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DOES A NON-LIPSCHITZ FUNCTION OPERATE ON A NON-TRIVIAL BANACH FUNCTION ALGEBRA?

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Abstract. We show a property of a normal Banach function algebra on which a non-Lipschitz function operates. An example of a non-trivial normal Banach function algebra such that the operating functions are not necessarily locally Lipschitzian is given. We also show a sufficient condition in terms of the operating functions for a normal Banach function algebra to coincide with the algebra of all complex-valued continuous functions.

1. Introduction. By a Banach function algebra on a compact Hausdorff space X we mean a subalgebra of the algebra $C(X)$ of all complex-valued continuous functions on X, which contains the constant functions and separates different points in X and is a Banach algebra with the norm $\|\cdot\|_A$. A complex-valued function φ defined on a domain D in the complex plane is said to operate on a Banach function algebra A if the composite function $\varphi\circ u$ belongs to A for all u in A with range in D. We consider the problem involving the functions which operate on A. When A is uniformly closed de Leeuw and Katznelson [9] showed that if a non-analytic continuous function φ defined on a domain D operates on A, then $A=C(X)$. Considering the case where $\varphi(z)=\bar{z}$ on the complex plane $\boldsymbol{C}$ we see that the theorem of de Leeuw and Katznelson above is a generalization of the Stone-Weierstrass theorem. Spraglin [19] showed that a function φ defined on a domain in D which operates on a uniformly closed Banach function algebra on an infinite compact Hausdorff space is continuous on D (cf. [7]). Obviously every function operates on a Banach function algebra on a finite compact Hausdorff space. There is a Banach function algebra on an infinite compact Hausdorff space on which a certain discontinuous function does operate (cf. [1], [7, pp. II12–II14], [8]).

This fact is compared with the previous results; for example, there exists a Banach function algebra such that every operating function is real analytic and a Banach function algebra such that every operating function is locally Lipschitzian (cf. [10], [16]).

On the other hand Katznelson [17] showed that if the square root function $\sqrt{\cdot}$ defined on the half-open interval $[0,1)$ operates on a conjugate-closed Banach function algebra A on X, then $A=C(X)$. In [14] we showed that if $|z|^p$ $(0<p<1)$ defined on

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the open unit disk $\{z\in C: |z|<1\}$ operates on A on X, then $A=C(X)$. We also proved in [14] that several non-Lipschitz functions never operate on a non-trivial Banach function algebra by using the results concerning operating functions on the real part of the algebra.

In this paper we consider the case where the algebra A is normal, which means that for every pair of disjoint compact subsets K_1 and K_2 of X there is $f\in A$ with $f=0$ on K_1 and $f=1$ on K_2. We show that there is a non-trivial Banach function algebra such that the operating functions are not necessarily locally Lipschitzian. We also show that a normal Banach function algebra on X on which a non-local-Lipschitz function operates coincides essentially with $C(X)$, that is, there is a finite subset K of X such that $A|F=C(F)$ for every compact subset F of $X\setminus K$. Furthermore we give a sufficient condition for a normal Banach function algebra A to coincide with $C(X)$. We prove these results by using an ultraseparation argument. The notion of ultra-separability was introduced by Bernard [3]. We say that a Banach function algebra A on X is ultraseparating if $\tilde{A}$ separates the points in $\tilde{X}$, where $\tilde{A}$ is the algebra of all bounded sequences in A and $\tilde{X}$ is the Stone-Čech compactification of the direct product $X\times N$, where N is the discrete space of all positive integers. Thus every sequence $\tilde{f}$ in $\tilde{A}$ is identified with a function defined on $\tilde{X}$. We begin by recalling some results on the ultraseparation argument which we need in this paper to prove the theorems.

THEOREM A (cf. [11, Theorem], [12, Corollary 1.2]). *Let A be an ultraseparating Banach function algebra on a compact Hausdorff space X and φ a complex-valued non-analytic continuous function defined on the unit disk $\Delta=\{z\in C: |z|<1\}$. Suppose that φ operates on A. Then $A=C(X)$.*

For a subset S of $\tilde{X}$, $[S]$ is the closure of S in $\tilde{X}$. For every $x\in X$ we denote the fiber $\bigcap[K\times N]$ by F_x, where K varies over all the compact neighborhoods of x.

THEOREM B (cf. [14, Lemma A]). *Suppose that A is a Banach function algebra on X. Let x be a point in X and let p and q be a pair of different points in F_x. Suppose that $\tilde{A}$ does not separate p and q. Then the following hold.*

(i) *If p and q are points in $F_x\setminus[\{x\}\times N]$, then there are two sequences $\{G_p^{(n)}\}$ and $\{G_q^{(n)}\}$ of nonvoid compact subsets in $X\setminus\{x\}$ such that*

$$G_\alpha^{(n)}\cap\left(\overline{\bigcup_{(\beta,m)\neq(\alpha,n)} G_\beta^{(m)}}\right)=\varnothing$$

for $\alpha=p$, q and for every n. In fact, let n be a positive integer. If a function f in E satisfies the inequalities

$$|f(y)|\leq\frac{1}{2},\qquad y\in G_p^{(n)}$$

and

$$|f(z)| \geq 1, \qquad z \in G_q^{(n)},$$

then we have

$$\|f\|_E > n.$$

(ii) *If q is in $F_x \setminus [\{x\} \times N]$ and p is in $[\{x\} \times N]$, then there is a sequence $\{G_q^{(n)}\}$ of nonvoid compact subsets of $X \setminus \{x\}$ which satisfies*

$$G_q^{(n)} \cap \left(\overline{\bigcup_{m \neq n} G_q^{(m)}} \right) = \varnothing$$

for every positive integer n. Let n be a positive integer. If a function f in E satisfies the inequalities $f(x)=0$ and $|f(y)| \geq 1$ for every y in $G_q^{(n)}$, then we have

$$\|f\|_E > n.$$

THEOREM C (cf. [13, Lemma 6]). *Let A be a Banach function algebra on a compact Hausdorff space X. Then A is ultraseparating if and only if $\tilde{A}$ separates the points in F_x for every x in X.*

In [4] and [5] Bernard studied the problems of involving the non-local-Lipschitz functions which operate on a certain space of real-valued continuous functions on a compact Hausdorff space. The author also treated similar problems in [14] and [15].

2. Continuity. Spraglin [19] proved that functions defined on a domain which operate on a uniformly closed Banach function algebra on an infinite compact Hausdorff space are continuous (cf. [7, Theorem 9 and the corollaries]). By a similar idea we see the following.

LEMMA 1. *Let A be a normal Banach function algebra on an infinite compact Hausdorff space X and φ a complex-valued function defined on the unit disk $\Delta = \{z \in C : |z| < 1\}$. Suppose that φ operates on A. Then φ is continuous on Δ.*

PROOF. Suppose that φ is not continuous. Without loss of generality we may assume that $\varphi(0)=0$ and there is a sequence $\{z_n\}_{n=1}^{\infty}$ in Δ with $z_n \to 0$ such that $\inf_n |\varphi(z_n)| = d > 0$. There is a sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $\overline{\{x_n\}_{n=1}^{\infty} \setminus \{x_m\}} \not\ni x_m$ for every $m \in N$. For each $m \in N$ there is a function u_m in A such that $u_m(x_m)=1$ and $u_m(x_n)=0$ for $n \neq m$ since A is normal. For each $m \in N$ there is $k_m \in N$ such that $\|u_m\| \, |z_{k_m}| < 2^{-m}$. Put

$$u = \sum_{n=1}^{\infty} z_{k_n} u_n.$$

Then we have $u \in A$. We may suppose that $u(X) \subset \Delta$. Let x_0 be a cluster point of $\{x_n\}_{n=1}^{\infty}$. Then $u(x_m) = z_{k_m}$ and $u(x_0)=0$ since $u_n(x_0)=0$. We see that

$$\inf_n |\varphi \circ u(x_n) - \varphi \circ u(x_0)| \geq d,$$

which is a contradiction since $\varphi \circ u$ is continuous and x_0 is a cluster point of $\{x_n\}_{n=1}^{\infty}$.

3. Operating functions which are not Lipschitzian.

THEOREM 2. *Let A be a normal Banach function algebra on a compact Hausdorff space X and φ a complex-valued non-local-Lipschitz function defined on Δ. Suppose that φ operates on A. Then there exists a finite subset K of X such that $A\,|\,F=C(F)$ for every compact subset F of $X\diagdown K$.*

PROOF. Suppose that φ is not continuous. Then X is a finite set by Lemma 1, so that $A=C(X)$. Suppose that φ is continuous on Δ. In the same way as in [3] (cf. [6, Lemma 4.22]) there is a finite subset K of X such that for every $x\in X\diagdown K$ there exist a compact neighborhood G_x of x and positive real numbers ε_x and δ_x such that $\varphi\circ u\in A\,|\,G_x$ and $\|\varphi\circ u\|_{A|G_x}<\varepsilon_x$ for $u\in A\,|\,G_x$ with $\|u\|_{A|G_x}<\delta_x$. Indeed, suppose that there is no such K. Then there are infinite sequences $\{x_n\}$ of X and $\{G_n\}$ of compact neighborhoods of x_n such that $G_n\cap(\overline{\bigcup_{m\neq n}G_m})=\varnothing$ and that for every $\varepsilon>0$, $\delta>0$ and a compact neighborhood G of x_n there exists $u\in A\,|\,G$ with $\|u\|_{A|G}<\delta$, $\varphi\circ u\in A\,|\,G$ and $\|\varphi\circ u\|_{A|G}>\varepsilon$. For every $n\in N$ there exists an $E_n\in A$ such that $E_n=1$ on G_n and $E_n=0$ on $\overline{\bigcup_{m\neq n}G_m}$. Then for every n there is $f_n\in A$ such that

$$\|f_n\|_A<2^{-n-1}\|E_n\|^{-1}\,, \qquad \|\varphi\circ f_n\,|\,G_n\|_{A|G_n}\geq n\,.$$

Put $g=\sum_{n=1}^{\infty}f_nE_n$. Then g converges in A. We see that $g\,|\,G_n=f_n$ for every n and $\|g\|_A<1$. Thus

$$\|\varphi\circ g\|_A\geq\|\varphi\circ g\,|\,G_n\|_{A|G_n}=\|\varphi\circ f_n\,|\,G_n\|_{A|G_n}\geq n$$

for every $n\in N$, which is a contradiction. We may assume that $\varphi(0)=0$. By multiplying φ by δ_x/ε_x, without loss of generality, we may assume that $\varepsilon_x=\delta_x$. We may also suppose that there are real numbers t_0 and η with $0\leq t_0<\delta_x/20$, $t_0<\eta$ such that $\varphi(t_0)=10t_0$ and $|\varphi(t)|>10t$ for $t_0<t<\eta$, since φ is a non-local-Lipschitz function. We will prove that $A\,|\,G_x=C(G_x)$ for every $x\in X\diagdown K$. Suppose not. Then by [2, Theorem 1.5] (cf. [6, Corollary 6.16]) there are two sequences $\{G_1^{(n)}\}$ and $\{G_2^{(n)}\}$ of nonvoid compact subsets of G_x such that $G_1^{(n)}\cap G_2^{(n)}=\varnothing$ for every $n\in N$ and $M_n=\inf\{\|u\|_A : u\in A,\ u(G_1^{(n)})=\{0\},\ u(G_2^{(n)})=\{1\}\}\to\infty$ as $n\to\infty$. For each $n\in N$ choose a function $u_n\in A$ such that $\|u_n\|_A<2M_n$, $u_n=0$ on $G_1^{(n)}$ and $u_n=1$ on $G_2^{(n)}$ and put

$$v_n=\frac{\delta_x}{4M_n}\times u_n+t_0\,.$$

Then $\|v_n\|_A<\delta_x$. Then $\|\varphi\circ v_n\,|\,G_x\|_{A|G_x}<\delta_x$. Put

$$w_n=\{\varphi\circ v_n\,|\,G_x-\varphi(t_0)\}\times\frac{2M_n}{5\delta_x}\,.$$

Then $\|w_n\|_{A|G_x}<3M_n/5$, $w_n=0$ on $G_1^{(n)}$ and

$$w_n=\left(\varphi\left(\frac{\delta_x}{4M_n}+t_0\right)-10t_0\right)\times\frac{2M_n}{5\delta_x}$$

on $G_2^{(n)}$. There is $n_0\in N$ such that $\delta_x/4M_n+t_0<\eta$ for every $n\geq n_0$, since $M_n\to\infty$ as $n\to\infty$. On the other hand

$$|w_n|\geq\frac{2M_n}{5\delta_x}\left\{\left|\varphi\left(\frac{\delta_x}{4M_n}+t_0\right)\right|-10t_0\right\}$$

on $G_2^{(n)}$. Thus $|w_n|>1$ on $G_2^{(n)}$ since $|\varphi(t)|>10t$ for $t_0<t<\eta$. It follows by the definition of the quotient norm that for an $n\in N$ with $n\geq n_0$ there is $\hat{w}_n\in A$ such that $\|\hat{w}_n\|_A<3M_n/5$, $\hat{w}_n=0$ on $G_1^{(n)}$ and $\hat{w}_n=c$ on $G_2^{(n)}$, where c is a real number greater than 1, which contradicts the definition of M_n. Thus $A\,|\,G_x=C(G_x)$. Suppose that F is a compact subset F of $X\setminus K$. Then by the fact above there are $x_1,\dots,x_n\in F$ and compact neighborhoods G_{x_i} of x_i such that $A\,|\,G_{x_i}=C(G_{x_i})$ for $i=1,\dots,n$, and $\bigcup_{i=1}^n G_{x_i}\supset F$. It follows by, for example, a decomposition of the unity argument that $A\,|\,F=C(F)$. $\square$

There is a normal Banach function algebra A on X such that $A\neq C(X)$ on which a non-local-Lipschitz function operates. In the same way as in the proof of Proposition 24 in [15] we see the following.

EXAMPLE. Let $X=\{0\}\cup\{1/n:n\in N\}$ and

$$A=\left\{f\in C(X):\ \sum_{n=1}^{\infty}\left|f\left(\frac{1}{n}\right)-f(0)\right|M_n<\infty\right\},$$

where $M_n=2^{n^2}$. A is a normal conjugate-closed Banach function algebra on X. Let $d_n=1/2M_{n+1}+1/2(M_{n+1}-1)$, $r_n=-1/2M_{n+1}+1/2(M_{n+1}-1)$ and $h_n=2^{-n^2-n}$. Let φ be a complex-valued continuous function on $\Delta=\{z\in C:|z|<1\}$ such that

$$\varphi(z)=\begin{cases}0, & |z-d_n|>r_n \quad \text{for } \forall n\in N\\ (r_n-|z-d_n|)h_n/r_n, & |z-d_n|\leq r_n \quad \text{for } \exists n\in N.\end{cases}$$

Then φ is a non-local-Lipschitz function on Δ operating on A, but $A\neq C(X)$.

4. A sufficient condition for $A=C(X)$.

THEOREM 3. *Let A be a normal Banach function algebra on a compact Hausdorff space X and φ a complex-valued function defined on the open unit disk such that $|(\varphi(z)-\varphi(0))z^{-1}|$ tends to infinity as z tends to* 0. *Suppose that φ operates on A. Then $A=C(X)$.*

PROOF. In the same way as in the proof of Theorem 2 we consider only the case where φ is continuous. We will prove that A is ultraseparating. If we prove that A is ultraseparating, then we see that $A=C(X)$ by Theorem A. By Theorem C it is enough to prove that $\tilde{A}$ separates different points in F_x for each $x \in X$. Let x be a point in X. Suppose that p and q are different points in F_x. We consider four cases:

(i) $p, q \in [\{x\} \times N]$;

(ii) $p, q \in F_x \setminus [\{x\} \times N]$;

(iii) $p \in F_x \setminus [\{x\} \times N]$, $q \in [\{x\} \times N]$;

(iv) $p \in [\{x\} \times N]$, $q \in F_x \setminus [\{x\} \times N]$.

In the case (i) it is easy to see that $\tilde{A}$ separates p and q, since A contains constant functions. We can prove the cases (iii) and (iv) in a way similar to the case (ii). We give a proof of (ii).

Suppose that p and q are different points in $F_x \setminus [\{x\} \times N]$. By Theorem B there are two sequences $\{G_p^{(n)}\}$ and $\{G_q^{(n)}\}$ of nonvoid compact subsets of $X \setminus \{x\}$ which satisfy

$$G_\alpha^{(n)} \cap \left(\overline{\bigcup_{(\beta,m) \neq (\alpha,n)} G_\beta^{(m)}} \right) = \emptyset$$

for every $(\alpha, n) \in \{p,q\} \times N$ and that for every $n \in N$ the inequality $\| f \|_A > n$ holds for every $f \in A$ such that $|f| \le 1/2$ on $G_p^{(n)}$ and $|f| \ge 1$ on $G_q^{(n)}$. Put

$$B = \left\{ f \in A : f(x) = 0\,,\ f\left(\bigcup_{n=1}^{\infty} G_p^{(n)} \right) = \{0\}\,,\ \text{and } f \text{ is constant on } G_q^{(m)} \text{ for every } m \right\}.$$

Put

$$M_n = \inf\{ \| f \|_A : f \in B,\ f = 1 \text{ on } G_q^{(n)} \}$$

for $n \in N$. Then $M_n < \infty$, since A is normal and $M_n \to \infty$ as $n \to \infty$. In fact we see that $M_n \ge n$ for all n. By the Baire category theorem (cf. Sidney [18]) there are positive real numbers δ and ε with $\delta < 1/2$, $u_0 \in B$ with $|u_0| < 1/2$ on X and a dense subset U of $\{u \in B : \| u - u_0 \|_A < \delta\}$ such that $\varphi \circ u \in A$ and $\| \varphi \circ u \|_A < \varepsilon$ for every $u \in U$. By the definition of M_n there is $u_n \in B$ such that $u_n = 1$ on $G_q^{(n)}$ and $\| u_n \|_A < 2M_n$ for each $n \in N$. Put $c_n = u_0(G_q^{(n)})$ and

$$v_n = \begin{cases} u_0 + \dfrac{\delta c_n}{2M_n |c_n|} u_n\,, & c_n \neq 0 \\[2ex] u_0 + \dfrac{\delta}{2M_n} u_n\,, & c_n = 0\,. \end{cases}$$

Note that $v_n \in \{u \in B : \| u - u_0 \|_A < \delta\}$. Without loss of generality we may suppose that $\varphi(0) = 0$. For every positive real number c there is a positive real number t_c such that $|\varphi(z)| > c|z|$ for all z with $0 < |z| < t_c$. Put $c = 3\varepsilon/\delta$. For n sufficiently large we see that

$$|v_n| = |c_n| + \frac{\delta}{2M_n} < t_c$$

on $G_q^{(n)}$. It follows that

$$|h\circ v_n|>c\left(|c_n|+\frac{\delta}{2M_n}\right)>\frac{3\varepsilon}{2M_n}$$

on $G_q^{(n)}$. Thus there is a $w_n \in U$ near v_n such that the inequalities $|h\circ w_n|>3\varepsilon/2M_n$ on $G_q^{(n)}$ and $\|h\circ w_n\|_A<\varepsilon$ hold for n large. It follows that $(2M_n/3\varepsilon)h\circ w_n$ is in B, constant on $G_q^{(n)}$ and $|(2M_n/3\varepsilon)h\circ w_n|\geq 1$ on $G_q^{(n)}$ and $\|(2M_n/3\varepsilon)h\circ w_n\|_A<2M_n/3$ for n sufficiently large, which is a contradiction. We have thus proved that $\tilde{A}$ separates p and q. $\square$

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