential equations. In this section, we write the system of differential equations in the last lemma of the previous section explicitly in terms of the components of ϕ.

Let $\lambda=(\lambda_1,\lambda_2)$ be the minimal K-type of the discrete series representation π_{Λ}. Then in V_{λ}, we choose a basis $\{v_k\,|\,0\le k\le d\}$ defined in Section 3. Here $d=\lambda_1-\lambda_2$. Then

$\phi: A \to V_\lambda$ is written as

$$\phi(a) = \sum_{k=0}^{d} c_k(a) v_k$$

with coefficients $c_k(a): A \to \boldsymbol{C}$.

The following lemma is a consequence of an easy computation.

LEMMA (7.1). (i) *The condition* $P^{(1,1)}(R(\nabla_{\eta,\lambda}^{+})\phi) = 0$ *is equivalent to the system*:

$$(\mathrm{C}_2^+)_k \qquad k(\mathscr{L}_1^- + \lambda_2 + d - k - 1)c_{k-1}(a) + (d-2k)\mathscr{S}^- c_k(a) + (k-d)(\mathscr{L}_2^- + \lambda_1 - k - 1)c_{k+1}(a) = 0 \qquad (0 \le k \le d).$$

(ii) *The condition* $P^{(-1,-1)}(R(\nabla_{\eta,\lambda}^{+})\phi) = 0$ *is equivalent to the system*:

$$(\mathrm{C}_2^-)_k \qquad (k-d)(\mathscr{L}_1^+ - \lambda_2 + k - d - 1)c_{k+1}(a) + (2k-d)\mathscr{S}^+ c_k(a) + k(\mathscr{L}_2^+ - \lambda_1 + k - 1)c_{k-1}(a) = 0 \qquad (0 \le k \le d).$$

(iii) *The condition* $P^{(0,2)}(R(\nabla_{\eta,\lambda}^{+})\phi) = 0$ *is equivalent to the system*:

$$(\mathrm{C}_3^+)_k \qquad (\mathscr{L}_1^- + \lambda_2 - k - 2)c_k(a) - 2\mathscr{S}^- c_{k+1}(a) + (\mathscr{L}_2^- + \lambda_1 - k - 2)c_{k+2}(a) = 0 \qquad (0 \le k \le d-2).$$

(iv) *The condition* $P^{(0,-2)}(R(\nabla_{\eta,\lambda}^{-})\phi) = 0$ *is equivalent to the system*:

$$(\mathrm{C}_3^-)_k \qquad (\mathscr{L}_1^+ - \lambda_2 - 2d + k)c_{k+2}(a) + 2\mathscr{S}^+ c_{k+1}(a) + (\mathscr{L}_2^+ - \lambda_1 + k)c_k(a) = 0 \qquad (0 \le k \le d-2).$$

Here we understand that, in the above formulas, $c_{-1}(a) = c_{d+1}(a) = 0$.

Since η is trivial on the commutator subgroup $[N, N]$, we have

$$\begin{cases} \mathscr{L}_1^+ = \mathscr{L}_1^- = \partial_1 = a_1(\partial/\partial a_1) \\ \mathscr{S}^+ = \mathscr{S}^- = (a_1/a_2)\eta(E_{(1,-1)}). \end{cases}$$

From now on we drop the supersripts $\pm$ from $\mathscr{L}_1^{\pm}$ and $\mathscr{S}^{\pm}$ to denote them simply by $\mathscr{L}_1$ and $\mathscr{S}$. Thus we have the following:

PROPOSITION (7.2). *Under the same assumption as in Lemma* (6.2) (i), $\phi(a) = \sum_{k=0}^{d} c_k(a) v_k$ *satisfies the following system of partial differential equations*:

$$(\mathrm{C}_3^+)_k \qquad (\mathscr{L}_1 + \lambda_2 - k - 2)c_k(a) - 2\mathscr{S} \cdot c_{k+1}(a) + (\mathscr{L}_2^- + \lambda_1 - k - 2)c_{k+2}(a) = 0 \qquad (0 \le k \le d-2).$$

$$(\mathrm{C}_3^-)_k \qquad (\mathscr{L}_1 - \lambda_2 - 2d + k)c_{k+2}(a) + 2\mathscr{S} \cdot c_{k+1}(a) + (\mathscr{L}_2^+ - \lambda_1 + k)c_k(a) = 0 \qquad (0 \le k \le d-2).$$

$(\mathrm{C}_2^-)_{k+1}$ $\quad (k+1-d)(\mathscr{L}_1-\lambda_2-d+k)c_{k+2}(a)+(2k+2-d)\mathscr{S}\cdot c_{k+1}(a)$
$\quad +(k+1)(\mathscr{L}_2^+-\lambda_1+k)c_k(a)=0 \qquad (-1\le k\le d-1)\,.$

8. Reduction of the system of partial differential equations. In this section we reduce the system of partial differential equations of the previous proposition to a simpler holonomic system, when η is non-degenerate.

In the first place, we see that the functions $c_k(a)$ is determined by the coefficient of the highest weight vector $c_d(a)$.

Indeed, when $k=0$, or $k=d$

$(\mathrm{C}_2^-)_0$ $\quad (\mathscr{L}_1-\lambda_2-d-1)c_1(a)+\mathscr{S}c_0(a)=0\,;$

$(\mathrm{C}_2^-)_d$ $\quad \mathscr{S}c_d(a)+(\mathscr{L}_2^+-\lambda_1+d-1)c_{d-1}(a)=0\,.$

Moreover for $1\leqq k\leqq d-1$, the computation of $(k+1)(\mathrm{C}_3^-)_k-(\mathrm{C}_2^-)_{k+1}$ yields

$$(\mathscr{L}_1-\lambda_2-d-1)c_{k+2}(a)+\mathscr{S}c_{k+1}(a)=0\,.$$

Noting $\lambda_2+d=\lambda_1$ together with $(\mathrm{C}_2^-)_0$, we have

$(\mathrm{E})_k$ $\quad (\mathscr{L}_1-\lambda_1-1)c_{k+2}(a)+\mathscr{S}c_{k+1}(a)=0 \qquad (-1\leqq k\leqq d-1)\,.$

Hence $c_0(a), c_1(a), \dots, c_{d-1}(a)$ are determined downward recursively by $c_d(a)$.

The system of the equations (C_2^-) are now replaced by the above $(\mathrm{E})_k$ and

$(\mathrm{C}_2^-)_{d-1}$ $\quad \mathscr{S}c_d(a)+(\mathscr{L}_2^+-\lambda_1+d-1)c_{d-1}(a)=0\,.$

Thus the system of the equations of Proposition (7.2) in Section 7 is equivalent to a system of equations:

(F-1) $\quad (\mathscr{L}_1-\lambda_1-1)c_d(a)+\mathscr{S}c_{d-1}(a)=0\,;$

(F-2) $\quad (\mathscr{L}_1-\lambda_1-1)c_{d-1}(a)+\mathscr{S}c_{d-2}(a)=0\,;$

(F-3) $\quad \mathscr{S}c_d(a)+(\mathscr{L}_2^+-\lambda_1+d-1)c_{d-1}(a)=0\,;$

(F-4) $\quad (\mathscr{L}_1+\lambda_2-d)c_{d-2}(a)-2\mathscr{S}c_{d-1}(a)+(\mathscr{L}_2^-+\lambda_1-d)c_d(a)=0\,.$

In order to make the above equations simpler, we replace unknown functions $c_k(a)$ by $h_k(a)$ defined by relations

$$c_k(a)=a_1^{\lambda_1+1-d}a_2^{\lambda_1}\left(\frac{a_1}{a_2}\right)^k e^{-i\eta(E_{(0,2)})a_2^2}h_k(a)\,.$$

Now we introduce the Euler operators ∂_i $(i=1, 2)$ by $\partial_i=a_i(\partial/\partial a_i)$ for each $i=1$, 2. Then the system of equations (F-1)–(F-4) is replaced by

(G-1) $\quad \partial_1 h_d(a)+\eta(E_{e_1-e_2})h_{d-1}(a)=0\,;$

(G-2) $(\partial_1-1)h_{d-1}(a)+\eta(E_{e_1-e_2})h_{d-2}(a)=0$;

(G-3) $\mathscr{S}\left(\frac{a_1}{a_2}\right)h_d(a)+\partial_2 h_{d-1}(a)=0$;

(G-4) $(\partial_1+2\lambda_2-1)h_{d-2}(a)-2\mathscr{S}\left(\frac{a_1}{a_2}\right)h_{d-1}(a)+\left(\frac{a_1}{a_2}\right)^2(\partial_2+2\lambda_1-2d-2\mathscr{S}')h_d(a)=0$.

Here $\mathscr{S}'=(\mathscr{L}_2^+-\mathscr{L}_2^-)/2=2\sqrt{-1}\,\eta(E_{2e_2})a_2^2$.

(G-1) and (G-3) are equivalent to a single equation:

(H-1)
$$(\partial_1\partial_2-\mathscr{S}^2)h_d(a)=0\,.$$

(G-1), (G-2) and (G-4) are equivalent to a single equation:

(∗)
$$(\partial_1+2\lambda_2-1)(\partial_1-1)\partial_1\left\{\left(\frac{a_1}{a_2}\right)^2 h_d(a)\right\}+2\left(\frac{a_1}{a_2}\right)^2\partial_1 h_d(a)$$
$$+\left(\frac{a_1}{a_2}\right)^2(\partial_2+2\lambda_2-2\mathscr{S}')h_d(a)=0\,.$$

Here we used the assumption that η is non-degenerate, i.e.

$$\eta(E_{e_1-e_2})=\eta_0\neq 0\,,\quad\text{and}\quad \eta(E_{2e_2})=\eta_3\neq 0\,.$$

Apply the operator ∂_2 to the above equation (∗), and use (H-1) to replace $\partial_1\partial_2 h_d(a)$ by $\mathscr{S}^2 h_d(a)$. Then we have

(H-2)
$$\{\partial_1^2+2\partial_1\partial_2+\partial_2^2+(2\lambda_2-2)(\partial_1+\partial_2)+(-2\lambda_2+1)-2\mathscr{S}'\partial_2\}h_d=0$$

Finally, we have the following:

LEMMA (8.1). *The system of equations of Proposition* (7.2) *is equivalent to*

(H-1) $(\partial_1\partial_2-\mathscr{S}^2)h_d(a)=0$

and

(H-2) $\{(\partial_1+\partial_2)^2+(2\lambda_2-2)(\partial_1+\partial_2)+(-2\lambda_2+1)-2\mathscr{S}'\partial_2\}h_d(a)=0$.

We can easily check that the system (H-1), (H-2) is a holonomic system of rank 4 defined over $(\boldsymbol{R}_{>0})^2=\{(a_1,a_2)\in\boldsymbol{R}^2\,|\,a_1,a_2>0\}$. Hence $\dim_{\boldsymbol{C}}\mathrm{Ker}(D_{\eta,\lambda})=4$. The contragredient representation π_Λ^* of π_Λ ($\Lambda\in\Xi_{\mathrm{II}}$) is written as $\pi_\Lambda^*=\pi_{\Lambda'}$ with some $\Lambda'\in\Xi_{\mathrm{III}}$. Using the difference-differential equations (C_2^+), (C_3^+) and (C_3^-), we can similarly show that $\dim_{\boldsymbol{C}}\mathrm{Ker}(D_{\eta,\lambda'})=4$ for the minimal K-type λ' of $\pi_{\Lambda'}$.

Since Kostant's result implies (cf. §5)

$$8=\dim\mathrm{Hom}_{(\mathfrak{g}_C,K)}(\pi_\Lambda^*,C_\eta^\infty(N\backslash G))+\dim\mathrm{Hom}_{(\mathfrak{g}_C,K)}(\pi_\Lambda,C_\eta^\infty(N\backslash G))$$
$$\leqq\dim_{\boldsymbol{C}}\mathrm{Ker}(D_{\eta,\lambda})+\dim_{\boldsymbol{C}}\mathrm{Ker}(D_{\eta,\lambda'})=8\,,$$

we have the following:

PROPOSITION (8.2). *Assume that η is generic, i.e.*

$$\eta_0=\eta(E_{e_1-e_2})\neq 0 \quad and \quad \eta_3=\eta(E_{2e_2})\neq 0 .$$

Then for the discrete series representation π_Λ corresponding to $\Lambda\in\Xi_{\mathrm{II}}\cup\Xi_{\mathrm{III}}$, we have

$$\dim_{\boldsymbol{C}}\mathrm{Hom}_{(\mathfrak{g}_C,K)}(\pi_\Lambda, C_\eta^\infty(N\backslash G))=4 .$$

9. Integral formula for the Whittaker function. Let us recall the multiplicity one theorem (cf. Shalika [Sh]). In the intertwining space

$$\mathrm{Hom}_{(\mathfrak{g},K)}(\pi_\Lambda^*, C_\eta^\infty(N\backslash G))$$

there is a subspace consisting of those intertwing operators which take values in the space $\mathscr{A}_\eta(N\backslash G)$ of functions with *moderate growth* (cf. [W, (8.1)]) in $C_\eta^\infty(N\backslash G)$. Then by the enhanced version of the multiplicity one theorem (Wallach [W, Theorem (8.8)] plus Kostant [K, Theorem (6.7.2)]), we have

$$\dim\mathrm{Hom}_{(\mathfrak{g},K)}(\pi, \mathscr{A}_\eta(N\backslash G))+\dim\mathrm{Hom}_{(\mathfrak{g},K)}(\pi^*, \mathscr{A}_\eta(N\backslash G))\leq 1 , \qquad \text{if} \quad \pi\in\Xi_{\mathrm{II}}\cup\Xi_{\mathrm{III}} .$$

We want to show that the above inequality is an equality. Namely there occur two cases:

(A) $\mathrm{Hom}_{(\mathfrak{g},K)}(\pi_\Lambda^*, \mathscr{A}_\eta(N\backslash G))\cong \boldsymbol{C}$, and $\mathrm{Hom}_{(\mathfrak{g},K)}(\pi_\Lambda, \mathscr{A}_\eta(N\backslash G))=\{0\}$

or

(B) $\mathrm{Hom}_{(\mathfrak{g},K)}(\pi_\Lambda^*, \mathscr{A}_\eta(N\backslash G))=\{0\}$, and $\mathrm{Hom}_{(\mathfrak{g},K)}(\pi_\Lambda, \mathscr{A}_\eta(N\backslash G))\cong \boldsymbol{C}$.

This dichotomy is controlled by the parity of the imaginary part of the purely imaginary number $\eta_3=\eta(E_{2e_2})\neq 0$. We show this by construction of an explicit integral formula for the image $F_W\in\mathrm{Ker}(D_{\eta,\lambda})\subset C_{\eta,\tau_\lambda}^\infty(N\backslash G/K)$ of the intertwining operator W with coefficients of moderate growth, which corresponds to a non-zero element W in $\mathrm{Hom}_{(\mathfrak{g},K)}(\pi_\Lambda^*, \mathscr{A}_\eta(N\backslash G))$.

Let us recall the confluent hypergeometric equation given by Whittaker ([W-W, Chap. 16]):

$$\frac{d^2}{dz^2}W+\left\{-\frac{1}{4}+\frac{k}{z}+\frac{1/4-m^2}{z^2}\right\}W=0 .$$

When $\mathrm{Re}(k-1/2-m)\leq 0$, for $z\notin(-\infty, 0)$, a unique solution, which rapidly decreases if $z\to+\infty$, is given by

$$W_{k,m}(z)=\frac{e^{-1/2z}\cdot z^k}{\Gamma(1/2-k+m)}\int_0^\infty t^{-k-1/2+m}\left(1+\frac{t}{z}\right)^{k-1/2+m}\cdot e^{-t}dt .$$

The following is the main result of this paper.

THEOREM (9.1). *Assume that* $\eta\colon N\to C^*$ *is non-degenerate, i.e.* $\eta_0=\eta(E_{e_1-e_2})\neq 0$ *and* $\eta_3=\eta(E_{2e_2})\neq 0$.

(i) *For* $\Lambda\in\Xi_{\mathrm{II}}$,

$$\begin{cases}\mathrm{Hom}_{(\mathfrak{g},K)}(\pi_\Lambda^*, \mathscr{A}_\eta(N\backslash G))\cong C & \text{if } \mathrm{Im}(\eta_3)<0\,;\\ \mathrm{Hom}_{(\mathfrak{g},K)}(\pi_\Lambda^*, \mathscr{A}_\eta(N\backslash G))=\{0\}\,, & \text{if } \mathrm{Im}(\eta_3)>0\,.\end{cases}$$

(ii) *Assume that* $\Lambda\in\Xi_{\mathrm{II}}$ *and* $\mathrm{Im}(\eta_3)<0$, *and let* W *be an intertwining operator in* $\mathrm{Hom}_{(\mathfrak{g},K)}(\pi_\Lambda^*, \mathscr{A}_\eta(N\backslash G))$ *unique up to scalar multiple. Then the function* $h_d(a_1, a_2)$ *associated to* $\phi(a)=F_{W|A}(a)=\sum_{i=0}^d c_i(a)v_i$ $(F_W\in\mathrm{Ker}(\mathscr{D}_{\eta,\tau_\lambda}))$ *has an integral representation*

$$h_d(a_1, a_2)=\int_0^\infty t^{\lambda_2-3/2}W_{0,-\lambda_2}(t)\exp\left(-\frac{t^2}{32\sqrt{-1}\cdot\eta_3 a_2^2}+\frac{8\sqrt{-1}\eta_0^2\eta_3 a_1^2}{t^2}\right)\frac{dt}{t}\,.$$

PROOF. It is easy to check that the integral represents a solution of the differential equations (H-1) and (H-2), by derivation of the integrand and partial integration.

Replace t by $a_1 t$ in the above integral expression of $h_d(a_1, a_2)$. Then

$$h_d(a_1, a_2)=\int_0^\infty\left(\frac{a_1}{a_2}\cdot a_2\cdot t\right)^{\lambda_2-3/2}W_{0,-\lambda_2}\left(\frac{a_1}{a_2}\cdot a_2\cdot t\right)$$
$$\times\exp\left\{-\frac{1}{32\sqrt{-1}\eta_3}\left(\frac{a_1}{a_2}\right)^2\cdot t^2+8\sqrt{-1}\eta_0^2\eta_3\cdot t^{-2}\right\}\frac{dt}{t}\,.$$

If $\mathrm{Im}(\eta_3)<0$, then $-1/32\sqrt{-1}\eta_3<0$ and $8\sqrt{-1}\eta_0^2\eta_3<0$. Also since $\Lambda\in\Xi_{\mathrm{II}}$, λ_2 is a negative integer. Hence the integrand is rapidly decreasing when $t\to+\infty$, and when $t\to 0$. Therefore the above integral converges, and as a function in (a_1, a_2), it is rapidly decreasing when $a_1/a_2\to\infty$ and $a_2\to\infty$. Put

$$c_d(a)=a_1^{\lambda_1+1-d}a_2^{\lambda_1}\cdot\left(\frac{a_1}{a_2}\right)^d\cdot e^{-i\eta_3\cdot a_2^2}\cdot h_d(a)\,,$$

and $c_k(a)$ for $0\le k\le d-1$ by the recurrence relation $(\mathrm{E})_k$ of §8.

Then $c_k(a)$ $(0\le k\le d)$ are also rapidly decreasing functions in $(a_1/a_2, a_2)$. Write $\phi(a)=\sum_{k=0}^d c_k(a)v_k\in C^\infty(A, V_\lambda)$. Then for any vector v^* of the dual space V_λ^*, $(\phi(a), v^*)$ is also a rapidly decreasing function. *A fortiori*, $\phi(a)$, i.e. $F(g)=\eta(n)\tau_\lambda(k)^{-1}\phi(a)$ is slowly increasing in $g=nak\in G$. This F defines an element W in $\mathrm{Hom}_{(\mathfrak{g},K)}(\pi_{\Lambda^*}, \mathscr{A}_\eta(N\backslash G))$.

Now Wallach's version of multiplicity one [W, §8] implies that the operators W in $\mathrm{Hom}_{(\mathfrak{g},K)}(\pi_\Lambda^*, \mathscr{A}_\eta(N\backslash G))$ such that $W(v)$ are slowly increasing on G for any $v\in\pi_\Lambda$, form a linear subspace of dimension at most one.

Hence $\mathrm{Hom}_{(\mathfrak{g},K)}(\pi_\Lambda^*, \mathscr{A}_\eta(N\backslash G))\neq\{0\}$, if $\mathrm{Im}(\eta_3)<0$. If $\mathrm{Im}(\eta_3)>0$, by a similar argument, we can show that $\mathrm{Hom}_{(\mathfrak{g},K)}(\pi_\Lambda, \mathscr{A}_\eta(N\backslash G))\neq\{0\}$. Since

$$\dim_{\boldsymbol{C}} \mathrm{Hom}_{(\mathfrak{g},K)}(\pi_\Lambda^*, \mathscr{A}_\eta(N\backslash G)) + \dim_{\boldsymbol{C}} \mathrm{Hom}_{(\mathfrak{g},K)}(\pi_\Lambda, \mathscr{A}_\eta(N\backslash G)) \leq 1$$

if η is non-degenerate, this proves (i). The part (ii) follows immediately from the uniqueness of the Whittaker model.

REMARK. In the general cases, the condition of (i) is described in terms of the wave front set by Matsumoto [M, §3].

When $G = SU(2, 2)$, we have a similar integral expression for the Whittaker function of the highest weight vector of the minimal K-type of a discrete series representation. Details will be discussed elsewhere is this case.

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DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF TOKYO
TOKYO 113
JAPAN