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SPECTRAL CONVERGENCE OF RIEMANNIAN MANIFOLDS, II

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Abstract. We prove a precompactness theorem concerning the spectral distance on the set of isometry classes of compact Riemannian manifolds and study the completion of a precompact family.

Introduction. For a compact connected Riemannian manifold $M=(M, g)$, we denote by $p_M(t, x, y)$ the heat kernel of the Laplace operator of M with respect ot the normarized Riemannian measure $\mu_M\,(=dv_g/\mathrm{Vol}(M))$. Given two compact connected Riemannian manifolds M and N, a mapping $f: M \to N$ is called an ε-spectral approximation if it satisfies

$$e^{-(t+1/t)}|p_M(t, x, y)-p_N(t, f(x), f(y))|<\varepsilon$$

for all $t>0$ and $x, y \in M$. The spectral distance $\mathrm{SD}(M, N)$ between M and N is by definition the lower bound of the positive numbers ε such that there exist ε-spectral approximations $f: M \to N$ and $h: N \to M$. The distance SD gives a uniform structure on the set $\mathcal{M}_c$ of isometry classes of compact connected Riemannian manifolds.

Riemannian manifolds are considered as metric spaces endowed with Riemannian distances. From this point of view, the set $\mathcal{M}_c$ has another uniform structure introduced by Gromov [18], called the Hausdorff distance HD. In [18], the conditions for a family of $\mathcal{M}_c$ to be HD-precompact are described and it is shown that the boundaries of such a family consist of certain metric spaces, called length spaces. This decade has seen intensive activities around the convergence theory of Riemannian manifolds with respect to the Gromov-Hausdorff distance. These includes some works from the viewpoint of spectral geometry, for instance, [14], [4], and [23]. In [25], motivated by these results, we introduced the spectral distance SD mentioned above and discussed some basic properties of the distance on a set of compact connected Riemannian manifolds of the same dimension with diameters uniformly bounded from above and Ricci curvatures uniformly bounded from below.

In the present paper, we are concerned with a certain precompact family of $\mathcal{M}_c$ and its compactification with respect to the spectral distance. More precisely, the main results are stated as follows.

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(1) (Theorems 2.3 and 2.6) Let positive constants a, v, and τ be given and let $\mathscr{S}(a, v, \tau)$ be a family in $\mathscr{M}_c$ such that the heat kernel $p_M(t, x, y)$ of each $M \in \mathscr{S}(a, v, \tau)$ satisfies

$$p_M(t, x, x) \leq \frac{a}{t^{v/2}} \tag{0.1}$$

for all $0 < t \leq \tau$ and $x \in M$. Then $\mathscr{S}(a, v, \tau)$ is precompact with respect to the spectral distance SD and also the Gromov-Hausdorff distance HD.

(2) (Theorem 3.1) Let $\{M_n : n = 1, 2, \dots\}$ be an SD-Cauchy sequence in $\mathscr{S}(a, v, \tau)$. Suppose that the diameter of M_n is bounded away from zero uniformly by a positive constant. Then there exist a compact metric space $X = (X, \Theta)$, a (positive) Radon measure μ on X, a nonnegative continuous function p on $(0, \infty) \times X \times X$, Borel measurable mappings $F_n : M_n \to X$, $H_n : X \to M_n$, and a sequence of positive numbers $\{\varepsilon_n\}$ converging to zero as $n \to \infty$, which satisfy the following properties:

(i) $p(t, x, y)$ is the heat kernel of a strongly continuous semigroup $\{T_t : t > 0\}$ on $L^2(X, \mu)$ associated with a regular Dirichlet form on $L^2(X, \mu)$;

(ii) the push-forward $F_{n*}\mu_{M_n}$ of the normalized Riemannian measure μ_{M_n} of M_n by the mapping F_n converges, as $n \to \infty$, to the measure μ with respect to the vague topology;

(iii) the i-th eigenvalue $\lambda_{n,i}$ of M_n for each i converges, as $n \to \infty$, to the i-th eigenvalue λ_i of the infinitesimal generator $\mathscr{L}$ of $\{T_t : t > 0\}$;

(iv) the mappings $F_n : M_n \to X$ and $H_n : X \to M_n$ are ε_n-spectral approximations, namely, they satisfy

$$e^{-(t+1/t)} | p_{M_n}(t, x, y) - p(t, F_n(x), F_n(y)) | \leq \varepsilon_n$$
$$e^{-(t+1/t)} | p_{M_n}(t, H_n(x'), H_n(y')) - p(t, x', y') | \leq \varepsilon_n$$

for all $t > 0$, $x, y \in M_n$, and $x', y' \in X$, and also one has

$$\Theta(F_n \circ H_n(x'), x') \leq \varepsilon_n$$

for $x' \in X$;

(v) letting a positive integer i be given, for each eigenfunction u of (M_n, μ_{M_n}) with eigenvalue $\lambda_{n,i}$ and unit norm in $L^2(M_n, \mu_{M_n})$, there is an eigenfunction v of $\mathscr{L}$ with eigenvalue λ_i and unit norm in $L^2(X, \mu)$, such that

$$\sup_{x \in M_n} | u(x) - v(F_n(x)) | \leq \varepsilon_{n.i}; \ \sup_{x \in X} | u(H_n(x)) - v(x) | \leq \varepsilon_{n,i},$$

where $\{\varepsilon_{n,i}\}$ is a sequence of positive constants tending to zero as $n \to \infty$.

We remark that the measure μ is not necessarily supported on the whole of X and the trivial eigenvalue $\lambda_0 = 0$ may be of multiplicity greater than one (cf. Sections 5, 7 and 8 for such examples).

Given a positive integer m and positive constants κ and D, we denote by $\mathcal{M}^*(m, \kappa, D)$ the subset in $\mathcal{M}_c$ consisting of compact connected Riemannian manifolds of dimension m whose diameters are not greater than D and whose Ricci curvatures are bounded from below by $-\kappa$. Then the main results of [25] are stated as follows.

(3) $\mathcal{M}^*(m, \kappa, D)$ is precompact with respect to the spectral distance SD, and the above assertions (1) and (2) certainly hold for an SD Cauchy sequence in this class. Moreover the semigroup $\{T_t : t>0\}$ on the limit space X satisfies the Feller property, namely, for each continuous function u on X,

$$\lim_{t\to 0} \sup_{x\in X} | T_t u(x) - u(x) | = 0$$

and in fact it possesses the Lindeberg type property, that is, for a compact subset K in X and an open subset G including K,

$$\lim_{t\to 0} \sup_{x\in K} \frac{1}{t} T_t(\mathcal{I}_{X-G}) = 0 ,$$

where $\mathcal{I}_A$ stands for the characteristic function of a set A.

(4) On $\mathcal{M}^*(m, \kappa, D)$, the uniform topology given by the spectral distance is finer than that of the Gromov-Hausdorff distance. In fact, if a sequence $\{M_n\}$ in $\mathcal{M}^*(m, \kappa, D)$ converges to a boundary element (X, μ, p) with respect to the spectral distance, then there is a distance d on X such that the metric space (M_n, d_{M_n}) endowed with the Riemannian distance d_{M_n} converges to (X, d) with respect to the Gromov-Hausdorff distance. Moreover one has

$$\lim_{t\to 0} 4t \log p(t, x, y) = -d(x, y)^2$$

for all $x, y \in X$.

The assertion (3) will be valid for larger classes considered in [32]. In fact, let $\Lambda > 1$ be given further and consider the set of equivalence classes of $M=(M, g)$, denoted by $\mathcal{M}^*(m, \kappa, D; \Lambda)$, which admit Riemannian metrics h such that $(M, h) \in \mathcal{M}^*(m, \kappa, D)$ and $\Lambda^{-1}h \le g \le \Lambda h$. Then we shall show

(5) (Theorem 4.3) the assertion (3) certainly holds for $\mathcal{M}(m, \kappa, D; \Lambda)$.

However the assertion (4) is not true in general for this class (cf. 4.3 Example).

These results will be verified in Sections 2 through 4. As the first step, in Section 1, we exhibit the method of embedding compact Riemannian manifolds into a certain metric space of infinite dimension in connection with the spectral distance defined above. It should be mentioned here that Bérard, Besson and Gallot [4] defined a family of spectral distances on the set of compact Riemannian manifolds by embedding them into the Hilbert space of real-valued, square integrable series. However their distances are different from ours. We consider a point of a compact Riemannian manifold as a curve in the Hilbert space, taking the Sturm-Liouville decomposition of the heat kernel into account.

In Sections 5 through 9, we shall exhibit some geometric classes which satisfy uniform diagonal estimates (0.1) for the heat kernels. It is well known that (0.1) is equivalent to inequalities involving the energy forms, Sobolev and Nash inequalities (cf. Theorem 2.1). In Section 5, we consider conformal classes of positive Yamabe invariants and derive Sobolev inequality for a class of conformal metrics in terms of the Yamabe invariant and a certain integral bound of scalar curvature. Section 6 concerns a family of Riemannian submanifolds in a complete manifold and a heat kernel bound (0.1) is shown with constants involving an upper bound of the volumes and a certain integral bound of the mean curvatures. Section 7 is devoted to exhibiting SD-precompact families of Riemannian manifolds with increasing topological type. In Section 8, we show a sequence of Riemannian metrics on a compact surface which degenerates along simply closed curves while keeping the heat kernels bounded uniformly. Riemannian submersions with totally geodesic fibers are taken up in Section 9 and a typical deformation of such metrics on a total space is discussed. In addition, some observations are made concerning Riemannian manifolds with certain integral bounds on curvatures.

As in [25], from the nature of the problem discussed here, we shall in fact investigate Riemannian manifolds endowed with weight functions and the associated operators rather than Riemannian manifolds and the Laplace operators. When we fix a compact connected differentiable manifold and consider a family of Riemannian metrics and weight functions, the topology of the spectral distance are closely related to those studied by many authors (see for instance, [27], [30], [20], [31] and references therein).

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1. Spectral embedding and spectral distance. In this section, we review some definitions introduced in [25].

1.1. Let $M=(M, g)$ be a compact, connected Riemannian manifold of dimension n and w be a positive smooth function on M. We consider an elliptic differential operator $\mathscr{L}_w$ of second order defined by

$$\mathscr{L}_w\phi=-\frac{1}{w}\operatorname{div}(w\nabla\phi)=-\Delta_M\phi-d\phi(\nabla\log w)\,,$$

where Δ_M stands for the Laplace operator of M acting on functions. This operator $\mathscr{L}_w$ is associated with the Dirichlet integral $\mathscr{E}$ on the space $C^\infty(M)$ of smooth functions defined by

$$\mathscr{E}(\phi, \psi) = \int_M \langle \nabla\phi, \nabla\psi \rangle_g d\mu_w ,$$

where we write μ_w for the (Radon) measure $w dv_M$ of density w with respect to the Riemannian measure dv_M. The operator $\mathscr{L}_w$ is essentially self-adjoint, and in this paper, its closure in $L^2(M, \mu_w)$ is also denoted by the same letter. Let $p_w(t, x, y)$ be the heat kernel of the operator $\mathscr{L}_w$. Since M is assumed to be compact, we have the eigenfunction expansion of the kernel:

$$p_w(t, x, y) = \sum_{i=0}^{\infty} e^{-\lambda_i t} u_i(x) u_i(y) .$$

Here $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \cdots$ are the eigenvalues of $\mathscr{L}_w$ and $\Phi = \{u_i\}$ is a complete orthonormal system of $L^2(M, \mu_w)$ consisting of eigenfunctions with u_i having eigenvalue λ_i.

Now let us recall a spectral embedding of M used in [25]. To begin with, we define two Hilbert spaces l_2 and h_1 by

$$l_2 = \left\{ (a_i)_{i=0,1,2,\ldots} : \sum_{i=0}^{\infty} a_i^2 < +\infty \right\} ,$$

$$h_1 = \left\{ (a_i)_{i=0,1,2,\ldots} : \sum_{i=0}^{\infty} (1+i^2) a_i^2 < +\infty \right\} .$$

We denote by $C_\infty([0, \infty), l_2)$ the space of continuous curves $\gamma : [0, \infty) \to l_2$ such that the l_2-norm $|\gamma(t)|_{l_2}$ of $\gamma(t)$ tends to zero as $t \to \infty$. This space is endowed with the distance

$$d_\infty(\gamma, \sigma) := \sup_{t>0} |\gamma(t) - \sigma(t)|_{l_2} .$$

We note that given a positive constant C and a nonnegative continuous function $\eta(t)$ on $[0, \infty)$ tending to zero as $t \to \infty$, if we set

$$\begin{aligned} K(C, \eta) := \{ \gamma \in C_\infty([0, \infty), l_2) : & \ |\gamma(t)|_{h_1} \leq \eta(t) \text{ for all } t \geq 0 , \\ & |\gamma(t) - \gamma(s)|_{l_2} \leq C|t-s| \text{ for all } t, s \geq 0 \} , \end{aligned}$$

then $K(C, \eta)$ becomes a compact metric subspace of $C_\infty([0, \infty), l_2)$, and hence the set of closed subsets of $K(C, \eta)$ is also compact with respect to the Hausdorff distance δ_H.

Let a complete orthonormal basis $\Phi = \{u_i\}$ of $L^2(M, \mu_w)$ as above be given. For a point $x \in M$, we shall define an element $F_\Phi[x]$ of $C_\infty([0, \infty), l_2)$ by

$$F_\Phi[x](t) = (e^{-(t+1/t)/2} e^{-\lambda_i t/2} u_i(x))_{i=0,1,2,\ldots} .$$

Then the map F_Φ of M into $C_\infty([0, \infty), l_2)$ gives rise to a continuous embedding of M, which will be called the spectral embedding of M with respect to the given basis $\Phi = \{u_i\}$. We observe that for all $x, y \in M$ and $t > 0$,

$$\langle F_\Phi[x](t), F_\Phi[y](t)\rangle_{l_2} = e^{-(t+1/t)} p_w(t, x, y) ,$$

and

$$d_\infty(F_\Phi[x], F_\Phi[y])^2 = \sup_{t>0} e^{-(t+1/t)}(p_w(t, x, x) + p_w(t, y, y) - 2p_w(t, x, y)) .$$

Putting

$$\Theta_{M,w}(x, y) = \left(\sup_{t>0} e^{-(t+1/t)}(p_w(t, x, x) + p_w(t, y, y) - 2p_w(t, x, y)) \right)^{1/2}$$

for $x, y \in M$, we have a distance $\Theta_{M,w}$ on M which induces the same topology of M. The spectral embedding $F_\Phi: (M, \Theta_{M,w}) \to (C_\infty([0, \infty), l_2), d_\infty)$ is thus distance preserving; the distance $\Theta_{M,w}$ will play an important role in this paper. The Riemannian distance between two points x, y of M is denoted by $d_M(x, y)$; (M, μ_w) will stand for a pair of a compact connected Riemannian manifold $M = (M, d_M)$ and a (Radon) measure $\mu_w = w dv_M$ with density w.

1.2. We are given two pairs $(M, \mu_v = v dv_M)$ and $(N, \nu_w = w dv_N)$. A mapping $f: M \to N$ is called an ε-spectral approximation for a positive number ε if

$$\sup\{e^{-(t+1/t)} |p_v(t, x, y) - p_w(t, f(x), f(y))| : t > 0, x, y \in M\} \le \varepsilon ,$$

where p_v and p_w are respectively the heat kernels of (M, μ_v) and (N, ν_w). The spectral distance between (M, μ_v) and (N, ν_w), denoted by $\mathrm{SD}((M, \mu_v), (N, \nu_w))$, is by definition the lower bound of the numbers $\varepsilon > 0$ such that they admit ε-spectral approximations $f: M \to N$ and $h: N \to M$. We observe that

$$\mathrm{SD}((M, \mu_v), (N, \nu_w)) = 0$$

if and only if there is a homeomorphism $f: M \to N$ which preserves the heat kernels and the measures, namely,

$$p_v(t, x, y) = p_w(t, f(x), f(y)) \qquad \text{for all} \quad t > 0, \ x, y \in M ; \quad f_* \mu_v = \nu_w .$$

Indeed, suppose first that $\mathrm{SD}((M, \mu_v), (N, \nu_w)) = 0$. Then there are ε_n-spectral approximations $f_n: M \to N$ with ε_n tending to zero as $n \to \infty$. Then by the definition of the distances $\Theta_{M,v}$ on M and $\Theta_{N,w}$ on N, we see that

$$|\Theta_{M,v}(x, y)^2 - \Theta_{N,w}(f_n(x), f_n(y))^2| \le 4\varepsilon_n$$

for all $x, y \in M$. Now we choose an increasing sequence of finite subsets A_k of M in such a way that A_k is $(1/k)$-dense with respect to $\Theta_{M,v}$. For each k, we take a subsequence $\{f_{n(k)}\}$ of $\{f_n\}$ such that $f_{n(k)}(x)$ converges for all $x \in A_k$ as $n(k) \to \infty$. We may assume that $\{f_{n(k+1)}\} \subset \{f_{n(k)}\}$. Then by the diagonal argument, we can find a subsequence $\{f_m\}$ which converges for every point $x \in \bigcup A_k$ as $m \to \infty$; we set $f(x) = \lim_{m\to\infty} f_m(x)$ ($x \in \bigcup A_k$). The mapping $f: \bigcup A_k \to N$ preserves the distances, namely,

$$\Theta_{M,v}(x, y)=\Theta_{N,w}(f(x), f(y))$$

for all $x, y\in\bigcup A_k$. This shows that f can be extended uniquely to a distance preserving mapping, denoted by the same letter f, of M onto N, because $\bigcup A_k$ is dense, and both of M and N are compact and connected. Moreover since f_n are ε-spectral approximations with $\lim_{n\to\infty}\varepsilon_n=0$, the mapping f actually preserves the heat kernels. It is easy to see that f also preserves the given measures, because, for a continuous function ψ on N,

$$\begin{aligned}\int_M \psi(f(x))d\mu_v(x) &= \lim_{t\to 0}\int_M\int_N p_w(t, f(x), b)\psi(b)dv_w(b)d\mu_v(x)\\ &= \lim_{t\to 0}\int_M\int_M p_w(t, f(x), f(y))\psi(f(y))d\mu_v(x)dv_w(f(y))\\ &= \lim_{t\to 0}\int_M\int_M p_v(t, x, y)\psi(f(y))d\mu_v(x)dv_w(f(y))\\ &= \int_M \psi(f(y))dv_w(f(y)).\end{aligned}$$

In our situation, the mapping $f: M\to N$ as above induces an isometry between M and N which preserves the given measures, since it is well known that

$$\lim_{t\to 0} 4t\log p_w(t, x, y)=-d_M(x, y)^2$$

for all $x, y\in M$ (see [34], [11], and [25]).

Now identifying two pairs (M, μ_v) and (N, v_w) when there is a mapping of M onto N which preserves the heat kernels and the measures, we obtain a metric space $\mathscr{M}_{c,w}:=\{(M, \mu_w)/\sim, \mathrm{SD}\}$.

1.3. We shall consider a triad of a compact, connected Riemannian manifold M, a measure $\mu_w=wdv_M$ as before and a complete orthonormal system $\Phi=\{u_i\}_{i=0,1,\dots}$ in the Hilbert space $L^2(M, \mu_w)$ such that u_i is an eigenfunction of the operator $\mathscr{L}_{M,w}$ having the i-th eigenvalue λ_i. Given such triads $\alpha=(M, \mu_v, \Phi=\{u_i\})$ and $\beta=(N, v_w, \Psi=\{v_i\})$, we set

$$\mathrm{SD}^*(\alpha, \beta):=\delta_H(F_\Phi[M], F_\Psi[N]),$$

where δ_H stands for the Hausdorff distance on the set of closed subsets of the metric space $C_\infty([0, \infty), l_2)$. Obviously $\mathrm{SD}^*(\alpha, \beta)=0$, that is, $F_\Phi[M]=F_\Psi[N]$, if and only if there exists a homeomorphism f of M onto N which preserves the heat kernels, the measures and further the given orthonormal systems, $f^*\Psi=\Phi$. In what follows, we shall identify these triads and denote by $\mathscr{F}\mathscr{M}_{c,w}$ the set of equivalence classes of elements $(M, \mu_w, \Phi=\{u_i\})$ endowed with the distance SD^*.

Now let $\pi: \mathscr{F}\mathscr{M}_{c,w}\to\mathscr{M}_{c,w}$ be the canonical projection from $\mathscr{F}\mathscr{M}_{c,w}$ onto $\mathscr{M}_{c,w}$

sending $(M, \mu_w, \Phi=\{u_i\})$ to (M, μ_w), and set

$$\rho(M, \mu_w) := \sup\{e^{-(t+1/t)} p_w(t, x, y) : t>0,\ x, y \in M\}\ .$$

Then for $\alpha, \beta \in \mathscr{F}\mathscr{M}_{c,w}$ we have

$$\mathrm{SD}(\pi(\alpha), \pi(\beta)) \le 2 \max\{\rho(\pi(\alpha)), \rho(\pi(\beta))\} \mathrm{SD}^*(\alpha, \beta)\ .$$

Therefore a subset $\mathscr{S}$ of $\mathscr{M}_{c,w}$ is SD-precompact, provided that $\rho(M, \mu_w)$ is uniformly bounded from above on $\mathscr{S}$ and $\pi^{-1}(\mathscr{S})$ is SD*-precompact. Thus we can deduce from the observations in 1.1 that $\mathscr{S}$ is SD-precompact, if there exist a positive constant C and a positive continuous function η on $[0, \infty)$, with $\eta(t)$ tending to zero as $t \to \infty$, such that

$$F_\Phi[M] \subset K(C, \eta)$$

for any $(M, \mu_w, \Phi) \in \pi^{-1}(\mathscr{S})$.

2. Upper bounds for heat kernels and precompactness. In this section, we shall give sufficient conditions for a given subset $\mathscr{S}$ of $\mathscr{M}_{c,w}$ to be precompact, in terms of uniform upper bounds for the heat kernels and also the measures.

2.1. We are given a pair (M, μ_w) of a compact connected Riemannian manifold $M=(M, g)$ and a Radon measure μ_w with density w. In what follows, for a function $\phi \in L^p(M, \mu_w)$, we write $\|\phi\|_p$ for the norm of ϕ:

$$\|\phi\|_p := \left(\int_M |\phi(x)|^p d\mu_w(x) \right)^{1/p}\ .$$

We shall first recall some basic results on bounds of the heat kernels $p_w(t, x, y)$ and inequalities involving the quadratic forms $\mathscr{E}$.

THEOREM 2.1. *Let (M, μ_w) and p_w be as above. Let $v \in (0, \infty)$ be given. If*

$$p_w(t, x, x) \le \frac{a}{t^{v/2}} \tag{2.1}$$

for some $a>0$ and $\tau>0$, and for all $t \in (0, \tau]$ and $x \in M$, then there is a constant A depending only on v and a such that

$$\|\phi\|_2^{2+4/v} \le A\{\mathscr{E}(\phi, \phi) + \tau^{-1}\|\phi\|_2^2\}\|\phi\|_1^{4/v} \tag{2.2}$$

for all $\phi \in C^\infty(M)$. Conversely (2.2) *implies that* (2.1) *holds for some $a>0$ depending only on v, A and τ. Moreover in the case when $v>2$,* (2.1) *is also equivalent to a bound of the form:*

$$\|\phi\|_{2v/(v-2)}^2 \le A'\{\mathscr{E}(\phi, \phi) + \tau^{-1}\|\phi\|_2^2\} \tag{2.3}$$

for all $f \in C^\infty(M)$, where A' (resp. a) is a constant depending only on v and a (resp. v, A'

and τ) *if* (2.1) (*resp.* (2.3)) *holds.*

Moreover we use local Sobolev inequalities and off-diagonal estimates for the heat kernels.

THEOREM 2.2. *Let* (M, μ_w) *and* p_w *be as above. Let* $v \in (0, \infty)$, *a point* $x \in M$ *and* $r \in (0, +\infty]$ *be given. Suppose that*

$$
(2.4) \qquad \begin{aligned} &\|\phi\|^2_{2v/(v-2)} \le A^*\{\mathscr{E}(\phi, \phi) + \tau^{-1}\|\phi\|^2_2\} \qquad (v>2)\,; \\ &\|\phi\|^4_4 \le A^*\{\mathscr{E}(\phi, \phi) + \tau^{-1}\|\phi\|^2_2\}\|\phi\|^2_2 \qquad (v=2) \end{aligned}
$$

for some A^* *and* τ, *and for all* $\phi \in C_0^\infty(B(x, r))$. *Then one has a diagonal estimate of the form*:

$$
(2.5) \qquad p_w(t, y, y) \le C(v)A^{*v/2}(1+\tau^{-1}t)^{v/2}(1+r^{-2}t)(r^{-2}+t^{-1})^{v/2}
$$

for all $t>0$ *and* $y \in B(x, r/2)$, *where* $C(v)$ *is a positive constant depending only on* v. *Moreover if* (2.4) *holds for another point* $x' \in M$, *then the following off-diagonal estimate holds*:

$$
(2.6) \quad p_w(t, x, x') \le C(v)A^{*v/2}(1+\tau^{-1}t)^{v/2}t^{-v/2}(1+t^{-1}d_M(x, x')^2)^{1+v/2}\exp\left(-\frac{d_M(x, x')^2}{4t}\right)
$$

for $0 < t < rd_M(x, x')$ $(\le \infty)$.

We refer the reader to [29], [35], and [8] for the proof of Theorem 2.1. See also [13]. So for as Theorem 2.2 is concerned, adapting the arguments in [29] (see also [9]) and [33] will yield the above estimates.

2.2. We shall now prove the following:

THEOREM 2.3. *Given a subset* $\mathscr{S}$ *of* $\mathscr{M}_{c,w}$, *suppose that* (2.1) *holds for some positive constants* v, a, *and* τ *and for all elements* $(M, \mu_w) \in \mathscr{S}$ *and moreover suppose that the total measure* $\mu_w(M)$ *of any* $(M, \mu_w) \in \mathscr{S}$ *is not greater than a positive constant* b. *Then* $\mathscr{S}$ *is precompact with respect to the spectral distance* SD.

PROOF. This follows from the lemma below. Indeed it can be deduced from the lemma that there exist a positive constant C and a positive continuous function $\eta(t)$ on $[0, \infty)$, tending to zero as $t \to \infty$ and depending only on the given constants v, a, τ and b, such that

$$
F_\Phi[M] \subset K(C, \eta)
$$

for any $(M, \mu_w, \Phi) \in \pi^{-1}(\mathscr{S})$. q.e.d.

LEMMA 2.4. *Let* $(M, \mu_w, \Phi = \{u_i : i=0, 1, 2 \dots\})$ *be an element of* $\mathscr{F}\mathscr{M}_{c,w}$ *such that* (2.1) *holds for some positive constants* v, a *and* τ. *Then the following assertions hold.*

(i) *The i-th eigenvalue* λ_i *satisfies*

$$
\lambda_i \ge \left(\frac{1+i}{ae\mu_w(M)}\right)^{2/v} \quad \text{if} \quad \lambda_i \ge \tau^{-1}\,; \quad 1+i \le ae\mu_w(M)\tau^{-v/2} \quad \text{if} \quad \lambda_i \le \tau^{-1}\,.
$$

(ii) *The eigenfunction u_i for each i has a bound of the form*:

$$|u_i|^2 \le ae \max\{\lambda_i^{\nu/2}, \tau^{-\nu/2}\} .$$

(iii) *Given $\sigma \ge 0$, one has*

$$\sum_{i=0}^{\infty} \lambda_i^{\sigma} e^{-\lambda_i t} u_i(x)^2 \le C\left(\frac{1}{t^{\sigma+\nu/2}} + \frac{1}{t^{\sigma}\tau^{\nu/2}}\right)$$

for all $t>0$ and $x \in M$, where C is a positive constant depending only on ν, a, and σ.

(iv) *Given $\sigma \ge 0$,*

$$e^{-(t+1/t)} \sum_{T \le \lambda_i} \lambda_i^{\sigma} e^{-t\lambda_i} u_i(x)^2 \le 2ae \int_T^{\infty} \lambda^{\sigma+\nu/2} e^{-2\sqrt{\lambda}} d\lambda$$

for all $T \ge \tau^{-1}$, $t>0$ and $x \in M$.

Proof. We first observe that for each $\lambda>0$, and for all $x \in M$,

$$\sum_{\lambda_i \le \lambda} u_i^2(x) \le ae \max\{\lambda^{\nu/2}, \tau^{-\nu/2}\} . \tag{2.7}$$

Indeed,

$$\sum_{\lambda_i \le \lambda} u_i^2(x) \le e \sum_{\lambda_i \le \lambda} e^{-\lambda_i/\lambda} u_i^2(x) \le e \sum_{i=0}^{\infty} e^{-\lambda_i/\lambda} u_i^2(x)$$
$$= e p_w(1/\lambda, x, x) \le ae \max\{\lambda^{\nu/2}, \tau^{-\nu/2}\} .$$

Integrating the both sides of (2.7) over M, we get

$$\#\{\lambda_i : \lambda_i \le \lambda\} \le ae \max\{\lambda^{\nu/2}, \tau^{-\nu/2}\} \mu_w(M) .$$

Thus the first two assertions follow.

Now for each point $x \in M$, we define a measure η_x on the real line $\boldsymbol{R}$ by

$$\eta_x = \sum_{i=0}^{\infty} u_i^2(x) \delta_{\lambda_i} ,$$

where δ_λ stands for the Dirac delta measure at λ. Since $\eta_x((-\infty, \lambda]) = \eta_x([0, \lambda]) \le ae \max\{\lambda^{\nu/2}, \tau^{-\nu/2}\}$, we have

$$\sum_{i=0}^{\infty} \lambda_i^{\sigma} e^{-t\lambda_i} u_i(x)^2 = \int_{-\infty}^{\infty} \lambda^{\sigma} e^{-t\lambda} d\eta_x(\lambda)$$
$$= \int_0^{\infty} (t\lambda^{\sigma} - \sigma\lambda^{\sigma-1}) e^{-t\lambda} \eta_x((-\infty, \lambda]) d\lambda$$
$$\le ae \int_0^{\infty} t\lambda^{\sigma} e^{-t\lambda} \max\{\lambda^{\nu/2}, \tau^{-\nu/2}\} d\lambda$$

$$\leq ae\{\Gamma(\sigma+v/2+1)+\Gamma(\sigma+1)\}\left(\frac{1}{t^{\sigma+v/2}}+\frac{1}{t^{\sigma}\tau^{v/2}}\right).$$

This shows the assertion (iii).

It remains to verify the last assertion. Observing that

$$te^{-(t+1/t)-t\lambda}\leq\frac{2}{-1+\sqrt{5+4\lambda}}e^{-\sqrt{5+4\lambda}}<2e^{-2\sqrt{\lambda}}$$

for all $t>0$ and $\lambda>0$, we have

$$\begin{aligned} e^{-(t+1/t)}\sum_{T\leq\lambda_i}\lambda_i^{\sigma}e^{-t\lambda_i}u_i(x)^2 &= e^{-(t+1/t)}\int_T^{\infty}\lambda^{\sigma}e^{t\lambda}d\eta_x(\lambda) \\ &< e^{-(t+1/t)}\int_T^{\infty}t\lambda^{\sigma}e^{-t\lambda}\eta_x((-\infty,\lambda])d\lambda \\ &< ae\int_T^{\infty}te^{-(t+1/t)-t\lambda}\lambda^{\sigma+v/2}d\lambda \\ &< 2ae\int_T^{\infty}\lambda^{\sigma+v/2}e^{-2\sqrt{\lambda}}d\lambda\,. \end{aligned}$$

q.e.d.

2.3. Now we shall derive some geometric estimates on $(M,\mu_w)\in\mathcal{M}_{c,w}$ from the Nash inequality (2.2).

LEMMA 2.5. *Let (M,μ_w) be an element of $\mathcal{M}_{c,w}$ such that* (2.2) *holds for some positive constants v, A and τ. Then the following assertions hold.*

(i) *The total measure $\mu_w(M)$ is bounded from below by $A^{-v/2}\tau^{v/2}$.*

(ii) *For all points $x\in M$ and any $r\in(0,\tau^{1/2}]$, the measure of the geodesic ball $B(x,r)$ of $M=(M,d_M)$ around x with radius r satisfies*

$$\mu_w(B(x,r))\geq C(v)A^{-v/2}r^{v}\,.$$

(iii) *The diameter of $M=(M,d_M)$ satisfies*

$$\operatorname{diam}M\leq C(v)A^{1/2}\mu_w(M)^{1/v}\qquad \textit{if}\quad \operatorname{diam}M\leq\tau^{1/2}\,;$$

$$\operatorname{diam}M\leq C(v)A^{v/2}\tau^{-(v-1)/2}\mu_w(M)\qquad \textit{if}\quad \operatorname{diam}M\geq\tau^{1/2}\,.$$

(iv) *The i-th eigenvalue λ_i has an upper bound of the form*:

$$\lambda_i\leq C(v)A^{v/2}\mu_w(M)(\operatorname{diam}M)^{-2-v}i^{2+v}$$

for $i\geq(4\tau^{1/2})^{-1}\operatorname{diam}M$.

Here $C(v)$ is a constant depending only on v.

PROOF. The inequality (2.2) applied to a constant function yields:

$$\mu_w(M)^{1+2/\nu} \le A\tau^{-1}\mu_w(M)^{1+4/\nu}.$$

This proves (i).

Now for a point $x \in M$ and a positive number r, we write ρ for the geodesic distance $d_M(*, x)$ to the point x and we choose a test function as follows:

$$\zeta_{x,r}(y) := \begin{cases} 1 & \text{if } 0 \le \rho(y) \le \dfrac{r}{2} \\ 2 - \dfrac{2}{r}\rho(y) & \text{if } \dfrac{r}{2} \le \rho(y) \le r \\ 0 & \text{if } r \le \rho(y). \end{cases}$$

Then $\zeta_{x,r}$ satisfies

$$|\nabla\zeta_{x,r}| \le \frac{2}{r}; \quad \mathrm{supp}|\nabla\zeta_{x,r}| \subset B(x, r) - B(x, r/2).$$

Hence applying (2.2) to the function $\zeta_{x,r}$, we see that

$$\mu_w(B(x, r/2))^{1+2/\nu} \le A[4r^{-2}\{\mu_w(B(x, r)) - \mu_w(B(x, r/2))\} + \tau^{-1}\mu_w(B(x, r))]\mu_w(B(x, r))^{4/\nu}.$$

This implies that for $r^2 \le \tau$,

$$\mu_w(B(x, r/2))^{1+2/\nu} \le 5Ar^{-2}\mu_w(B(x, r))^{1+4/\nu}.$$

Now we put $V(t) := \mu_w(B(x, t))$, $\alpha := \nu/(\nu+4)$, $\beta := (\nu+2)/(\nu+4)$, and $r_m := r/2^{m-1}$ $(m = 1, 2, \ldots)$. Then it follows that

$$V(r_m) \ge (5A)^{-\alpha} r_m^{2\alpha} V(r_{m+1})^\beta \qquad (m = 1, 2, \ldots)$$

and hence

$$\begin{aligned} \mu_w(B(x, r)) &\ge (5A)^{-\alpha} r_1^{2\alpha} V(r_2)^\beta \ge \cdots \\ &\ge (5A)^{-\alpha(1-\beta^m)/(1-\beta)} r^{2\alpha(1-\beta^m)/(1-\beta)} \prod_{j=1}^{m} 2^{-2\alpha(j-1)\beta^{j-1}} V(r_{m+1})^{\beta^m}. \end{aligned}$$

Since $V(r_{m+1})^{\beta^m}$ tends to 1 as $m \to \infty$, we obtain

$$\mu_w(B(x, r)) \ge C(\nu)A^{-\nu/2}r^\nu$$

for some constant $C(\nu)$ depending only on ν. This proves the second assertion (ii). In particular, in the case when the diameter of M is not greater than $\tau^{1/2}$, we see that

$$\mu_w(M) \ge C(\nu)A^{-\nu/2}(\mathrm{diam}\, M)^\nu.$$

Now we shall consider the case when $\mathrm{diam}\, M > \tau^{1/2}$. We put first $r_0 := \tau^{1/2}$ and also

$n_0 := [\operatorname{diam} M/2r_0] + 1$, where $[x]$ stands for the greatest integer n satisfying $n \leq x$. Let $\gamma : [0, \operatorname{diam}(M)] \to M$ be a distance minimizing curve joining two points $x, y \in M$ such that $d_M(x, y) = \operatorname{diam} M$. Set $x_k := \gamma(2kr_0)$ $(k = 0, 1, \dots, n_0 - 1)$. Then we get a disjoint family of geodesic balls $\{B(x_k, r_0) : k = 0, 1, \dots, n_0 - 1\}$. Therefore we see that

$$\mu_w(M) \geq \sum_{k=0}^{n_0-1} \mu_w(B(x_k, r_0)) \geq n_0 C(v) A^{-v/2} r_0^v .$$

Since $\operatorname{diam} M \leq 2r_0 n_0$, we obtain

$$\operatorname{diam} M \leq 2C(v)^{-1} A^{v/2} \tau^{(1-v)/2} \mu_w(M) .$$

This shows (iii).

It remains to prove (iv), for which we use the Rayleigh principle. Let $x, y \in M$ and $\gamma : [0, \operatorname{diam} M] \to M$ be as above. Given an integer i greater than $\operatorname{diam} M/4\tau^{1/2}$, we have a disjoint family of $(i+1)$ geodesic balls $B(x_j, r)$ and linearly independent test functions $\zeta_{x_j,r}$ defined as above, where we set $x_j := \gamma(2rj)$ $(j = 0, 1, \dots, i)$ and $r := \operatorname{diam} M/2i$. Taking the variational characterization of the i-th eigenvalue λ_i into account, we see that for some $(a_0, \dots, a_i) \in \boldsymbol{R}^{i+1} - \{0\}$,

$$\lambda_i \leq \frac{\mathscr{E}(\phi, \phi)}{\|\phi\|_2^2} \quad \left(\phi := \sum_{j=0}^{i} a_j \zeta_{x_j,r}\right) .$$

Hence we can deduce the last assertion (iv). Indeed, we have

$$\begin{aligned} \frac{\mathscr{E}(\phi, \phi)}{\|\phi\|_2^2} &\leq \frac{4\sum_{j=0}^{i} a_j^2 \mu_w(B(x_j, r))}{r^2 \sum_{j=0}^{i} a_j^2 \mu_w(B(x_j, r/2))} \\ &\leq \frac{4\mu_w(M)}{r^2 C(v) A^{-v/2} (r/2)^v} \qquad \text{(by (ii))} \\ &= C'(v) A^{v/2} (\operatorname{diam} M)^{-(2+v)} \mu_w(M) i^{2+v} \end{aligned}$$

where $C'(v)$ is a constant depending only on v. q.e.d.

It should be mentioned that the main estimate (ii) of Lemma 2.5 is essentially due to Akutagawa [1] and we have adapted the method employed in [1, Theorem 2.1] for proving our estimates (cf. Remarks (ii) below).

As an application of Lemma 2.5 (ii) and (iii), we have the following:

THEOREM 2.6. *Given positive numbers v, a, τ and b, the set of isometry classes of compact Riemannian manifolds $M = (M, g)$, which admit weight functions w such that* (2.1) *with these constants v, a, τ holds for (M, μ_w) and the total measure $\mu_w(M)$ is bounded from above by b, is precompact with respect to the Gromov-Hausdorff distance* HD.

PROOF. In view of Theorem 2.1 and Lemma 2.5 (iii), we see that the diameter of M is bounded from above by a positive constant depending only on the given v, a, τ

and b. Moreover let a positive number r be given and take a family of disjoint geodesic balls $B(x_i, b)$ around points x_i of radius r. Then the number k of such balls is not greater than a constant depending only on the given ones v, a, τ and b. Indeed, it follows from Lemma 2.5 (ii) that

$$\begin{aligned} b \geq \mu_w(M) &\geq \sum_{i=1}^{k} \mu_w(B(x_i, r)) \\ &\geq \sum_{i=1}^{k} \mu_w(B(x_i, r_*)) \quad (r_* := \min\{r, \tau^{1/2}\}) \\ &\geq kC(v)A^{-v/2} r_*^v . \end{aligned}$$

Hence a result due to Gromov [18] shows that the set of isometry classes as above is HD-precompact. q.e.d.

REMARKS. (i) Let v, A, and τ be given and (2.2) hold for a pair (M, μ_w). In the case when $v = \dim M$, it follows from Lemma 2.5 (ii) that the density w must be uniformly bounded away from zero.

$$w(x) \geq C'(n)A^{-n/2} > 0 \quad (n = \dim M)$$

for all $x \in M$, where $C'(n)$ is a positive constant depending only on n.

(ii) Let $N = (N, h)$ be a (not necessarily complete) Riemannian manifold and μ a Radon measure on N. Let A'', R, τ, v and p be given positive constants such that $R^p \leq \tau$ and $v > p$. Suppose that the closure of the geodesic ball $B(x, R)$ around a point $x \in N$ of radius R is compact, and moreover

$$\left(\int_{B(x,R)} |\phi|^{vp/(v-p)} d\mu \right)^{(v-p)/v} \leq A'' \left(\int_{B(x,R)} |d\phi|_h^p d\mu + \tau^{-1} \int_{B(x,R)} |\phi|^p \mu \right)$$

for all $\phi \in C_0^\infty(B(x, R))$. Then we have

$$\mu(B(y, r)) \geq A''^{-v/p} 2^{-v(v+1)/p} r^v$$

for all geodesic balls $B(y, r)$ included in $B(x, R)$ such that

$$\limsup_{\varepsilon \to 0} \frac{\log \mu(B(y, \varepsilon))}{\log \varepsilon} < +\infty .$$

This can be verified by the same argument as in the proof of [1, Theorem 2.1] or Lemma 2.5 (ii). See 9.3 for a related result.

3. Convergence of eigenvalues and eigenfunctions. In this section, we shall study a certain SD-Cauchy sequence and the limit element. In addition, we shall discuss the resolvent convergence in some sense.

3.1. The main result is stated as follows:

THEOREM 3.1. *Let* $\mathscr{S}=\{(M_n, \mu_{w_n}=w_n dv_{M_n}): n=1, 2, \dots)$ *be a sequence of* $\mathscr{M}_{c,w}$ *such that for some positive constants* a, v, τ *and* b, *the heat kernel* $p_{w_n}(t, x, y)$ *of* (M_n, μ_{w_n}) *satisfies* (2.1) *and the total measure* $\mu_{w_n}(M_n)$ *of* M_n *is bounded from above by* b. *Suppose that* $\mathscr{S}$ *is an* SD-*Cauchy sequence and further the diameter of* $M_n=(M_n, d_{M_n})$ *is bounded away from zero uniformly by a positive constant. Then there exist a compact connected metric space* $X=(X, \Theta)$, *a (positive) Radon measure* μ *on* X, *and a nonnegative continuous function* p *on* $(0, \infty)\times X\times X$, *Borel measurable mappings* $F_n: M_n\to X$ *and* $H_n: X\to M_n$, *and a sequense of positive numbers* $\{\varepsilon_n\}$ *converging to zero as* $n\to\infty$, *which satisfy the following properties*:

(i) $p(t, x, y)$ *is the heat kernel of a strongly continuous semigroup* $\{T_t: t>0\}$ *on* $L^2(X, \mu)$ *associated with a regular Dirichlet form on* $L^2(X, \mu)$.

(ii) *The push-forward* $F_{n*}\mu_{w_n}$ *of the measure* μ_{w_n} *by the mapping* F_n *converges, as* $n\to\infty$, *to the measure* μ *with respect to the vague topology.*

(iii) *The total measure* $\mu_{w_n}(M_n)$ *of* μ_{w_n} *and the i-th eigenvalue* $\lambda_{n,i}$ *of* (M_n, μ_{w_n}) *for each i converge respectively, as* $n\to\infty$, *to* $\mu(X)$ *and the i-th eigenvalue* λ_i *of the infinitesimal generator* $\mathscr{L}$ *of* $\{T_t: t>0\}$.

(iv) *The mappings* $F_n: M_n\to X$ *and* $H_n: X\to M_n$ *are* ε_n-*spectral approximations, namely, they satisfy*

$$e^{-(t+1/t)}|p_{w_n}(t, x, y)-p(t, F_n(x), F_n(y))|\le\varepsilon_n\,;$$

$$e^{-(t+1/t)}|p_{w_n}(t, H_n(x'), H_n(y'))-p(t, x', y')|\le\varepsilon_n$$

for all $t>0$, $x, y\in M_n$, *and* $x', y'\in X$, *and also one has*

$$\Theta(F_n\circ H_n(x'), x')\le\varepsilon_n$$

for all $x'\in X$.

(v) *Letting an positive integer i be given, for each eigenfunction u of* (M_n, μ_{w_n}) *with eigenvalue* $\lambda_{n,i}$ *and unit norm in* $L^2(M_n, \mu_{w_n})$, *there is an eigenfunction v of* $\mathscr{L}$ *with eigenvalue* λ_i *and unit norm in* $L^2(X, \mu)$, *such that*

$$\sup_{x\in M_n}|u(x)-v(F_n(x))|\le\varepsilon_{n,i}\,;\quad \sup_{x\in X}|u(H_n(x))-v(x)|\le\varepsilon_{n,i}\,,$$

where $\{\varepsilon_{n,i}\}$ *is a sequence of positive constants tending to zero as* $n\to\infty$.

PROOF. The proof of the theorem will be divided into three steps.

Step 1. We set $\mathscr{S}^*:=\pi^{-1}(\mathscr{S})$, which is an SD*-precompact subset of $\mathscr{F}\mathscr{M}_{c,w}$, and choose a sequence $\alpha_n=(M_n, \mu_{w_n}, \Phi_n=\{u_{n,i}\})$ of $\mathscr{S}^*$. In addition, we put

$$v:=\limsup_{n\to\infty}\mu_{w_n}(M_n)\,;\quad \lambda_i:=\limsup_{n\to\infty}\lambda_{n,i}\qquad (i=1, 2, \dots)\,.$$

Since the diameter of M_n is assumed to be bounded away from zero uniformly by a positive constant, we see by Lemma 2.5 (iv) that λ_i is finite for each i.

Now we shall assume in this step that α_n is an SD*-Cauchy sequence, and further that as $n\to\infty$, $\mu_{w_n}(M_n)$ and $\lambda_{n,i}$ converge to v and λ_i, respectively. We denote by X_n the image of the embedding $F_{\Phi_n}: M_n\to C_\infty([0, \infty), l_2)$. Then by the definition of the distance SD*, X_n converges, as $n\to\infty$, to a compact subspace X of $C_\infty([0, \infty), l_2)$ with respect to the Hausdorff distance δ_H there. Hence we we can define (not necessarily continuous) mappings $f_n: X_n\to X$ and $h_n: X\to X_n$ in such a way that

$$d_\infty(\gamma, f_n(\gamma))\le\delta_n$$

for all $\gamma\in X_n$;

$$d_\infty(\gamma', h_n(\gamma'))\le\delta_n \tag{3.1}$$

for all $\gamma'\in X$, where δ_n is a sequence of positive constants tending to zero as $n\to\infty$. Moreover these mappings have the following properties:

$$|d_\infty(\gamma, \sigma)-d_\infty(f_n(\gamma), f_n(\sigma))|\le 2\delta_n$$

$$|d_\infty(\gamma', \sigma')-d_\infty(h_n(\gamma'), h_n(\sigma'))|\le 2\delta_n$$

$$d_\infty(\gamma, h_n\circ f_n(\gamma))\le 2\delta_n\,; \quad d_\infty(\gamma', f_n\circ h_n(\gamma'))\le 2\delta_n$$

for all $\gamma, \sigma\in X_n$, $\gamma', \sigma'\in X$. In particular, the $2\delta_n$-neighborhood of the image $f_n(X_n)$ covers X and also the $2\delta_n$-neighborhood of the image $h_n(X)$ covers X_n.

Now for an element γ of X, we denote by $\gamma_i(t)$ the i-th component of $\gamma(t)$ in l_2, and set $\bar{h}_n(\gamma):=F_{\Phi_n}^{-1}(h_n(\gamma))$. Then inequality (3.1) reads as follows:

$$\sum_{i=0}^{\infty}(\gamma_i(t)-e^{-(t+1/t)/2}e^{-\lambda_{n,i}t/2}u_{n.i}(\bar{h}_n(\gamma)))^2\le\delta_n^2$$

for all $t>0$ and $\gamma\in X$. In particular, it follows that

$$|\gamma_0(t)-e^{-(t+1/t)/2}\mu_{w_n}(M_n)^{-1/2}|\le\delta_n;$$

$$|\gamma_i(t)-e^{-(t+1/t)/2}e^{-\lambda_{n,i}t/2}u_{n,i}(\bar{h}_n(\gamma))|\le\delta_n$$

for any i and for all $t>0$ and $\gamma\in X$. Now we define $\bar{\gamma}_0(t)$ and $\bar{\gamma}_i(t)$ respectively by the identities

$$\bar{\gamma}_0(t)=e^{(t+1/t)/2}v^{1/2}\gamma_0(t)\,;$$

$$\bar{\gamma}_i(t)=e^{(t+1/t)/2}e^{\lambda_i t/2}\gamma_i(t)\,.$$

Then the last two inequalities are written as

$$e^{-(t+1/t)/2}|v^{-1/2}\bar{\gamma}_0(t)-\mu_{w_n}(M_n)^{-1/2}|\le\delta_n\,;$$

$$e^{-(t+1/t)/2}|e^{-\lambda_i t/2}\bar{\gamma}_i(t)-e^{-\lambda_{n,i}t/2}u_{n,i}(\bar{h}_n(\gamma))|\le\delta_n\,.$$

Therefore in view of Lemma 2.4 (ii), by letting n tend to infinity, we see that $\bar{\gamma}_0\equiv 1$ and

$u_{n,i}(\bar{h}_n(\gamma))$ converges to $\bar{\gamma}_i(t)$. This implies in particular that $\bar{\gamma}_i$ does not depend on the parameter t, and hence we may write $u_{X,i}(\gamma)$ for $\bar{\gamma}_i(t)$. These $u_{X,i}$ $(i=1, 2, \dots)$ are continuous functions on X, which separate the points of X, namely, for any pair (γ, σ) of distinct points of X, $u_{X,i}(\gamma) \neq u_{X,i}(\sigma)$ for some i.

Now if we define a function $p_X(t, \gamma, \sigma)$ on $(0, \infty) \times X \times X$ by

$$p_X(t, \gamma, \sigma) = e^{(t+1/t)} \langle \gamma(t), \sigma(t) \rangle_{l_2},$$

then this can be decomposed as follows:

$$p_X(t, \gamma, \sigma) = \sum_{i=0}^{\infty} e^{-\lambda_i t} u_{X,i}(\gamma) u_{X,i}(\sigma), \tag{3.2}$$

where the convergence holds uniformly on $[s, \infty) \times X \times X$ for each $s > 0$. Moreover the mappings $\bar{f}_n \colon M_n \to X$ and $\bar{h}_n \colon X \to M_n$ defined respectively by $\bar{f}_n = f_n \circ F_{\Phi_n}$ and $\bar{h}_n = F_{\Phi_n}^{-1} \circ h_n$ provide formally spectral approximations between $(M_n, \mu_{w_n}, p_{w_n})$ and (X, p_X). Namely we have

$$e^{-(t+1/t)} | p_{w_n}(t, x, y) - p_X(t, \bar{f}_n(x), \bar{f}_n(y)) | \leq C_1 \delta_n \tag{3.3}$$

for all $t > 0$ and $x, y \in M_n$;

$$e^{-(t+1/t)} | p_{w_n}(t, \bar{h}_n(\gamma), \bar{h}_n(\sigma)) - p_X(t, \gamma, \sigma) | \leq C_1 \delta_n \tag{3.4}$$

for all $t > 0$ and $\gamma, \sigma \in X$, where C_1 is a positive constant depending only on v, a and τ. We recall also that the distance of X, denoted by Θ_X for consistency, is given by

$$\Theta_X(\gamma, \sigma) = \left(\sup_{t>0} e^{-(t+1/t)} (p_X(t, \gamma, \gamma) + p_X(t, \sigma, \sigma) - 2 p_X(t, \gamma, \sigma)) \right)^{1/2} \tag{3.5}$$

and the metric space (M_n, Θ_{M_n, w_n}) converges, as $n \to \infty$, to $X = (X, \Theta_X)$ with respect to the Gromov-Hausdorff distance via the mappings $\bar{f}_n \colon M_n \to X$ and $\bar{h}_n \colon X \to M_n$, namely,

$$\begin{aligned} &| \Theta_{M_n, w_n}(x, y) - \Theta_X(\bar{f}_n(x), \bar{f}_n(y)) | \leq \delta_n ; \\ &| \Theta_{M_n, w_n}(\bar{h}_n(\gamma), \bar{h}_n(\sigma)) - \Theta_X(\gamma, \sigma) | \leq \delta_n \end{aligned} \tag{3.6}$$

for all $x, y \in M_n$ and $\gamma, \sigma \in X$. These mappings $\bar{f}_n$ and $\bar{h}_n$ may be assumed to be Borel measurable.

Step 2. Let $X, p_X, v, \{\lambda_i\}, \{u_{X,i}\}, \bar{f}_n \colon M_n \to X$ and $\bar{h}_n \colon X \to M_n$ be as in Step 1. Then we have a family of Radon measures $\{\bar{f}_{n*} \mu_{w_n}\}$ on X with uniformly bounded total measure, $\bar{f}_{n*} \mu_{w_n}(X) = \mu_{w_n}(M_n) \leq b$. This sequence contains a subsequence which converges to a Radon measure μ on X with respect to the vague topology. For simplicity, we suppose that this is the case for the sequence $\{\bar{f}_{n*} \mu_{w_n}\}$ itself. Then the family of continuous functions $\{u_{X,i}\}$ is orthonormal in $L^2(X, \mu)$:

$$(u_{X,i}, u_{X,j})_\mu = \delta_{ij},$$

where $(\ ,\ )_\mu$ stands for the inner product of $L^2(X, \mu)$.

Now we have a contraction semigroup of symmetric operators $\{T_t^{(\mu)}: t>0\}$ on $L^2(X, \mu)$ with continuous kernel $p_X(t, x, y)$ defined by

$$\begin{aligned} T_t^{(\mu)}\phi(x) &= \int_X p_X(t, x, y)\phi(y)d\mu(y) \\ &= \sum_{i=0}^{\infty} e^{-\lambda_i t}(u_{X,i}, \phi)_\mu u_{X,i}(x), \qquad \phi \in L^2(X, \mu). \end{aligned}$$

The semigroup $\{T_t^{(\mu)}: t>0\}$ is Markovian and conservative:

$$0 \le T_t^{(\mu)}\phi \le 1 \quad \text{if} \quad 0 \le \phi \le 1; \quad T_t^{(\mu)}1 = 1.$$

Each $u_{X,i}$ is an eigenfunction of $T_t^{(\mu)}$ with eigenvalue $e^{-\lambda_i t}$:

$$T_t^{(\mu)}u_{X,i} = e^{-\lambda_i t}u_{X,i}.$$

Associated with the semigroup $\{T_t^{(\mu)}\}$, we have a symmetric closed form $\mathscr{E}^{(\mu)}$ with domain $D[\mathscr{E}^{(\mu)}]$ in $L^2(X, \mu)$ which are given by

$$D[\mathscr{E}^{(\mu)}] = \left\{\phi \in L^2(X, \mu): \lim_{t\to 0} \frac{1}{t}(\phi - T_t^{(\mu)}\phi, \phi)_\mu < +\infty\right\}$$

$$\mathscr{E}^{(\mu)}(\phi, \psi) = \lim_{t\to 0} \frac{1}{t}(\phi - T_t^{(\mu)}\phi, \psi)_\mu, \quad \phi, \psi \in D[\mathscr{E}^{(\mu)}].$$

We observe that for two bounded elements $\phi, \psi \in D[\mathscr{E}^{(\mu)}] \cap L^\infty(X)$,

$$\begin{aligned} &\mathscr{E}^{(\mu)}(\phi\psi, \phi\psi)^{1/2} \\ &= \left(\lim_{t\to 0} \frac{1}{2t} \iint (\phi(y)\psi(y) - \phi(x)\psi(x))^2 p_X(t, x, y)d\mu(y)d\mu(x)\right)^{1/2} \\ &\le \left(\lim_{t\to 0} \frac{1}{2t} \iint (|\phi|_\infty |\psi(y) - \psi(x)| + |\psi|_\infty |\phi(y) - \phi(x)|)^2 p_X(t, x, y)d\mu(x)d\mu(y)\right)^{1/2} \\ &\le |\phi|_\infty \left(\lim_{t\to 0} \frac{1}{2t} \iint |\psi(y) - \psi(x)|^2 p_X(t, x, y)d\mu(x)d\mu(y)\right)^{1/2} \\ &\quad + |\psi|_\infty \left(\lim_{t\to 0} \frac{1}{2t} \iint |\phi(y) - \phi(x)|^2 p_X(t, x, y)d\mu(x)d\mu(y)\right)^{1/2} \\ &= |\phi|_\infty \mathscr{E}(\psi, \psi)^{1/2} + |\psi|_\infty \mathscr{E}(\phi, \phi)^{1/2} \end{aligned}$$

(see e.g., [15, Theorem 1.4.2 (ii)]). It follows that the subalgebra $\mathscr{C}(\{u_{X,i}\})$ generated by the family $\{u_{X,i}\}$ in $C(X)$ is contained in $D[\mathscr{E}^{(\mu)}]$ and also dense there with respect to the norm $\mathscr{E}^{(\mu)}(\ ,\ ) + (\ ,\ )_\mu$. Moreover since $\mathscr{C}(\{u_{X,i}\})$ separates points of X, we can deduce from the Stone-Weierstrass theorem that $\mathscr{C}(\{u_{X,i}\})$ is also dense in $C(X)$ with respect to the C^0 norm. In particular, $\mathscr{C}(\{u_{X,i}\})$ is dense in $L^2(X, \mu)$. Thus $\mathscr{C}(\{u_{X,i}\})$ is a core of the form $\mathscr{E}^{(\mu)}$ and hence $\mathscr{E}^{(\mu)}$ is a regular Dirichlet form on $L^2(X, \mu)$ (see [15, Chap. 1]). This implies that the corresponding semigroup $\{T_t^{(\mu)}: t>0\}$ on $L^2(X, \mu)$ is strongly continuous. As a result, $\{u_{X,i}: i=0, 1, \dots\}$ is a complete orthonormal system of $L^2(X, \mu)$, and a limit measure μ is in fact unique. Thus the push-forward $\bar{f}_{n*}\mu_{w_n}$ of μ_{w_n} converges to μ as $n\to\infty$ in the vague topology.

Step 3. In Steps 1 and 2, we have assumed that the given sequence $\{\alpha_n\}$ in $\mathscr{F}\mathscr{M}_{c,w}$ is SD*-Cauchy, and also that the total measure $\mu_{w_n}(M_n)$ and the i-th eigenvalue $\lambda_{n,i}$ of (M_n, μ_{w_n}) converge as $n\to\infty$. But as seen from the above discussions, the latter assumption is a consequence of the former.

Now suppose that we have two SD*-Cauchy subsequence, say, $\{\alpha_{n'}\}$ and $\{\alpha_{n''}\}$, of $\{\alpha_n\}$. Then for the sequence $\{\alpha_{n'}\}$, we have a compact connected metric space $X'=(X', \Theta_{X'})$, a (positive) Radon measure μ', a nonnegative continuous function $p_{X'}(t, x', y')$ on $(0, \infty)\times X'\times X'$, a divergent sequence of nonnegative numbers $\{\lambda_i'\}$, a sequence of continuous functions $\{u_{X',i}\}$ on X', a sequence of positive numbers $\{\delta_{n'}\}$ with $\lim_{n'\to\infty}\delta_{n'}=0$, and Borel measurable mappings $\bar{f}_{n'}: M_{n'}\to X'$ and $\bar{h}_{n'}: X'\to M_{n'}$, which satisfy the assertions (i) through (iv) in the theorem. For the other $\{\alpha_{n''}\}$, we have correspondingly X'', μ'', $p_{X''}$, $\{\lambda_i''\}$, $\{u_{X'',i}\}$, $\{\delta_{n''}\}$, $\bar{f}_{n''}: M_{n''}\to X''$ and $\bar{h}_{n''}: X''\to M_{n''}$. Since the given sequence $\{\alpha_n\}$ is SD-Cauchy, we have $\delta_{n'n''}$-spectral approximations $\xi_{n'n''}: (M_{n'}, \mu_{w_{n'}})\to(M_{n''}, \mu_{w_{n''}})$ with $\delta_{n'n''}$ tending to zero as $n', n''\to\infty$. Using these mappings, we get a family of mappings between X' and X'' as follows:

$$F_{n'n''}:=\bar{f}_{n''}\circ\xi_{n'n''}\circ\bar{h}_{n'}: X'\to X'' ,$$

$$H_{n'n''}:=\bar{f}_{n'}\circ\xi_{n''n'}\circ\bar{h}_{n''}: X''\to X' .$$

Then these mappings give spectral approximations between $(X', \mu', p_{X'})$ and $(X'', \mu'', p_{X''})$. To be precise, these mappings satisfy

$$e^{-(t+1/t)}|p_{X'}(t, x', y')-p_{X''}(t, F_{n'n''}(x'), F_{n'n''}(y'))|\le C_1\delta_{n'}+C_1\delta_{n''}+\delta_{n'n''}$$

for all $t>0$ and $x', y'\in X'$;

$$e^{-(t+1/t)}|p_{X''}(t, x'', y'')-p_{X'}(t, H_{n'n''}(x''), H_{n'n''}(y''))|\le C_1\delta_{n'}+C_1\delta_{n''}+\delta_{n''n'}$$

for all $t>0$ and $x'', y''\in X''$.

Now taking subsequences of $\{F_{n'n''}\}$ and $\{H_{n'n''}\}$ respectively if necessarily, we may assume that they converge to mappings $F: X'\to X''$ and $H: X''\to X'$, respectively, which preserve the functions $p_{X'}$ and $p_{X''}$:

$$p_{X'}(t, x', y')=p_{X''}(t, F(x'), F(y')) \tag{3.7}$$

for all $t>0$ and $x', y'\in X'$;

$$p_{X''}(t, x'', y'')=p_{X'}(t, H(x''), H(y''))$$

for all $t>0$ and $x'', y''\in X''$. In particular, these mappings preserve the distances and hence they give isometries between X' and X''.

Now we arrange the nondecreasing sequence $\{\lambda_i'\}$ so that we have a strictly increasing sequence, say $\{\lambda_i^{*\prime}\}$, together with a sequence $\{m_i'\}$ of multiplicities. We note that m_0' may be greater than one, though it is bounded from above by a constant depending only on the given constants a, v, b and τ (cf. Lemma 2.4 (i)). Similarly we obtain such sequences, say $\{\lambda_i^{*\prime\prime}\}$ and $\{m_i''\}$. Then (3.7) reads

$$\sum_{i=0}^{\infty} e^{-\lambda_i^{*\prime}t} \sum_{j=1}^{m_i'} u_{i_j}'(x)u_{i_j}'(y) = \sum_{i=0}^{\infty} e^{-\lambda_i^{*\prime\prime}t} \sum_{k=1}^{m_i''} F^*u_{i_k}''(x)F^*u_{i_k}''(y) \tag{3.8}$$

for all $t>0$ and $x, y \in X'$.

Now we claim that $\lambda_i^{*\prime}=\lambda_i^{*\prime\prime}$ and $m_i'=m_i''$ for all $i=0, 1, 2, \ldots$, and further the functions u_{i_j}' $(j=1, \ldots, m_i')$ span the same vector space as the functions $F^*u_{i_k}''$ $(k=1, \ldots, m_i'')$. To see this, let us put $v_i=F^*u_{X'',i}$ and $\nu=(F^{-1})_*\mu''$ for simplicity. Then $\{u_{X',i}\}$ (resp. $\{v_i\}$) is a complete orthonormal system of $L^2(X', \mu')$ (resp. $L^2(X', \nu)$). Now for any j, multiplying $u_{X',j}(y)$ by the both sides of (3.8), and integrating them on X' with respect to the measure $d\mu'(y)$, we obtain

$$e^{-\lambda_j't}u_{X',j}(x) = \sum_{i=0}^{\infty} e^{-\lambda_i^{*\prime\prime}t} \sum_{k=1}^{m_i''} v_{i_k}(x) \int_{X'} v_{i_k}(y)u_{X',j}(y)d\mu'(y) .$$

Since this holds for all $t>0$, we see that

$$\sum_{k=1}^{m_i''} v_{i_k}(x) \int_{X'} v_{i_k}(y)u_{X',j}(y)d\mu'(y) = \begin{cases} 0 , & (\lambda_i^{*\prime\prime} \neq \lambda_j') \\ u_{X',j}(x) , & (\lambda_i^{*\prime\prime} = \lambda_j') . \end{cases} \tag{3.9}$$

This shows that if $\lambda_i^{*\prime\prime} \neq \lambda_j'$ for all i, then $u_{X',j}$ must vanish identically. This is a contradiction, because $u_{X',j}$ has unit norm in $L^2(X', \mu')$. Thus we see that $\{\lambda_i^{*\prime\prime}\} \subset \{\lambda_i^{*\prime}\}$. By the same way, we can deduce that $\{\lambda_i^{*\prime}\} \subset \{\lambda_i^{*\prime\prime}\}$, and hence these sets coincide. Moreover the argument above proves that $m_i'=m_i''$ for all i and the functions u_{i_k}' $(k=1, \ldots, m_i')$ span the same vector space as the functions v_{i_k} $(k=1, \ldots, m_i'')$. Thus our claim is verified. In addition, examining the case $j=0$, we easily see that $u_{X',0}=u_{X'',0}$, namely, $\mu'(X')=\nu(X')=\mu''(X'')$.

Finally multiplying the both sides of (3.9) by $v_i(x)$ and integrating them with respect to the measure ν, we obtain

$$\int_{X'} v_i(x)u_{X',j}(x)d\mu'(x) = \int_{X'} v_i(x)u_{X',j}(x)d\nu(x) .$$

This shows that $\{v_i\}$ is also a complete orthonormal system of $L^2(X', \mu')$ and hence these measures μ' and ν are actually identical. Thus the mapping F: $X' \to X''$ preserves the measures μ' and μ'' on X' and X'' respectively and also the heat kernels $p_{X'}$ and $p_{X''}$ of the symmetric Markov semigroups $\{T_t^{(\mu')}\}$ and $\{T_t^{(\mu'')}\}$ respectively. Moreover we can interpret the mapping F as the spectral embedding of $(X', \mu', p_{X'})$ with respect to a complete orthonormal basis $\{v_i\}$ of $L^2(X', \mu')$ with v_i being eigenfunction of the generator of the semigroup $\{T_t^{(\mu')}: t>0\}$, and also $(X', \mu', p_{X'}, \{v_i\})$ as the boundary element of $\mathscr{F}\mathscr{M}_{c,w}$ to which $\alpha''=(M_{n''}, \mu_{w_{n''}}, \{u_{n''}\})$ converges as $n''\to\infty$. We note that the basis $\{v_i\}$ itself may depend on the choice of a limit mapping F: $X'\to X''$ of $\{F_{n'n''}\}$, but as a boundary element of $\mathscr{F}\mathscr{M}_{c,w}$, $(X', \mu', p_{X'}, \{v_i\})$ is uniquely determined, because of the definition of $\mathscr{F}\mathscr{M}_{c,w}$.

Thus we have found the limit elements, say (X, μ, p), of the given SD Cauchy sequence $\mathscr{S}$ and mappings F_n: $M_n\to X$, H_n: $X\to M_n$ having the properties (i) through (iv) described in the theorem. It is easy to verify the last assertion (v). q.e.d.

REMARKS. (i) As seen from the proof of Theorem 3.1, the theorem certainly holds without the assumption that the diameter of M_n is bounded away from zero uniformly by a positive constant. But if we drop this assumption, it may occur the case that as $n\to\infty$, the i-th eigenvalue $\lambda_{n,i}$ of (M_n, μ_{w_n}) diverges for some i (and hence for all $j\geq i$); as a consequence of Lemma 2.4 (iv), the Hilbert space $L^2(X, \mu)$ is of finite dimension. For example, this is the case when the limit space X itself consists of a single point.

(ii) In Theorem 3.1, the limit measure μ is not necessarily supported on the whole of X and the trivial eigenvalue $\lambda_0=0$ may be of multiplicity greater than one. See Sections 5, 7 and 8 for such examples.

3.2. Let us observe that in Theorem 3.1, the resolvent of $\mathscr{L}_{w_n}$ converges to that of the generator $\mathscr{L}$ of the semigroup T_t as $n\to\infty$ in some sense. To be precise, given $\sigma>0$, we define a bounded linear mapping $\mathscr{R}^*_{n,\sigma}$ of the space of continuous functions $C(X)$ into that of bounded measurable function $L^\infty(X)$ by

$$\mathscr{R}^*_{n,\sigma}\phi=H_n^*\circ(\mathscr{L}_{w_n}+\sigma I)^{-1}\circ F_n^*(\phi)\,, \qquad \phi\in C(X)\,,$$

where H_n and F_n are (Borel) measurable mappings as in Theorem 3.1. Then we have the following

COROLLARY 3.2. *Let $\mathscr{R}^*_{n,\sigma}$: $C(X)\to L^\infty(X)$ be a bounded linear operator defined as above. Then for each $\phi\in C(X)$ and $x\in X$,*

$$(\mathscr{L}_X+\sigma I)^{-1}\phi(x)=\lim_{n\to\infty}\mathscr{R}^*_{n,\sigma}\phi(x)\,.$$

PROOF. Let $\{T_t^{(n)}: t>0\}$ be the semigroup of (M_n, μ_{w_n}) in $L^2(M_n, \mu_{w_n})$ with kernel p_{w_n}. The resolvent $(\mathscr{L}_{w_n}+\sigma I)^{-1}$ is given by

$$(\mathscr{L}_{w_n}+\sigma I)^{-1}=\int_0^\infty e^{-\sigma t}T_t^{(n)}dt\,.$$

Since $T_t^{(n)}$ defines a contraction semigroup on $L^\infty(X)$, we have

$$e^{-\sigma t}H_n^*\circ T_t^{(n)}\circ F_n^*(\phi)\leq e^{-\sigma t}|\phi|_\infty\,.$$

Hence it suffices to show that

$$\lim_{n\to\infty} H_n^* \circ T_t^{(n)} \circ F_n^* \phi(x) = T_t \phi(x) .$$

This follows from (iv) of Theorem 3.1. Indeed,

$$\begin{aligned}
&| H_n^* \circ T_t^{(n)} \circ F_n^* \phi(x) - T_t \phi(x) | \\
&= \left| \int_{M_n} p_{w_n}(t, H_n(x), a)\phi(F_n(a))d\mu_{w_n}(a) - \int_X p(t, x, y)\phi(y)d\mu(y) \right| \\
&\le \left| \int_{M_n} (p_{w_n}(t, H_n(x), a) - p(t, F_n \circ H_n(x), F_n(a)))\phi(F_n(a))d\mu_{w_n}(a) \right| \\
&\quad + \left| \int_{M_n} (p(t, F_n \circ H_n(x), F_n(a)) - p(t, x, F_n(a)))\phi(F_n(a))d\mu_{w_n}(a) \right| \\
&\quad + \left| \int_X p(t, x, y)\phi(y)dF_{n*}\mu_{w_n}(y) - \int_X p(t, x, y)\phi(y)d\mu(y) \right| .
\end{aligned}$$

The first two terms of the right side are bounded from above by

$$e^{(t+1/t)}\varepsilon_n' | \phi |_\infty \mu_{w_n}(M_n)$$

because of (iv) of Theorem 3.1, where $\{\varepsilon_n'\}$ is a sequense of positive constants tending to zero as $n\to\infty$. Since $F_{n*}\mu_{w_n}$ converges to μ with respect to the weak* topology, the last term of the right side also tends to zero as $n\to\infty$. q.e.d.

3.3. Before concluding this section, we consider a Riemannian manifold $M=(M, g)$ which is not necessarily compact nor complete. Let $\mu_w = w dv_g$ be a measure with smooth density $w>0$. The energy form $\mathscr{E}$ is defined on the space of smooth functions compactly supported, $C_0^\infty(M)$, by

$$\mathscr{E}(u, v) = \int_M \langle du, dv \rangle_g d\mu_w , \qquad u, v \in C_0^\infty(M) .$$

This form $\mathscr{E}$ is closable and the domain of its smallest closed extension, denoted by the same letter $\mathscr{E}$, is the Sobolev space $H_0^1(M, \mu_w)$, i.e., the completion of $C_0^\infty(M)$ with respect to the norm $\mathscr{E}_1(u, u) = (u, u)_{\mu_w} + \mathscr{E}(u, u)$ (see, e.g., [15]). Let $\{T_t = e^{-t\mathscr{L}} : t>0\}$ be the strongly continuous Markovian semigroup on $L^2(M, \mu_w)$ associated with $\mathscr{E}$, which has a kernel $p_w(t, x, y)$, called the minimal heat kernel of (M, μ_w), and whose generator $\mathscr{L}$ is the Friedrichs extension of the elliptic differential operator $-w^{-1}\mathrm{div}(w \,\mathrm{grad} *)$ acting on $C_0^\infty(M)$. In what follows, we assume that

$$\mu_w(M) < +\infty , \quad \text{i.e.,} \quad 1 \in L^2(M, \mu_w) . \tag{3.10}$$

Then $0 < T_t 1 \le 1$ and $T_t 1 \in H_0^1(M, \mu_w)$. It is not hard to see the following

ASSERTION 3.3. *Under the condition* (3.10), *the following two conditions are equivalent*:

(i) $T_t 1 = 1$, *i.e.*, $\int_M p(t, x, y) d\mu_w(y) = 1$;

(ii) $1 \in H_0^1(M, \mu_w)$, *i.e.*, *there is a sequence of functions* $\{\rho_i\}$ *in* $C_0^\infty(M)$ *such that*

$$\lim_{i \to \infty} \mathscr{E}_1(\rho_i - 1, \rho_i - 1) = 0 .$$

(*In this case,* $\mathscr{L}$ *is a unique self-adjoint extension of the operator* $-w^{-1}\mathrm{div}(w \,\mathrm{grad} *)$ *acting on* $C_0^\infty(M)$.)

We observe that Theorem 2.1 is valid for (M, μ_w), if we replace the space $C^\infty(M)$ with $C_0^\infty(M)$ in (2.2) and (2.3). Suppose, in addition to (3.10), that the minimal heat kernel $p_w(t, x, y)$ satisfies (2.1). Then the spectrum of $\mathscr{L}$ is discrete and the eigenfunction decomposition for the heat kernel $p_w(t, x, y)$ holds (see e.g., [13, Chap. 2]). In particular, in this case, Lemma 2.4 remains true for (M, μ_w) and we have a spectral embedding of (M, μ_w) and hence we can find a compact metric space $\tilde{M}$ which includes M as an open dense subset in such a way that the heat kernel $p_w(t, x, y)$ continuously extends to $(0, \infty) \times \tilde{M} \times \tilde{M}$ and so does any eigenfunction to $\tilde{M}$. In addition, the second assertion (ii) of Lemma 2.5 certainly hold, if we replace $\tau^{1/2}$ there with $\min\{\tau^{1/2}, \mathrm{in.rad}(x)\}$. Here $\mathrm{in.rad}(x)$ stands for the inscribed radius of a point $x \in M$, which is by definition the least upper bound of positive numbers r such that the geodesic ball $B(x, r)$ around x with radius r is relatively compact in M. The assertions (iii) and (iv) of Lemma 2.5 also hold if we use the inscribed radius of M, $\mathrm{in.rad}\, M = \sup_{x \in M} \mathrm{in.rad}(x)$, instead of the diameter of M. When $1 \in H_0^1(M, \mu_w)$, the first assertion (i) of the lemma is true. Thus we are able to derive similar results to Theorems 2.3 and 3.1 for a family of pairs (M, μ_w) as above.

Now we shall close this section with the following

THEOREM 3.4 (Li-Tian [28]). *Let M be the regular points of an n-dimensional algebraic subvariety in a complex projective space* $\boldsymbol{CP}^{n+l}$. *Then the restriction of the standard Fubini-Study metric of* $\boldsymbol{CP}^{n+l}$ *to M gives a smooth metric g called the Bergman metric of M and the heat kernel $p_M(t, x, y)$ enjoys the following properties*:

$$\int_M p_M(t, x, y) dv_g(y) = 1 ;$$

$$p_M(t, x, y) \le \bar{p}(t, r(x, y)) ,$$

where we denote $\bar{p}(t, r(\bar{x}, \bar{y})) = \bar{p}(t, \bar{x}, \bar{y})$ *to be the rotationally symmetric heat kernel on the standard* $\boldsymbol{CP}^n$.

4. Uniform continuity of heat kernels and limit spaces. In the preceding section, as the limit for a certain SD-Cauchy sequence, we have obtained a triad of a compact connected metric spaces X, a positive Radon measure μ on X, and a nonnegative continuous function p on $(0, \infty)\times X\times X$ such that p is the heat kernel of a symmetric Markov semigroup $\{T_t: t>0\}$ on $L^2(X, \mu)$. The purpose of this section is to study further the boundary elements of an SD-precompact family under some conditions discussed in Saloff-Coste [33] and show that the semigroups of the boundary elements satisfy the Feller condition and in fact the Lindeberg type condition. The main result of this section is given in Theorem 4.1.

4.1. Let $(M, \mu_w = w dv_g)$ be a pair of $\mathscr{M}_{c,w}$, and let positve constants r_0, η_1 and η_2 be given. We shall assume that (M, μ_w) satisfies the following properties discussed in [33]:

$$\mu_w(B(x, 2r)) \leq \eta_1 \mu_w(B(x, r)) \tag{4.1}$$

for all $x\in M$ and $0<r\leq r_0$;

$$\int_{B(x,r)} |\phi-\phi_{x,r}|^2 d\mu_w \leq \eta_2 r^2 \int_{B(x,2r)} |d\phi|_g^2 d\mu_w \tag{4.2}$$

for all $x\in M$, $0<r\leq r_0$, and $\phi\in C^\infty(M)$, where $\phi_{x,r}$ stands for the average of ϕ over the geodesic ball $B(x, r)$ around x with radius r:

$$\phi_{x,r} = \frac{1}{\mu_w(B(x, r))} \int_{B(x,r)} \phi \, d\mu_w \, .$$

According to [33], (4.1) and (4.2) imply a family of Sobolev inequalities on geodesic balls. To be precise, there exists constants $\nu>2$ and $C_1>0$ depending only on η_1 and η_2 such that

$$\|\phi\|^2_{2\nu/(\nu-2)} \leq C_1 \mu_w(B(x, r))^{-2/\nu} r^2 (\mathscr{E}(\phi, \phi) + r^{-2}\|\phi\|_2^2)\,, \qquad \phi\in C_0^\infty(B(x, r))\,, \tag{4.3}$$

for all $x\in M$ and $0<r\leq r_0$. Moreover it is shown in [33] that a parabolic Harnack inequality is equivalent to the properties (4.1) and (4.2), and as a corollary, the Hölder continuity of solutions of parabolic equation $(\partial/\partial t + \mathscr{L}_w)u = 0$ is shown. In fact, applying Theorem 4.1 in [33] to the heat kernel p_w of (M, μ_w) will yield the following estimate:

$$|p_w(t, x, x) - p_w(t, x, y)| \leq C_2 \frac{d_M(x, y)^\alpha}{t^{\nu/2}} \sup\{p_w(t/2, x, z) : z\in B(x, \sqrt{t})\}$$

for all $x\in M, y\in B(x, \sqrt{t})$ and $0<t<r_0^2$, where $\alpha\in(0, 1)$ and $C_2>0$ are constants depending only on η_1 and η_2. This implies that

$$\begin{aligned} &e^{-(t+1/t)}|p_w(t, x, x) - p_w(t, x, y)| \\ &\quad \leq C_3 d_M(x, y)^\alpha \sup\{e^{-(s+1/s)} s^{-\nu/2} p_w(s/2, z, z) : 0<s<\infty, z\in M\} \end{aligned} \tag{4.4}$$

for all $x, y\in M$, where $C_3>0$ is a constant depending only on η_1, η_2 and r_0.

Now suppose further that (M, μ_w) satisfies

(4.5) $$\mu_w(B(x, r_0)) \geq \eta_3$$

for some $\eta_3 > 0$ and all $x \in M$. Then in view of (4.3) and Theorem 2.2, we can derive a diagonal estimate for p_w:

(4.6) $$p_w(t, x, x) \leq \frac{a}{t^{\nu/2}}$$

for all $x \in M$, $0 < t \leq r_0^2$, where $a > 0$ is a constant depending only on η_k $(k = 1, 2, 3)$. Hence by (4.4), we have

$$e^{-(t+1/t)} |p_w(t, x, x) - p_w(t, x, y)| \leq \frac{K}{2} d_M(x, y)^\alpha$$

for all $x, y \in M$ and $t > 0$, where $K > 0$ is a constant depending only on r_0 and η_k $(k = 1, 2, 3)$. In other words, it holds that

(4.7) $$\Theta_{M,w}(x, y) \leq K^{1/2} d_M(x, y)^{\alpha/2}$$

for all $x, y \in M$.

4.2. Let $\mathscr{S} = \{(M_n, \mu_{w_n} = w_n dv_{M_n}) : n = 1, 2, \dots\}$ be a sequence in $\mathscr{M}_{c,w}$ such that for some positive constants r_0, η_k $(k = 1, 2, 3)$, (4.1), (4.2) and (4.5) hold uniformly for $\mathscr{S}$, and further the total measure $\mu_{w_n}(M_n)$ is also bounded from above uniformly by a constant b. Then as we have seen, $\mathscr{S}$ is precompact with respect to the spectral distance SD. In this case, we note that the (Riemannian) diameter of M_n tends to zero as $n \to \infty$ if and only if the first nonzero eigenvalue $\lambda_{n,1}$ of (M_n, μ_{w_n}) diverges to infinity as $n \to \infty$; the limit element (X, μ, p) of $\mathscr{S}$ is in this case trivial, $X = \{a\ point\}$. In what follows, we assume that the diameter of M_n is bounded away from zero uniformly, and the sequence $\mathscr{S}$ itself is an SD-Cauchy sequense. Let (X, μ, p) be the limit triad of $\mathscr{S}$ described in Theorem 3.1, and let Θ be a distance on X defined by (3.5). Then we first claim that the limit measure μ is supported on the whole of X, namely, $\operatorname{supp} \mu = X$. Indeed, if we denote by $D_n(x, r)$ (resp., $B_n(x, r)$) the metric ball of the metric space (M_n, Θ_{M_n,w_n}) (resp., the geodesic ball of $M_n = (M_n, d_{M_n})$) around a point $x \in M_n$ with radius r, we have by (4.7)

$$D_n(x, r) \supseteq B_n(x, K^{-1/\alpha} r^{2/\alpha})$$

for $0 < r \leq r_0$ and for all $x \in M_n$, where K and α are positive constants depending only on the given constants r_0 and η_k $(k = 1, 2, 3)$. This implies

$$\mu_{w_n}(D_n(x, r)) \geq \mu_{w_n}(B_n(x, K^{-1/\alpha} r^{2/\alpha}))$$

and hence from Lemma 2.5 (ii)

$$\mu_{w_n}(D_n(x, r)) \geq C_4 r^{2\nu/\alpha}$$

for all $x \in M_n$ and $0 < r \leq r_0$, where C_4 is a positive constant depending only on the given constants r_0 and η_k $(k=1, 2, 3)$. Since the metric space $(M_n, \Theta_{M_n,p_{w_n}})$ converges to (X, Θ) with respect to the Gromov-Hausdorff distance (cf. (3.6)), the same inequality holds for X, namely, the metric ball of (X, Θ) around a point x of radius r satisfies

$$\mu(D(x, r)) \geq C_4 r^{2\nu/\alpha}$$

for all $r \in (0, r_0]$. This shows that μ is supported on the whole of X and the claim is verified.

Now for each $t>0$, the bounded operator T_t on $L^2(X, \mu)$ defined by

$$T_t\phi(x) = \int_X p(t, x, y)\phi(y)d\mu(y), \qquad \phi \in L^2(X, \mu)$$

acts on the Banach space $C(X)$ with the uniform convergence topology and $\{T_t : t>0\}$ is strongly continuous on $C(X)$. Indeed, applying the off-diagonal estimate (2.6) to the heat kernel p_{w_n} and using (4.7), we have

$$p_{w_n}(t, x, y) \leq \frac{C_5}{t^{1+\nu}} \exp\left(-\frac{K^{-2/\alpha}\Theta_{M_n,w_n}(x, y)^{4/\alpha}}{4t}\right)$$

for all $x, y \in X$ and $0<t\leq r_0^2$, where C_5 is a positive constant depending only on r_0 and η_k $(k=1, 2, 3)$. Hence taking (3.6) into account and passing through the limit as $n\to\infty$, we obtain

$$p(t, x, y) \leq \frac{C_5}{t^{1+\nu}} \exp\left(-\frac{K^{-2/\alpha}\Theta(x, y)^{4/\alpha}}{4t}\right)$$

for all $x, y \in X$ and $0<t\leq r_0^2$. Then it is easy to see that $\{T_t : t>0\}$ is strongly continuous on $C(X)$, and also it possesses the property that

$$\lim_{t\to 0} \sup_{x\in X} \frac{1}{t} T_t(\chi_{X-D(x;r)})(x) = 0$$

for any $r>0$, where χ_B stands for the characteristic function of a subset B of X.

What we have observed is summarized in the following

THEOREM 4.1. *Let $\mathscr{S} = \{(M_n, \mu_{w_n} = w_n dv_{M_n}) : n=1, 2, \ldots\}$ be an* SD-*Cauchy sequence in $\mathscr{M}_{c,w}$ satisfying* (4.1), (4.2) *and* (4.5) *uniformly for some positive constants. Let (X, μ, p) be the limit of $\mathscr{S}$ as in Theorem* 3.1. *Then in addition to the properties* (i) *through* (v) *in Theorem* 3.1, *the following holds*:

(ii) *The semigroup $\{T_t : t>0\}$ with kernel p on the Banach space $C(X)$ with the uniform norm is strongly continuous, namely, for any continuous function $u \in C^0(X)$,*

$$\lim_{t\to 0} \| T_t u - u\|_{C^0} = \lim_{t\to 0} \sup_{x\in X} \left| \int_X p(t, x, y)u(y)d\mu_X(y) - u(x) \right| = 0 .$$

Moreover it possesses the property that

$$\limsup_{t\to 0}\ \sup_{x\in X} \frac{1}{t} T_t(\chi_{X-D(x;r)})(x)=0$$

for any $r>0$.

We notice that in the discussions above, the condition (4.4) is essential. In fact, Theorem 4.1 remains true for an SD-Cauchy sequence as in Theorem 2.3 satisfying further (4.4) uniformly.

4.3. Now we shall discuss geometric conditions for (4.1), (4.2) and (4.5), or (4.4) and (4.6). Given a pair $(M, \mu_w = w dv_g)$ and a positive integer k, a symmetric tensor $R_{w,k}$ on M is defined by

$$R_{w,k} = \mathrm{Ric}_M - \frac{1}{k} d\log w \otimes d\log w - Dd\log w\,,$$

where Ric_M stands for the Ricci tensor of M. For $k=0$, we set $R_{w,0}=\mathrm{Ric}_M$; in this case, w is always assumed to be a constant.

Following [25], we consider first the case where (M, μ_w) satisfies

$$R_{w,k} \geq -(m-1)\kappa^2 \quad (m = \dim M)\,; \tag{4.8}$$

$$\operatorname{diam} M \leq D \tag{4.9}$$

for some constants $\kappa>0$ and $D>0$, and further

$$\mu_w(M)=1\,. \tag{4.10}$$

Then (4.1), (4.2) and (4.5) certainly hold with constants $r_0=D$, some $\eta_1=\eta_1(m+k, \kappa, D)$ depending only on the quantities in the parenthesis, $\eta_2=\exp(1+D\kappa)$ and also $\eta_3=1$ (see [ibid., Propositions 2.1 and 2.6]). Thus if we denote by $\mathcal{M}_w^*(m, k, \kappa, D)$ the set of equivalence classes of pairs (M, μ_w) satisfying (4.8), (4.9) and (4.10), then we have the following

THEOREM 4.2 ([25]). (i) *$\mathcal{M}_w^*(m, k, \kappa, D)$ is precompact with respect to the spectral distance* SD, *and the assertions of Theorem* 4.1 *hold for an* SD-*Cauchy sequence in this class* (*see* [*ibid., Theorems* 3.6, 4.4, 4.5 *and* 5.1]).

(ii) *If a sequence $\mathcal{S}=\{(M_n, \mu_{w_n})\}$ in $\mathcal{M}_w^*(m, k, \kappa, D)$ converges to a boundary element (X, μ, p), then the metric space (M_n, d_{M_n}) also converges to X endowed with another distance d_X with respect to the Gromov-Hausdorff distance* HD, *and moreover one has*

$$\lim_{t\to 0} 4t \log p(t, x, y) = -d_X(x, y)^2$$

for all $x, y\in X$ (*see* [*ibid., Theorems* 3.5, 3.8]).

The first assertion (i) of this theorem will be valid for larger classes. Let $\Lambda>1$ be given further and consider the set of equivalence classes of pairs (M, g, μ_w), denoted by

$\mathcal{M}_w^*(m, k, \kappa, D; \Lambda)$, which admit Riemannian metrics h and positive smooth functions v such that $(M, h, \mu_v) \in \mathcal{M}_w^*(m, k, \kappa, D)$, $\Lambda^{-1}h \leq g \leq \Lambda h$ and $\Lambda^{-1}v \leq w \leq \Lambda v$. Then (4.3) and (4.5) obviously hold with appropriate constants. Moreover by virtue of a result of [32], (4.4) is certainly satisfied. Thus we have the following

THEOREM 4.3. *$\mathcal{M}_w^*(m, k, \kappa, D; \Lambda)$ is precompact with respect to the spectral distance* SD, *and the assertions of Theorem* 4.1 *hold for an* SD-*Cauchy sequence in this class.*

However in this case, the second assertion of Theorem 4.2 is not true in general, as shown in the following simple

EXAMPLE. Let $\{g_n : n=1, 2, \dots\}$ be a sequence of metrics on the product $\boldsymbol{R}/\boldsymbol{Z} \times \boldsymbol{R}/\boldsymbol{Z}$ given by

$$g_n = dt^2 + a_n(t)^2 d\theta^2, \qquad (t, \theta) \in \boldsymbol{R}/\boldsymbol{Z} \times \boldsymbol{R}/\boldsymbol{Z},$$

with

$$a_{2n}(t) = \begin{cases} 2, & \text{for } 0 \leq t \leq \frac{1}{2} - \frac{1}{n}, \text{ or } \frac{1}{2} + \frac{1}{n} \leq t \leq 1 \\ nt + 3 - \frac{n}{2}, & \text{for } \frac{1}{2} - \frac{1}{n} \leq t \leq \frac{1}{2} \\ -nt + 3 + \frac{n}{2}, & \text{for } \frac{1}{2} \leq t \leq \frac{1}{2} + \frac{1}{n} \end{cases}$$

$$a_{2n-1}(t) = \begin{cases} 2, & \text{for } 0 \leq t \leq \frac{1}{2} - \frac{1}{n}, \text{ or } \frac{1}{2} + \frac{1}{n} \leq t \leq 1 \\ -nt + 1 + \frac{n}{2}, & \text{for } \frac{1}{2} - \frac{1}{n} \leq t \leq \frac{1}{2} \\ nt + 1 - \frac{n}{2}, & \text{for } \frac{1}{2} \leq t \leq \frac{1}{2} + \frac{1}{n}. \end{cases}$$

Then the sequence of Riemannian manifolds, $\{(\boldsymbol{R}/\boldsymbol{Z} \times \boldsymbol{R}/\boldsymbol{Z}, g_n)\}$, converges to the Riemannian product $\boldsymbol{R}/\boldsymbol{Z} \times \boldsymbol{R}/\boldsymbol{Z}$ with respect to the spectral distance. However this does not hold with respect to the Gromov-Hausdorff distance. Indeed, $(\boldsymbol{R}/\boldsymbol{Z} \times \boldsymbol{R}/\boldsymbol{Z}, d_{g_n})$ converges to the product metric as n is even and tends to infinity; it converges to a different metric as n is odd and tends to infinity. The length of a cycle $\gamma_t : \theta \to (t, \theta)$ measured by the distance is equal to 2 for $t \neq 1/2$ and 1 for $t = 1/2$.

4.4. In this subsection, we shall investigate more closely the spectral distance on certain restricted classes of pairs of metrics and measures on a fixed Riemannian manifold.

Let $M = (M, g_0)$ be a compact connected Riemannian manifold. We write $L^2(M)$

for the Hilbert space of square integrable functions and denote the inner product by $(u, v)_0$ $(u, v \in L^2(M, dv_0))$. Let $H^1(M)$ be the Sobolev space consisting of L^2 functions with derivatives in $L^2(M)$. The inner product of $H^1(M)$ is as usual given by

$$(u, v)_1 = \mathscr{E}(u, v) + (u, v)_0 = \int_M \langle du, dv \rangle_{g_0} dv_{g_0} + \int_M uv dv_{g_0} .$$

We write $H^{-1}(M)$ for the dual space of $H^1(M)$ and also $\langle\ ,\ \rangle$ for the pairing on $H^{-1}(M) \times H^1(M)$.

Now we are given two constants $\alpha > 1$ and $\beta > 1$. We denote by $E(\alpha, \beta)$ the set of pairs (g, μ_w) which consist of metric tensors g with coefficients in $L^\infty(M)$ and measures $\mu_w = w dv_g$ with density w in $L^\infty(M)$ such that

$$\frac{1}{\alpha} g_0 \leq g \leq \alpha g_0 ; \quad \frac{1}{\beta} \leq w \leq \beta .$$

For each (g, μ_w), we have two operators A_{g,μ_w}: $H^1(M) \to H^{-1}(M)$ and I_{μ_w}: $L^2(M) \to H^{-1}(M)$ respectively defined by the identities:

$$\langle A_{g,\mu_w} u, v \rangle = \int_M \langle du, dv \rangle_g d\mu_w , \qquad u, v \in H^1(M) ;$$

$$\langle I_{\mu_w} u, v \rangle = \int_M uv d\mu_w , \qquad u, v \in L^2(M) .$$

Given a sequence $\{(g_n, \mu_n = w_n dv_{g_n}) : n = 1, 2, \dots\}$ in $E(\alpha, \beta)$, we say that the sequence $\{A_{g_n,\mu_n}\}$ is *G-convergent* as $n \to \infty$, if $\langle \psi, (A_{g_n,\mu_n} + \sigma I_{\mu_n})^{-1} \phi \rangle$ converges for some $\sigma > 0$ and all $\phi, \psi \in H^{-1}(M)$. We notice that in our previous notation,

$$(A_{g,\mu_w} + \sigma I_{\mu_w})^{-1} (I_{\mu_w} u) = (\mathscr{L}_{g,\mu_w} + \sigma I)^{-1} u$$

for $u \in L^2(M)$. The definition of G-convergence is actualy independent of the choice of $\sigma > 0$ and moreover if $\{A_{g_n,\mu_n}\}$ is G-convergent, then there exists a unique (g_∞, μ_∞) in $E(\alpha', \beta')$ with some $\alpha' > 1$ and $\beta' > 1$, such that $(A_{g_n,\mu_n} + \sigma I_{\mu_n})^{-1} \phi$ converges weakly to $(A_{g_\infty,\mu_\infty} + \sigma I_{\mu_\infty})^{-1} \phi$ for any $\phi \in H^{-1}(M)$. We also note that any sequence $\{(g_n, \mu_n)\}$ in $E(\alpha, \beta)$, $\{A_{g_n,\mu_n}\}$ contains a G-convergent subsequence. See [27] and the references therein for these facts and related ones on the G-convergence of differential operators.

Thus so far as the restricted classes of pairs as above are concerned, we have the following

PROPOSITION 4.4. *Let $M = (M, g_0)$ be a compact connected Riemannian manifold, and let $\alpha > 1$, $\beta > 1$ be given. For a sequence $\{(g_n, \mu_n = w_n dv_{g_n}) : n = 1, 2, \dots\}$ in $E(\alpha, \beta)$, the following are mutually equivalent*:

(i) *the sequence of operators A_{g_n,μ_n}: $H^1(M) \to H^{-1}(M)$ is G-convergent as $n \to \infty$;*

(ii) *as $n \to \infty$, the measure μ_n converges with respect to the weak* topology, and* *also for some* $\sigma>0$ *and all* $u\in L^2(M)$, $(\mathscr{L}_{g_n,\mu_n}+\sigma I)^{-1}u$ *converges weakly in* $H^1(M)$;

(iii) *the heat kernel of the operator* $\mathscr{L}_{g_n,\mu_n}$ *multiplied by* $e^{-(t+1/t)}$, $e^{-(t+1/t)}p_{w_n}(t, x, y)$, *converges uniformly on* $[0, \infty)\times M\times M$.

Furthermore suppose one of these conditions (and hence all of them) hold and let A_{g_∞,μ_∞} *be the G-limit of* $\{A_{g_n,\mu_n}\}$, *where* $(g_\infty, \mu_\infty)\in E(\alpha', \beta')$ *for some* $\alpha'>1$ *and* $\beta'>1$. *Then the heat kernel* p_{w_n} *and the resolvent kernel* $G^{(\sigma)}_{w_n}$ $(\sigma>0)$ *of* (g_n, μ_n) *converges respectively to those of* $\mathscr{L}_{g_\infty,\mu_\infty}$:

$$p_\infty(t, x, y)=\lim_{n\to\infty} p_{w_n}(t, x, y)\,;$$

$$G^{(\sigma)}_\infty(x, y)=\lim_{n\to\infty} G^{(\sigma)}_{w_n}(x, y)\,.$$

The convergence of the heat kernels (resp., the resolvent kernels) occurs in the C^0*-norm on* $[\delta, \infty)\times M\times M$ *(resp.,* $\{(x, y)\in M\times M: d_{g_0}(x, y)\geq\delta\}$*) for each* $\delta>0$.

We remark that the condition (iii) of this proposition is equivalent to saying that $\{(M, g_n, \mu_n)\}$ is a Cauchy sequence with respect to the spectral distance SD whose spectral approximations are given by the identity mapping of M itself. In general, given an SD-Cauchy sequence $\{(M, g_n, \mu_n)\}$ in $E(\alpha, \beta)$, if we take two subsequences, say, $\{(g_{n'}, \mu_{n'})\}$ and $\{(g_{n''}, \mu_{n''})\}$ in such a way that the sequences $\{A_{g_{n'},\mu_{n'}}\}$ and $\{A_{g_{n''},\mu_{n''}}\}$ are G-convergent, then $SD((M, g'_\infty, \mu'_\infty), (M, g''_\infty, \mu''_\infty))=0$, namely, there is a homeomorphism $f: M\to M$ which preserves the heat kernels and the measures.

5. Conformal changes of metrics. In this section, we shall discuss a family of metrics in the conformal class of a metric with positive Yamabe invariant.

5.1. Let M be a compact connected smooth manifold of dimension $n\geq 3$. Given a conformal class $\mathscr{C}$ of M, the Yamabe invariant, denoted by $Q(M, \mathscr{C})$, is by definition the largest lower bound for the Yamabe functional, namely,

$$Q(M, \mathscr{C})=\inf\left\{\frac{\int_M S_g dv_g}{\mathrm{Vol}(M, g)^{(n-2)/n}} : g\in\mathscr{C}\right\},$$

where S_g stands for the scalar curvature of a Riemannian metric $g\in\mathscr{C}$. If we fix a metric $g\in\mathscr{C}$, then the Yamabe invariant is also given by

$$Q(M, \mathscr{C})=\inf\left\{\frac{4\dfrac{n-1}{n-2}\int_M |d\phi|_g^2 dv_g+\int_M S_g\phi^2 dv_g}{(\int_M |\phi|^{2n/(n-2)}dv_g)^{(n-2)/n}} : \phi\in C^\infty(M), \phi\not\equiv 0\right\}.$$

In this section, we shall first prove the following

PROPOSITION 5.1. *Given positive constants* q, γ *and* $p>n/2$ $(\geq 3/2)$, *suppose a Riemannian metric* g *of* M *satisfies*

$$Q(M,[g])\geq q$$

and

$$\int_M (S_g)_+^p dv_g \leq \gamma^p ,$$

where $[g]$ *denotes the conformal class of* M *to which* g *belongs, and* $(S_g)_+ := \max\{S_g, 0\}$. *Then the Sobolev inequality* (2.3) *holds with constants* $v=n$, $A'=8(n-1)/q(n-2)$, *and* τ *depending only on* n, q, γ *and* p.

PROOF. We first observe that

$$q\left(\int_M |\phi|^{2n/(n-2)} dv_g\right)^{(n-2)/n} \leq 4\frac{n-1}{n-2}\int_M |d\phi|_g^2 dv_g + \int_M (S_g)_+\phi^2 dv_g .$$

Now for $t\geq 0$, we set $A_t := \{x\in M : (S_g)_+(x)\geq t\}$. Then applying Hölder's inequality, we get

$$\begin{aligned}\int_M (S_g)_+\phi^2 dv_g &= \int_{M-A_t} (S_g)_+\phi^2 dv_g + \int_{A_t} (S_g)_+\phi^2 dv_g \\ &\leq t\int_M \phi^2 dv_g + \left(\int_{A_t} (S_g)_+^{n/2} dv_g\right)^{2/n}\left(\int_M |\phi|^{2n/(n-2)} dv_g\right)^{(n-2)/n} \\ &\leq t\int_M \phi^2 dv_g + t^{(n-2p)/n}\left(\int_{A_t} (S_g)_+^p dv_g\right)^{2/n}\left(\int_M |\phi|^{2n/(n-2)} dv_g\right)^{(n-2)/n} \\ &\leq t\int_M \phi^2 dv_g + t^{(n-2p)/n}\gamma^{2p/n}\left(\int_M |\phi|^{2n/(n-2)} dv_g\right)^{(n-2)/2} .\end{aligned}$$

Hence we have

$$\begin{aligned} q&\left(\int_M |\phi|^{2n/(n-2)} dv_g\right)^{(n-2)/n} \\ &\leq 4\frac{n-1}{n-2}\int_M |d\phi|_g^2 dv_g + t\int_M \phi^2 dv_g + t^{(n-2p)/n}\gamma^{2p/n}\left(\int_M |\phi|^{2n/(n-2)} dv_g\right)^{(n-2)/n}\end{aligned}$$

Thus choosing t in such a way that $t^{(n-2p)/n}\gamma^{2p/n} = q/2$, we see that the Sobolev inequality (2.3) holds with constants $A=8(n-1)/q(n-2)$ and $\tau=4(n-1)/(n-2)t$. q.e.d.

5.2. In what follows, we assume M admits a Riemannian metric g_0 such that $q_0 = Q(M,[g_0])$ is positive, and consider a sequence of metrics $g_k = \phi_k^{4/(n-2)} g_0$ which belong to the conformal class $[g_0]$, such that for some positive constants $p>n/2$, γ and b

$$\int_M (S_{g_k})_+^p dv_{g_k} \leq \gamma^p ; \quad \mathrm{Vol}(M, g_k)\leq b .$$

Then the sequence $\{(M, g_k, dv_{g_k})\}$ is precompact with respect to the spectral distance SD.

Now we shall suppose, in addition, that as $k \to \infty$, ϕ_k converges to a continuous function ϕ_∞ uniformly on M. Set

$$\Sigma = \{x \in M : \phi_\infty(x) = 0\},$$

which is a proper compact subset of M, since the volume of (M, g_k) is bounded away from zero uniformly (cf. Theorem 2.1 and Lemma 2.5 (i)). We shall discuss the case where the limit metric $g_\infty = \phi_\infty^{4/(n-2)} g_0$ is degenerate somewhere, namely, Σ is not empty. Note that $M - \Sigma$ may be disconnected.

We denote by $p_\infty(t, x, y)$ $(t > 0,\ x, y \in M - \Sigma)$ the minimal heat kernel of the Laplace operator of the metric g_∞ acting on $C_0^\infty(M - \Sigma)$ (cf. Subsection 3.3). Then p_∞ satisfies

$$p_\infty(t, x, x) \le \frac{a}{t^{n/2}}$$

on $(0, \tau] \times (M - \Sigma)$ for some positive constant a (resp. τ) depending only on n and q_0 (resp. n, p, q_0 and γ). Indeed, for some positive constants A' and τ, A' depending only on n and q_0, and τ depending only on n, q_0, p and γ, the Sobolev inequality (2.3) holds on $M - \Sigma$ uniformly for the metrics g_k, namely,

$$\left(\int_{M-\Sigma} |\phi|^{2n/(n-2)} dv_{g_k}\right)^{(n-2)/n} \le A'\left(\int_{M-\Sigma} |d\phi|_{g_k}^2 dv_{g_k} + \tau^{-1} \int_{M-\Sigma} \phi^2 dv_{g_k}\right)$$

for all $\phi \in C_0^\infty(M - \Sigma)$. Since ϕ_k converges to ϕ_∞ uniformly, we see that the inequality holds on $M - \Sigma$ for the metric g_∞ with the same constants. Thus we have an upper bound for the heat kernel $p_\infty(t, x, y)$ as above. Since the volume of $(M - \Sigma, g_\infty)$ is finite, the upper bound for p_∞ as above implies, as mentioned in 3.3, that the Laplacian of g_∞ defined on $C_0^\infty(M - \Sigma)$ has a discrete spectrum, say $\{\lambda_{\infty,i} : i = 0, 1, 2, \dots\}$ and the kernel p_∞ has the eigenfunction expansion. Thus making use of spectral embeddings of $(M - \Sigma, g_\infty, dv_\infty)$ as in the compact case, we obtain the following

ASSERTION 5.2. *There exists a compact metric space $(M - \Sigma)^\sim$ which includes $M - \Sigma$ as an open dense subset and to which the minimal heat kernel p_∞ and also eigenfunctions of $(M - \Sigma, g_\infty, dv_{g_\infty})$ extend continuously. This metric space coincides with the completion of $M - \Sigma$ with respect to the distance Θ_∞ on $M - \Sigma$ defined by*

$$\Theta_\infty(x, y)^2 := \sup_{t > 0} e^{-(t + 1/t)}(p_\infty(t, x, x) + p_\infty(t, y, y) - 2p_\infty(t, x, y)).$$

Now we shall assume further that

$$\text{(5.1)} \qquad \begin{gathered} \Sigma \text{ is a compact submanifold of codimension} \ge 2, \quad \text{or} \\ \phi_\infty(x) \le O(\rho_\Sigma(x)^{1/2}) \end{gathered}$$

in a neighborhood of Σ, where ρ_Σ stands for the distance to Σ with respect to the fixed metric g_0. Then we choose a function $f_\varepsilon \in C_0^\infty(M-\Sigma)$ in such a way that $0 \le f_\varepsilon \le 1$, $f_\varepsilon(x)=1$ if $\rho_\Sigma(x) \ge \varepsilon$ and $|df_\varepsilon|_{g_0} \le c\varepsilon^{-1}$ for some positive constant c. Then for any $0<t_1<t_2$, we have

$$\begin{aligned}
&\left|\int_{M-\Sigma} f_\varepsilon(y)p_\infty(t_1,x,y)dv_{g_\infty}(y)-\int_{M-\Sigma} f_\varepsilon(y)p_\infty(t_2,x,y)dv_{g_\infty}(y)\right| \\
&\quad \le \int_{t_1}^{t_2}\left|\frac{\partial}{\partial t}\int_{M-\Sigma} f_\varepsilon(y)p_\infty(t,x,y)dv_{g_\infty}(y)\right|dt \\
&\quad = \int_{t_1}^{t_2}\left|\int_{M-\Sigma}\langle df_\varepsilon, dp_\infty(t,x,y)\rangle_{g_\infty}dv_{g_\infty}\right|dt \\
&\quad \le \int_{t_1}^{t_2}\left(\int_{M-\Sigma}|df_\varepsilon|_{g_\infty}^2 dv_{g_\infty}\right)^{1/2}\left(\int_{\{\rho_\Sigma\le\varepsilon\}}|dp_\infty(t,x,y)|_{g_\infty}^2 dv_{g_\infty}(y)\right)^{1/2}dt\,.
\end{aligned}$$

By the assumption (5.1),

$$\limsup_{\varepsilon\to 0}\int_{M-\Sigma}|df_\varepsilon|_{g_\infty}^2 dv_{g_\infty}=\limsup_{\varepsilon\to 0}\frac{c^2}{\varepsilon^2}\int_{\{\rho_\Sigma\le\varepsilon\}}\phi_\infty^2 dv_{g_0}<+\infty$$

and moreover, since $p_\infty(t,x,*)\in H_0^1(M-\Sigma, g_\infty)$ (the Sobolev space with respect to g_∞),

$$\lim_{\varepsilon\to 0}\int_{\{\rho_\Sigma\le\varepsilon\}}|dp_\infty(t,x,y)|_{g_\infty}^2 dv_{g_\infty}(y)=0\,.$$

Thus we see that

$$\int_{M-\Sigma} p_\infty(t_1,x,y)dv_{g_\infty}(y)=\int_{M-\Sigma} p_\infty(t_2,x,y)dv_{g_\infty}(y)\,.$$

Namely, we have

ASSERTION 5.3. *Under the above assumptions, $p_\infty(t,x,y)$ is conservative,*

$$\int_{M-\Sigma} p_\infty(t,x,y)dv_{g_\infty}(y)=1\,.$$

Hence $p_\infty(t,x,y)$ is the unique heat kernel of $(M-\Sigma, g_\infty, dv_{g_\infty})$.

In view of the uniqueness of the heat kernel of the limit metric, we see that for a compact set K of $M-\Sigma$,

$$\lim_{k\to\infty}\sup\{e^{-(t+1/t)}|p_k(t,x,y)-p_\infty(t,x,y)|: t>0,\, x,y\in K\}=0\,, \tag{5.2}$$

where p_k denotes the heat kernel of g_k. Hence if we set

$$\Theta_k(x, y)=\left(\sup_{t>0} e^{-(t+1/t)}(p_k(t, x, x)+p_k(t, y, y)-2p_k(t, x, y))\right)^{1/2},$$

then as $k\to\infty$,

(5.3) Θ_k converges to Θ_∞ uniformly on a compact subset of $(M-\Sigma)\times(M-\Sigma)$.

Now we shall take a subsequence $\{k'\}$ in such a way that $\{(M, g_{k'}, dv_{g_{k'}})\}$ is an SD-Cauchy sequence. Let (X, Θ), p, $\{\lambda_i\}$, $F_{k'}: M\to X$, $H_{k'}: X\to M$ and $\{\varepsilon_{k'}\}$ be as in Theorem 3.1. Taking a subsequence if necessarily, we may assume that $F_{k'}$ converges pointwise to a mapping F_∞ defined on some dense subset A of M. Then it follows from (3.3), (3.6), (5.2) and (5.3) that

$$\Theta_\infty(x, y)=\Theta(F_\infty(x), F_\infty(y))\,; \quad p_\infty(t, x, y)=p(t, F_\infty(x), F_\infty(y)) \tag{5.4}$$

for all $t>0$ and $x, y\in A\cap(M-\Sigma)$. Hence F_∞ uniquely extends to a mapping of $M-\Sigma$ into X, which is denoted by the same letter F_∞, in such a way that (5.4) holds on $M-\Sigma$ and as $k'\to\infty$, $F_{k'}$ converges pointwise to F_∞ on $M-\Sigma$. As a consequence, for any $\phi\in C(X)$

$$\int_{M-\Sigma} F_\infty^*\phi dv_{g_\infty}=\lim_{k'\to\infty}\int_M F_{k'}^*\phi dv_{g_{k'}}=\int_X \phi d\mu\,,$$

which shows that $F_{\infty *}dv_{g_\infty}=\mu$. Hence the support $X^{(\mu)}$ of μ coincides with the closure of the image $F_\infty(M-\Sigma)$. Thus $(X^{(\mu)}, \mu, p)$ is independent of the choice of subsequences as above. In particular, $\{\lambda_i\}$ are the eigenvalues of the Laplacian of g_∞ and the i-th eigenvalue $\lambda_{k,i}$ of g_k converges to λ_i. Namely we have the following

ASSERTION 5.4. *Under the assumptions above, the i-th eigenvalue $\lambda_{k;i}$ of (M, g_k, dv_{g_k}) converges to the i-th eigenvalue λ_i of $(M-\Sigma, g_\infty, dv_{g_\infty})$ as $k\to\infty$.*

EXAMPLE. Let M and g_0 be as above and let D be a proper open set of M whose complement has interior points. We take an interior point x_0 of $M-D$ and a smooth positive function ψ on M in such a way that on $\bar{D}$, ψ coincides with the Green function of the positive operator $-4(n-1)/(n-2)\Delta_{g_0}+S_{g_0}$ with pole x_0. Then the scalar curvature of the metric $g'_0=\psi^{4/(n-2)}g_0$ vanishes on $\bar{D}$. Let ϕ_∞ be a nonnegative continuous function on M such that ϕ_∞ is subharmonic on a neighborhood of $\bar{D}$ with respect to the metric g'_0 and

$$\Sigma:=\{x\in M: \phi_\infty(x)=0\}\subset D\,.$$

Then for each positive integer k, we take a smooth approximation ϕ_k of the function $\max\{\phi_\infty, 1/k\}$ in such a way that ϕ_k is positive on M and subharmonic on D (see e.g., [17]). Then the scalar curvature of the metric $g_k=\phi_k^{4/(n-2)}g'_0$ is kept nonpositive on D as $k\to\infty$. Thus Assertion 5.2 certainly holds for the limit metric $g'_\infty=\phi_\infty^{4(n-2)}g'_0$.

Moreover Assertions 5.3 and 5.4 are also true in case Σ is a submanifold of codimension ≥ 2 or $\phi_\infty \leq O(\rho_\Sigma^{1/2})$ near Σ.

6. Submanifolds of bounded mean curvatures. In this section, making use of some geometric inequalities due to Croke [12] and Hoffman-Spruck [19], we shall show upper bounds for the heat kernels of compact Riemannian (sub)manifolds.

6.1. To begin with, we shall prove the following

PROPOSITION 6.1. *Let M be a compact Riemannian manifold of dimension n. Suppose that the injectivity radius of M is bounded from below by a positive constant ι. Then the heat kernel of (M, dv_M) has a bound of the form* (2.1) *with constants $\nu = n$, a depending only on n, and $\tau = \iota^2$.*

PROOF. According to a result in [12], we see that

$$\left(\int_M |\phi|^{n/(n-1)} dv_M\right)^{(n-1)/n} \leq C(n) \int_M |d\phi|_g dv_M$$

for any $\phi \in C^\infty(M)$ supported in a geodesic ball of radius $\iota/2$, where $C(n)$ is a constant depending only on n. Therefore replacing ϕ with $\phi^{2(n-1)/(n-2)}$ in the case $n \geq 3$ and also with ϕ^2 in the case $n = 2$, and then using Hölder inequality, we can deduce that

$$\left(\int_M |\phi|^{2n/(n-2)} dv_M\right)^{(n-2)/n} \leq A(n) \int_M |d\phi|_g^2 dv_M \quad (n \geq 3);$$

$$\int_M \phi^4 dv_M \leq A(n) \int_M |d\phi|_g^2 dv_M \int_M \phi^2 dv_M \quad (n = 2)$$

for all $\phi \in C^\infty(M)$ as above. Hence the proposition follows from Theorem 2.2. q.e.d.

6.2. Let us now consider compact submanifolds in certain Riemannian manifolds.

PROPOSITION 6.2. *Let $M = (M, g)$ be a compact Riemannian manifold of dimension n isometrically immersed into a complete Riemannian manifold $\bar{M}$. Suppose that the sectional curvature of $\bar{M}$ is bounded from above by a constant $\kappa \geq 0$ and the injectivity radius of $\bar{M}$ is bounded from below by a constant $\iota > 0$. Moreover suppose that for some $b > 0$, $\gamma \geq 0$ and $p > n$, the volume of M is not greater than b and the mean curvature H_M of the immersion satisfies*

$$\int_M |H_M|^p dv_M \leq \gamma^p .$$

Then the heat kernel p_M of (M, dv_M) satisfies (2.1) *with constants $\nu = n$, a and τ depending only on the given n, κ, ι, b, γ and p.*

PROOF. We first recall a result in [19] (see also [7]) stated as follows: for an open subset Ω in M satisfying

$$\text{(6.1)} \qquad \kappa\left(\frac{2}{\omega(n)}\operatorname{Vol}(\Omega)\right)^{1/n} \leq 1\,; \quad \frac{2}{\kappa}\sin^{-1}\left(\kappa\left(\frac{2}{\omega(n)}\operatorname{Vol}(\Omega)\right)^{1/n}\right) \leq \frac{\iota}{2}\,,$$

one has

$$\left(\int_M |\phi|^{(n/n-1)} dv_M\right)^{(n-1)/n} \leq C(n)\left(\int_M |d\phi|_g dv_M + \int_M |H_M||\phi| dv_M\right)$$

for all $\phi \in C^\infty(M)$ supported on Ω, where $\omega(n)$ stands for the volume of the Euclidean unit n-sphere and $C(n)$ is a constant depending only on n. Therefore replacing ϕ with $\phi^{2(n-1)/(n-2)}$ in the case $n \geq 3$ and also with ϕ^2 in the case $n=2$, and then using Hölder's inequality, we can deduce that

$$\left(\int_M |\phi|^{2n/(n-2)} dv_M\right)^{(n-2)/n} \leq A(n)\left(\int_M |d\phi|_g^2 dv_M + C(n,p)\gamma^{2p/(p-n)}\int_M \phi^2 dv_M\right) \quad (n \geq 3)\,;$$

$$\int_M \phi^4 dv_M \leq A(n)\left(\int_M |d\phi|_g^2 dv_M + C(p)\gamma^{2p/(p-2)}\int_M \phi^2 dv_M\right)\int_M \phi^2 dv_M \quad (n=2)$$

for all $\phi \in C^\infty(M)$ supported on Ω (cf. the proof of Proposition 5.1).

Now in view of Theorem 2.2, it suffices to prove that the intersection of M with a geodesic ball of $\bar{M}$ of radius r satisfies (6.1) if r is sufficiently small. Let $\psi: M \to \bar{M}$ be the isometric immersion and set

$$M(r) = \psi^{-1}(B_{\bar{M}}(x, r))\,; \quad V(r) = \operatorname{Vol}(M(r))\,,$$

where $B_{\bar{M}}(x, r)$ stands for the geodesic ball of $\bar{M}$ around a point $x \in \bar{M}$ with radius r. Then we have

$$\frac{d}{dr}\left[-(\sin \kappa r)^{-n} V(r)\right] \leq (\sin \kappa r)^{-n}\int_{M(r)} |H_M| dv_M$$

for almost all $r \in (0, R)$ $(R := \min\{\iota, \pi/2\kappa\})$ (cf. e.g., [7, Chap. 6 Lemma 36.5.7]). It follows from the assumption and Hölder inequality that

$$\int_{M(r)} |H_M| dv_M \leq \gamma V(r)^{1-1/p}\,.$$

These show that

$$\frac{d}{dr}\left[-(\sin \kappa r)^{-n/p} V(r)^{1/p}\right] \leq \frac{\gamma}{p}(\sin \kappa r)^{-n/p}\,.$$

Hence integrating the both sides from r to R, we have

$$(\sin \kappa r)^{-n/p} V(r)^{1/p} \leq (\sin \kappa R)^{-n/p} V(R)^{1/p} + \frac{\gamma}{p} \int_0^R (\sin \kappa t)^{-n/p} dt .$$

This implies that

$$V(r) \leq C(n, \iota, \kappa)(b + \gamma^p) r^n$$

for all $r \in (0, R]$. Thus if we take a sufficiently small r depending only on the given constants, we see that $M(r)$, namely, $\psi^{-1}(B_{\bar{M}}(x, r))$ satisfies the condition (6.1). q.e.d.

REMARK. In a very recent paper [36], Yoshikawa shows the continuity of the spectrum of a certain degenerating family of algebraic manifolds in a complex projective space (cf. Theorem 3.4).

7. Families of Riemannian manifolds of increasing topological type. In this section, we shall construct SD-precompact families of Riemannian manifolds with increasing topological type.

Let U be a compact connected n-dimensional Riemannian manifold with boundary ∂U such that ∂U has v connected components $\{\partial_i U : i = 1, \ldots, v\}$ and each of the components has a neighborhood which is isometric to that of the boundary of the unit n-cube $I^n = [0, 1] \times \cdots \times [0, 1]$ in Euclidian n-space $\boldsymbol{R}^n$. We first take v copies of $\boldsymbol{R}^n$, say $\boldsymbol{R}^n_1, \ldots, \boldsymbol{R}^n_v$, and for each element $\gamma = (\gamma_1, \ldots, \gamma_n) \in \boldsymbol{Z}^n$ and $\alpha \in \{1, 2, \ldots, v\}$, we denote the unit n-cube $[\gamma_1, \gamma_1 + 1] \times \cdots \times [\gamma_n, \gamma_n + 1]$ in $\boldsymbol{R}^n_\alpha$ by $I_\alpha(\gamma)$. Secondly for each element $\gamma \in \boldsymbol{Z}^n$, we replace the disjoint union of $\{I_\alpha(\gamma) : \alpha = 1, 2, \ldots, v\}$ with U in such a way that $\partial_\alpha U$ is just glued on the boundary of $I_\alpha(\gamma)$. Repeating this process for every $\gamma \in \boldsymbol{Z}^n$, we obtain a complete noncompact Riemannian manifold $\bar{M} = (\bar{M}, \bar{g})$. It is easy to see that $\bar{M}$ is rough isometric to $\boldsymbol{R}^n$. In fact, there exist positive constants, a, b and mappings $\phi : \bar{M} \to \boldsymbol{R}^n$, $\psi : \boldsymbol{R}^n \to \bar{M}$ such that

$$\frac{1}{a} d_{\bar{M}}(x, y) - b \leq |\phi(x) - \phi(y)| \leq a d_{\bar{M}}(x, y) + b$$

for all $x, y \in \bar{M}$;

$$\frac{1}{a} |x' - y'| - b \leq d_{\bar{M}}(\psi(x'), \psi(y')) \leq a |x' - y'| + b$$

for all $x', y' \in \boldsymbol{R}^n$, and

$$d_{\bar{M}}(x, \psi \circ \phi(x)) \leq a$$

for all $x \in \bar{M}$. Then by virtue of a result of Kanai [22], we have a Sobolev inequality on $\bar{M}$ of the form:

$$\left(\int_{\bar{M}} |\phi|^{n/(n-1)} dv_{\bar{g}}\right)^{(n-1)/n} \leq C \int_{\bar{M}} |d\phi|_{\bar{g}} dv_{\bar{g}}, \qquad \phi \in C_0^{\infty}(\bar{M})$$

for some $C>0$. Moreover $\boldsymbol{Z}^n$ acts on $\bar{M}$ in a natural manner as isometries of $\bar{M}$. For each positive integer i, we denote by $i\boldsymbol{Z}^n$ the subgroup of $\boldsymbol{Z}^n$ consisting of elements $(i\gamma_1, \ldots, i\gamma_n)$ $(\gamma_1, \ldots, \gamma_n \in \boldsymbol{Z})$, and then we obtain the quotient space $\bar{M}/i\boldsymbol{Z}^n$ endowed with the induced metric $\bar{g}_i$.

Now scaling the metric $\bar{g}_i$ by i^{-2}, we have a sequence of compact Riemannian manifolds $\{M_i=(\bar{M}/i\boldsymbol{Z}^n, i^{-2}\bar{g}_i): i=1, 2, \ldots\}$. We first observe that the volume of M_i is equal to that of U for any i. Secondly we notice that for some positive constants r and C independent of i, a Sobolev inequality of the following form holds uniformly for all geodesic ball $B(x, r)$ in M_i with radius r:

$$\left(\int_{B(x,r)} |\phi|^{n/(n-1)} dv_{M_i}\right)^{(n-1)/n} \leq C \int_{B(x,r)} |d\phi| dv_{M_i}, \qquad \phi \in C_0^{\infty}(B(x, r)).$$

Therefore in view of Theorems 2.2, 2.3, and 2.6, we get the following

ASSERTION 7.1. *The family $\{M_i=(M_i, dv_{M_i})\}$ obtained as above is precompact with respect to the spectral distance* SD *and also the Gromov-Hausdorff distance* HD.

We remark that the mappings $\phi\colon \bar{M}\to\boldsymbol{R}^n$ and $\psi\colon \boldsymbol{R}^n\to\bar{M}$ respectively induce mappings $\phi_i\colon M_i\to\boldsymbol{R}^n/\boldsymbol{Z}^n$ of M_i into the flat torus $\boldsymbol{R}^n/\boldsymbol{Z}^n$ and $\psi_i\colon \boldsymbol{R}^n/\boldsymbol{Z}^n\to M_i$ of $\boldsymbol{R}^n/\boldsymbol{Z}^n$ into M_i. These mappings satisfy

$$\frac{1}{a} d_{M_i}(x, y) - \frac{b}{i} \leq d_0(\phi_i(x), \phi_i(y)) \leq a d_{M_i}(x, y) + \frac{b}{i}$$

for all $x, y \in M_i$;

$$\frac{1}{a} d_0(x', y') - \frac{b}{i} \leq d_{M_i}(\psi_i(x'), \psi_i(y')) \leq a d_0(x', y') + \frac{b}{i}$$

for all $x', y' \in \boldsymbol{R}^n/\boldsymbol{Z}^n$, and

$$d_{M_i}(x, \psi_i \circ \phi_i(x)) \leq \frac{a}{i}$$

for all $x \in M_i$, where d_0 stands for the distance of $\boldsymbol{R}^n/\boldsymbol{Z}^n$. Suppose that a subsequence, $\{M_j\}$, of $\{M_i\}$ converges, as $j\to\infty$, to a compact connected length space $X=(X, d_X)$ with respect to the Gromov-Hausdorff distance HD. Then there exist $\varepsilon(j)$-Hausdorff approximations $f_j\colon M_j\to X$ and $h_j\colon X\to M_j$ such that $d_X(x, f_j\circ h_j(x)) \leq 2\varepsilon(j)$ for all $x\in X$, where $\varepsilon(j)$ tends to zero as $j\to\infty$. If we set $\Phi_j=\phi_j\circ h_j$ and $\Psi_j=f_j\circ\psi_j$, then these mappings satisfy

$$\frac{1}{a}\,d_X(x,y)-\frac{1}{a}\,\varepsilon(j)-\frac{b}{j}\le d_0(\Phi_j(x),\Phi_j(y))\le ad_X(x,y)+a\varepsilon(j)+\frac{b}{j} \tag{7.1}$$

for all $x, y\in X$;

$$\frac{1}{a}\,d_0(x',y')-\frac{1}{a}\,\varepsilon(j)-\frac{b}{j}\le d_X(\Psi_j(x'),\Psi_j(y'))\le ad_0(x',y')+a\varepsilon(j)+\frac{b}{j}$$

for all $x', y'\in \boldsymbol{R}^n/\boldsymbol{Z}^n$; further

$$d_X(x,\Psi_j\circ\Phi_j(x))\le 3\varepsilon(j)+\frac{a}{j} \tag{7.2}$$

for all $x\in X$. Now taking a subsequence of $\{M_j\}$ if necessarily, we may assume that Φ_j converges pointwise to a mapping Φ_∞ defined on a dense subset X_0 of X. Then letting j go to infinity in (7.1), we see that Φ_∞ is a Lipschitz mapping of X_0 into $\boldsymbol{R}^n/\boldsymbol{Z}^n$ such that

$$\frac{1}{a}\,d_X(x,y)\le d_0(\Phi_\infty(x),\Phi_\infty(y))\le ad_X(x,y) \tag{7.3}$$

for all $x, y\in X$. Since X_0 is a dense subset of X, Φ_∞ extends uniquely to a Lipschitz mapping, denoted by the same letter Φ_∞, of X into $\boldsymbol{R}^n/\boldsymbol{Z}^n$ with the property (7.3), to which Φ_j converges pointwise as $j\to\infty$. In a similar fashion, we have a Lipschitz mapping $\Psi_\infty\colon \boldsymbol{R}^n/\boldsymbol{Z}^n\to X$, to which Ψ_j converges pointwise. Taking (7.2) into account, we see that $\Psi_\infty\circ\Phi_\infty=\mathrm{id}_X$. Thus $\Phi_\infty\colon X\to\boldsymbol{R}^n/\boldsymbol{Z}^n$ induces a bi-Lipschitz homeomorphism satisfying (7.3). As a summary, we have the following

ASSERTION 7.2. *For a limit $X=(X, d_X)$ of the metric spaces $\{M_i\}$ with respect to the Gromov-Hausdorff distance, there exists a bi-Lipschitz homeomorphism Φ of X onto the flat torus $\boldsymbol{R}^n/\boldsymbol{Z}^n$ such that*

$$\frac{1}{a}\,d_X(x,y)\le d_0(\Phi(x),\Phi(y))\le ad_X(x,y)$$

for all $x, y\in X$, where a is a positive constant depending only on U.

In the construction of the family $\{M_i\}$ above, we have placed infinitely many copies of U for the disjoint unions $\bigcup_{\alpha=1}^{\nu} I_\alpha(\gamma)$ with γ running over all elements of $\boldsymbol{Z}^n$. We shall now show another kind of example. We fix first an integer k, $1\le k\le n-1$, and then replace the disjoint union $\bigcup_{\alpha=1}^{\nu} I_\alpha(\gamma)$ with the copies of U for all $\gamma=(\gamma_1,\dots,\gamma_n)\in\boldsymbol{Z}^n$ such that $\gamma_1=\cdots=\gamma_k=0$. Then by the same way as in the preceding construction, we obtain a sequence of compact Riemannian manifolds $\{M_i^{(k)}\}$. In this case, a limit metric space $X^{(k)}$ of $\{M_i^{(k)}\}$ is unique and obtained from the disjoint union of ν copies of $\boldsymbol{R}^n/\boldsymbol{Z}^n$ by identifying the subspaces $\boldsymbol{R}^{n-k}/\boldsymbol{Z}^{n-k}$. Moreover Assertion 5.2 certainly holds and if in addition, $k\ge 2$, then Assertions 5.3 and 5.4 are also true.

8. Isoperimetric inequality for some metrics on surfaces. In this section, we exhibit a certain family of Riemannian metrics on a compact surface which degenerates along simple closed curves while keeping the heat kernels bounded uniformly.

8.1. Let us begin with the following:

PROPOSITION 8.1. *Let g be a Riemannian metric on $\Omega=[-1, 1]\times \boldsymbol{R}/\boldsymbol{Z}$ of the form*:

$$g=E(t)^2dt^2+G(t)^2d\theta^2\,, \qquad (t,\theta)\in\Omega\,,$$

where $E(t)>0$ and $G(t)>0$ are smooth functions on $[-1, 1]$. Suppose that

$$G'\geq 0 \text{ on } [0,1] \quad \text{and} \quad G'\leq 0 \text{ on } [-1,0]\,; \tag{8.1}$$

for some $a>0$,

$$\inf_{-1\leq t\leq 1}\frac{G(t)^2}{|\int_0^t E(s)G(s)ds|}\geq a\,. \tag{8.2}$$

Then

$$\left(\int_\Omega f^2dv_g\right)^{1/2}\leq\left(2+\frac{2}{a}\right)^{1/2}\int_\Omega |df|_g dv_g$$

for all $f\in C_0^\infty(\Omega)$.

PROOF. As is well known, the above Sobolev inequality is equivalent to the following isoperimetric inequality: for any compact domain D of Ω with piecewise C^1 boundary ∂D which does not intersect that of Ω,

$$\mathrm{A}(D)\leq 2\left(1+\frac{1}{a}\right)\mathrm{L}(\partial D)^2\,,$$

where $\mathrm{A}(D)$ and $\mathrm{L}(\partial D)$ respectively denote the area of D and the length of the boundary ∂D. In what follows, we shall show this isoperimetric inequality. We first set

$$\rho(t)=\int_0^t E(s)ds$$

and write the metric g of the above form as follows:

$$g=d\rho^2+F(\rho)^2d\theta^2\,, \quad (\rho,\theta)\in[\rho(-1),\rho(1)]\times\boldsymbol{R}/\boldsymbol{Z}\,,$$

where $F(\rho)=G(t(\rho))$. Let α be a given number in $[-1, 1]$ and define a smooth function $S_\alpha(\rho)$ on Ω by

$$S_\alpha(\rho)=\int_0^\rho\frac{\int_\alpha^u F(s)ds}{F(u)}du\,.$$

Then the Laplacian of $S_\alpha(\rho)$ is identically equal to 1. Therefore applying Stokes theorem to a domain D with piecewise C^1 boundary, we obtain

$$\mathrm{A}(D)=\int_D \Delta_g S_\alpha(\rho)dv_g=\sum_{i=1}^{k}\int_{\gamma_i} S'_\alpha(\rho)\langle\nabla\rho,\, \nu_{\partial D}\rangle\,, \tag{8.3}$$

where $\{\gamma_i\}$ denotes the connected components of ∂D and $\nu_{\partial D}$ stands for the outer unit normal of ∂D. We notice here that if $\rho=\alpha$ on γ_i for some i, then

$$\int_{\gamma_i} S'_\alpha(\rho)\langle\nabla\rho,\, \nu_{\partial D}\rangle=0\,,$$

since $S'_\alpha(\rho)=0$ on γ_i.

To prove the assertion of the proposition, it suffices to consider either of the case $k=1$, namely, D is homeomorphic to a disk, or the case $k=2$, namely, D is homeomorphic to an annulus. In the sequel, we put $\Omega^+=(0,1]\times \boldsymbol{R}/\boldsymbol{Z}$ and $\Omega^-=[-1,0)\times \boldsymbol{R}/\boldsymbol{Z}$.

Now we assume that γ_1 intersects Ω^+. Set $\beta_*=\inf\{\rho(x): x\in\gamma_1\cap\Omega^+\}$ and $\beta^*=\max\{\rho(x): x\in\gamma_1\}$. We observe that

$$\beta^*-\beta_*\le \mathrm{L}(\gamma_1)\,.$$

Let Ω_0 be the connected component of $\Omega^+-\gamma_1$ whose boundary contains the circle $\{1\}\times\boldsymbol{R}/\boldsymbol{Z}$, and set $\Omega_1=\Omega^+-\Omega_0$. Then in view of (8.1) and (8.3) with $\alpha=0$, if $\beta_*=0$, we see that

$$\begin{aligned}\mathrm{A}(D\cap\Omega^+)\le \mathrm{A}(\Omega_1)&=\int_{(\partial\Omega_1)\cap\Omega^+}\frac{\int_0^\rho F(s)ds}{F(\rho)}\langle\nabla\rho,\, \nu_{\partial\Omega_1}\rangle\\ &\le\int_{(\partial\Omega_1)\cap\mathrm{int}(\Omega^+)}\rho\le\beta^*\mathrm{L}((\partial\Omega_1)\cap\mathrm{int}(\Omega^+))\le \mathrm{L}(\gamma_1)^2\,.\end{aligned}$$

Similarly, in case $\beta_*>0$ and γ_1 is null homotopic, it follows from (8.3) with $\alpha=\beta_*$ that

$$\begin{aligned}\mathrm{A}(D\cap\Omega^+)\le \mathrm{A}(\Omega_1)&=\int_{\gamma_1}\frac{\int_{\beta_*}^\rho F(s)ds}{F(\rho)}\langle\nabla\rho,\, \nu_{\partial\Omega_1}\rangle\\ &\le(\beta^*-\beta_*)\mathrm{L}(\gamma_1)\le \mathrm{L}(\gamma_1)^2\,.\end{aligned}$$

Moreover in case $\beta_*>0$ and γ_1 is homotopic to the circle $\{1\}\times\boldsymbol{R}/\boldsymbol{Z}$, it follows from (8.3) with $\alpha=0$ that

$$\begin{aligned}\mathrm{A}(D\cap\Omega^+)\le \mathrm{A}(\Omega_1)&=\int_{\gamma_1} S'_0(\rho)\langle\nabla\rho,\, \nu_{\partial\Omega_1}\rangle\le\int_{\gamma_1}\left(\frac{\int_0^{\beta_*}F(s)ds}{F(\rho)}+\frac{\int_{\beta_*}^\rho F(s)ds}{F(\rho)}\right)\\ &\le\left(\frac{\int_0^{\beta_*}F(s)ds}{F(\beta_*)}+(\beta^*-\beta_*)\right)\mathrm{L}(\gamma_1)\le\left(\frac{\int_0^{\beta_*}F(s)ds}{F(\beta_*)^2}+1\right)\mathrm{L}(\gamma_1)^2\,.\end{aligned}$$

Since assumption (8.2) implies that

$$\frac{\int_0^{\beta_*} F(s)ds}{F(\beta_*)^2} \leq \frac{1}{a},$$

we have

$$\mathrm{A}(D \cap \Omega^+) \leq \left(1 + \frac{1}{a}\right) \mathrm{L}(\gamma_1)^2 .$$

The arguments just above are available for Ω^- instead of Ω^+, or γ_2 instead of γ_1 (if D is an annular domain). Therefore it is not hard to see that

$$\mathrm{A}(D) \leq 2\left(1 + \frac{1}{a}\right) \mathrm{L}(\partial D)^2$$

for any compact domain D with piecewise C^1 boundary which does not interesect the boundary of Ω. q.e.d.

Let M be a compact 2-dimensional manifold and Ω a domain of M which is diffeomorphic to $[-1, 1] \times \boldsymbol{R}/\boldsymbol{Z}$. Let g be a Riemannian metric on M which has the same expression on Ω as in Proposition 8.1 and further satisfies

$$\int_0^1 E(t)dt \geq r ; \quad \int_{-1}^0 E(t)dt \geq r$$

for a constant $r > 0$. Then in view of Theorem 2.2, the heat kernel $p_g(t, x, y)$ of (M, g, dv_g) has an upper bound of the form:

$$p_g(t, x, x) \leq \frac{c(1+a)}{at}$$

for all $0 < t \leq r^2$ and $x \in \Omega$ with $|\rho(x)| \leq r/2$, where c is some numerical constant.

8.2. Let us now exibit some families of metrics with the properties described in Proposition 8.1.

EXAMPLE. Let $a > 0$, $b > 0$, $0 < \alpha \leq 1$ and $0 < \beta \leq 1$ be given, and set

$$g_\varepsilon = dt^2 + G_\varepsilon(t)^2 d\theta^2 , \quad (t, \theta) \in [-1, 1] \times \boldsymbol{R}/\boldsymbol{Z} ,$$

where G_ε is given by

$$G_\varepsilon(t) = \begin{cases} at^\alpha + \varepsilon & 0 \leq t < 1 \\ b(-t)^\beta + \varepsilon & -1 < t \leq 0 . \end{cases}$$

Then direct computation shows that

$$\inf_{|t|<1} \frac{G_\varepsilon(t)^2}{|\int_0^t G_\varepsilon(s)ds|} \geq c$$

for some $c>0$ depending only on a, b, α, β.

EXAMPLE. Let $a>0$ and $b>0$ be given, and set

$$g_\varepsilon = dt^2 + G_\varepsilon(t)^2 d\theta^2, \quad (t, \theta) \in [-1, 1] \times \boldsymbol{R}/\boldsymbol{Z},$$

where G_ε is given by

$$G_\varepsilon(t) = \begin{cases} \min\left\{\dfrac{a-\varepsilon}{\varepsilon} t + \varepsilon, a\right\} & 0 \le t \le 1 \\ \min\left\{-\dfrac{b-\varepsilon}{\varepsilon} t + \varepsilon, a\right\} & -1 \le t \le 0. \end{cases}$$

Then we have

$$\inf_{|t|<1} \frac{G_\varepsilon(t)^2}{\left|\int_0^t G_\varepsilon(s) ds\right|} \ge c$$

for some $c>0$ depending only on a, b.

EXAMPLE. Let $a>0$, $b>0$, $0<\alpha\le 3/2$ and $0<\beta\le 3/2$ be given, and set

$$g_\varepsilon = G_\varepsilon(t)^2(dt^2 + d\theta^2), \quad (t, \theta) \in [-1, 1] \times \boldsymbol{R}/\boldsymbol{Z}$$

where G_ε is given by

$$G_\varepsilon(t) = \begin{cases} at^\alpha + \varepsilon & 0 \le t \le 1 \\ b(-t)^\beta + \varepsilon & -1 \le t \le 0. \end{cases}$$

Then direct computation shows that

$$\inf_{|t|<1} \frac{G_\varepsilon(t)^2}{\left|\int_0^t G_\varepsilon(s)^2 ds\right|} \ge c$$

for some $c>0$ depending only on α, β.

EXAMPLE. Let us consider a family of rotationally symmetric surfaces around the z-axis in Euclidean 3-space $\boldsymbol{R}^3 = \{(x, y, z) : x, y, z \in \boldsymbol{R}\}$. We start with a simply closed curve $\mathscr{C}_1$ in yz-plane parametrized by $t \in \boldsymbol{R}/\boldsymbol{Z} \to (0, r(t), z(t))$. We assume that $r(t)>1$ for $t \ne 0$, $z(t)=t$ for $|t| \le 1/2$, and further

$$r(t) = \begin{cases} at^\alpha + 1 & 0 \le t \le \dfrac{1}{2} \\ b(-t)^\beta + 1 & -\dfrac{1}{2} \le t \le 0 \end{cases}$$

for some constants $a>0$, $b>0$, $0<\alpha\le 1$ and $0<\beta\le 1$. Now rotating the curves given by $\mathscr{C}_n = \mathscr{C}_1 + (0, -1+1/n, 0)$ around the z-axis, we obtain a family of surfaces M_n in $\boldsymbol{R}^3$, namely,

$$M_n = \{((r(t)-1+1/n)\cos\theta, (r(t)-1+1/n)\sin\theta, z(t)) : t, \theta \in \boldsymbol{R}/\boldsymbol{Z}\} .$$

Then the local Sobolev inequality (2.4) holds uniformly for all M_n, and moreover the same results as in Assertions 5.2, 5.3, and 5.4 can be derived. In this case, the completion of $M_\infty - \{o\}$ as in Assertion 5.2 can be described as the sphere S^2 equipped with a Riemannian metric g_∞ which is only continuous at the north and/or south poles if $\alpha = 1$ and/or $\beta = 1$.

See [9], [10] and [21] for related results.

9. Further discussions. In this section, we shall first consider a Riemannian submersion with totally geodesic fibers and recall a result due to Besson [6] on a domination of the heat kernel of the total space by those of the base space and the fiber. An example of an SD-convergent family of such metrics on a total space is exhibited. Secondly, we shall review a result of Gallot [16] and discuss a question on Albanese tori in relation to the spectral distance.

9.1. Let M be a compact connected Riemannian manifold. We are given a Riemannian submersion $\pi: M \to B$ of M onto another manifold B with totally geodesic fibers $\{F_b : b \in B\}$. Then the horizontal lift of a piecewise smooth curve $\gamma: [\alpha, \beta] \to B$ in the base manifold B gives rise to an isometry $\mathscr{P}_\gamma$ between the fibers $F_{\gamma(\alpha)}$ and $F_{\gamma(\beta)}$ over the end points $\gamma(\alpha)$ and $\gamma(\beta)$. Hence the holonomy group of the fibration is included in the isometry group of the corresponding fiber. See, e.g., [5, Chap. 9] for some basic results on Riemannian submersions with totally geodesic fibers.

Now we fix a point b_0 of B and denote by F the fiber over b_0. Let $p_M(t, x, x')$, $p_B(t, b, b')$, and $p_F(t, u, u')$ be respectively the heat kernels of M, B and F with respect to the normarized Riemannian measures. Then we have

$$p_M(t, x, x') \le p_B(t, b, b') p_F(t, \mathscr{P}_\gamma(x), \mathscr{P}_\gamma(x))^{1/2} p_F(t, \mathscr{P}_{\gamma'}(x'), \mathscr{P}_{\gamma'}(x'))^{1/2}$$

where $b = \pi(x)$, $b' = \pi(x')$ and $\gamma: [0, 1] \to B$ (resp. $\gamma': [0, 1] \to B$) is a piecewise smooth curve joining b (resp. b') to b_0. This inequality is due to Besson [6] and it enables us to construct SD precompact families of such metrics on the total space in conjunction with (principal) connections.

Let us here exhibit a degenerating family of metrics on the total space M of the fibration $\pi: M \to B$ as above. We denote by g_M, g_B and g_F, respectively, the metrics of M, B and F. For a tangent vector X of M at a point x, X^V and X^H stand respectively for the vertical component of X which is tangent to the fiber through x and the horizontal one which is orthogonal to the fiber. A one-parameter family of metrics $\{g_\varepsilon : \varepsilon > 0\}$ on M is defined by

$$g_\varepsilon(X, Y) = g_M(X^V, Y^V) + \varepsilon^2 g_M(X^H, Y^H) .$$

Then $\pi: (M, g_\varepsilon) \to \varepsilon B = (B, \varepsilon^2 g_B)$ remains to be a Riemannian submersion with the totally geodesic fibers $\{F_b : b \in B\}$. The normalized Riemannian measure of g_ε is independent of ε and we write μ_M for it. Then the heat kernel p_ε of $(M, g_\varepsilon, \mu_M)$ is estimated by

$$p_\varepsilon(t, x, x') \leq p_B(\varepsilon^{-2}t, b, b')p_F(t, \mathscr{P}_\gamma(x), \mathscr{P}_\gamma(x))^{1/2} p_F(t, \mathscr{P}_{\gamma'}(x'), \mathscr{P}_{\gamma'}(x'))^{1/2} .$$

We shall now describe the SD-limit of $(M, g_\varepsilon, \mu_M)$ as $\varepsilon \to 0$. For this, we denote by $\bar{G}$ the closure of the holonomy group of the fibration at a fixed point $b_0 \in B$ in the isometry group of the fiber F. A canonical mapping Π of M onto the quotient space $F/\bar{G}$ is given by

$$\Pi(x) = \rho(\mathscr{P}_\gamma(x)) ,$$

where $\rho : F \to F/\bar{G}$ is the natural projection and γ is a piecewise smooth path joining $\pi(x)$ with a fixed point $b_0 \in B$. The quotient space $F/\bar{G}$ is endowed with the image measure $\mu = \rho_* \mu_F$ of the normarized Riemannian measure μ_F on F. Then the Hilbert space $L^2(F/\bar{G}, \mu)$ on $F/\bar{G}$ is identified with the closure of the space $C^\infty(F; \bar{G})$ of $\bar{G}$-invariant smooth functions on F in the Hilbert space $L^2(F, \mu_F)$. We have now a strongly continuous symmetric Markovian semigroup $\{T_t^{(\bar{G})} : t > 0\}$ on $L^2(F; \bar{G})$ which is associated with the closure of the energy form defined on $C^\infty(F; \bar{G})$. The kernel of $T_t^{(\bar{G})}$ is the pull-back of a positive continuous function $p(t, u, v)$ on $(0, \infty) \times F/\bar{G} \times F/\bar{G}$. Then $(F/\bar{G}, \mu, p)$ is the SD-limit of $(M, g_\varepsilon, \mu_M)$ as $\varepsilon \to 0$. In fact, $\Pi : M \to F/\bar{G}$ and any mapping $\Gamma : F/\bar{G} \to M$ with $\Pi \circ \Gamma = \mathrm{id}.$ provide spectral approximations between $(M, g_\varepsilon, \mu_M)$ and $(F/\bar{G}, \mu, p)$, namely, for all $t > 0$, $x, y \in M$ and $u, v \in F/\bar{G}$,

$$e^{-(t+1/t)} | p_\varepsilon(t, x, y) - p(t, \Pi(x), \Pi(y)) | \leq \delta(\varepsilon) ;$$

$$e^{-(t+1/t)} | p_\varepsilon(t, \Gamma(u), \Gamma(v)) - p(t, u, v) | \leq \delta(\varepsilon) ,$$

where $\lim_{\varepsilon \to 0} \delta(\varepsilon) = 0$. We observe finally that the metric space (M, d_ε) endowed with the Riemannian distance of g_ε converges to the metric space $(F/\bar{G}, d_{\bar{G}})$ with respect to the Gromov-Hausdorff distance, where the distance $d_{\bar{G}}$ on $F/\bar{G}$ is defined by

$$d_{\bar{G}}(u, v) = d_F(\rho^{-1}(u), \rho^{-1}(v)) ,$$

and further the following property holds:

$$\lim_{t \to 0} 4t \log p(t, u, v) = -d_{\bar{G}}(u, v)^2 \qquad u, v \in F/\bar{G} .$$

We have just discussed an SD-convergent example of metrics on a fixed compact manifold endowed with a measure, which features the geometric structure of a Riemannian submersion with totally geodesic fibers. Relevantly to the subjects of [20], [31], and [30], we shall study in [26] the convergence of heat kernels of metrics on a compact manifold endowed with a measure, including the above example as a special case.

9.2. We shall now recall some results by Gallot [16], which are stated as follows. Let $M = (M, g)$ be a compact connected Riemannian manifold of dimension n and μ_M

denote the normalized Riemannian measure of M as before. For a point $x \in M$, we set

$$r(x)=\inf\left\{\frac{1}{n-1}\operatorname{Ric}_M(X, X): X \in T_xM, g(X, X)=1\right\}; \quad r_-(x)=\sup\{0, -r(x)\}.$$

Let α and D be any positive constants and p be any element of $(n, +\infty)$. Suppose that the diameter of M is bounded by D,

$$\operatorname{diam} M \leq D \tag{9.1}$$

and the Ricci curvature satisfies

$$\int_M \left(\frac{r_-}{\alpha^2}-1\right)_+^{p/2} d\mu_M \leq \frac{1}{2}\left(e^{B(p)\alpha D}-1\right)^{-1}, \tag{9.2}$$

where

$$\left(\frac{r_-}{\alpha^2}-1\right)_+=\sup\left\{\frac{r_-}{\alpha^2}-1, 0\right\}; \quad B(p)=\left(\frac{2(p-1)}{p}\right)^{1/2}(n-1)^{1-1/p}\left(\frac{p-2}{p-n}\right)^{1/2-1/p}.$$

Then the i-th eigenvalue λ_i, the heat kernel $p_M(t, x, y)$, and the first Betti number $b_1(M)$ of M are respectively estimated by

$$\lambda_i \geq A(p, \alpha, D)i^{2/p};$$

$$p_M(t, x, y) \leq B(p, \alpha, D)t^{-p/2}+1;$$

$$b_1(M) \leq nZ(p, \alpha, D), \tag{9.3}$$

where $A(p, \alpha, D)$, $B(p, \alpha, D)$ and $Z(p, \alpha, D)$ are respectively computable constants depending only on the given constants. See [16] for details.

We claim here that *under the assumptions* (9.1) *and* (9.2), *the Albanese torus* $\mathscr{A}(M)$ *of* M *satisfies*

$$\operatorname{diam} \mathscr{A}(M) \leq C(n, p, \alpha, D) \operatorname{diam} M$$

for some constant $C(n, p, \alpha, D)$ *depending only on the given constants.*

The Albanese torus $\mathscr{A}(M)$ of a compact connected Riemannian manifold $M=(M, g)$ is defined as follows. Let us first denote by $\mathscr{H}^1(M, \boldsymbol{R})$ the space of harmonic one forms on M equipped with an inner product $(\ ,\ )_{\mu_M}$,

$$(\omega, \eta)_{\mu_M}=\int_M \langle\omega, \eta\rangle_g d\mu_M.$$

Let $\mathscr{H}^1(M, \boldsymbol{Z})$ be a lattice of $\mathscr{H}^1(M, \boldsymbol{R})$ which consists of harmonic one forms with integral periods. Dividing the dual space $\mathscr{H}^1(M, \boldsymbol{R})^*$ by the dual lattice $\mathscr{H}^1(M, \boldsymbol{Z})^*$, we obtain a flat torus, called the Albanese torus of M:

$$\mathscr{A}(M)=\mathscr{H}^1(M, \boldsymbol{R})^*/\mathscr{H}^1(M, \boldsymbol{Z})^*.$$

The result claimed as above was proved in [24] under (9.1) and the condition that the Ricci curvature of M is bounded from below by $-(n-1)\alpha^2$, and in fact the same argument there is valid under the assumptions (9.1) and (9.2). The estimate above implies particularly that the Albanese tori of compact connected Riemannian n-manifolds M satisfying (9.1) and (9.2) form a precompact family of flat tori with dimension less than or equal to $nZ(p, \alpha, D)$, where the topologies of the spectral distance and the Gromov-Hausdorff distance coincide. For an SD-Cauchy sequence $\{M_i : i=1, 2, \dots\}$ of compact connected Riemannian n-manifolds satisfying (9.1) and a stronger condition that the sectional curvature of M_i is uniformly bounded in its absolute values, it was also proved in [24] that the corresponding sequence of the Albanese tori $\mathscr{A}(M_i)$ converges to a point or a flat torus of positive dimension as $i\to\infty$. It might be asked whether this would be true under (9.1) and a much weaker condition (9.2). Relevantly, a question could be raised concerning the continuity of the energy spectrum of harmonic mappings into nonpositively curved manifolds with respect to the spectral distance. See [23, Section 4] for related results.

As seen in Section 7, a bound for the Betti numbers as in (9.3) can not be expected in general for an SD-precompact family. We remark also that an estimate for the diameters of Albanese tori as in the claim above does not hold in general. Indeed, we can see such examples in Section 5 as follows. Let M be a compact, connected and oriented manifold of dimension $n\geq 3$ such that the first Betti number $r=b_1(M)$ is greater than or equal to one. Let $\{c_i : i=1, \dots, r\}$ be a basis of the first de Rham cohomology group $H^1_{\mathrm{deR}}(M)$. Then we choose $(n-1)$-cycles $\{s_i : i=1, \dots, r\}$ in such a way that s_i is the Poincaré dual to the class c_i for each i and fix an open subset U of M which includes the union of the cycles s_i. We may assume by the localization principle that each c_i is represented by a closed 1-form η_i, the support of which is contained in U. Now we suppose that M admits a Riemannian metric g_0 such that the scalar curvature of g_0 is positive. Then as in 5.2, we can find an SD-Cauchy sequence of conformal metrics $g_k=\phi_k^{4/(n-2)}g_0$ such that ϕ_k converges uniformly to a continuous function ϕ_∞ which is positive outside U and vanishes on the union of the supports of the forms η_i. Then the norm $(\eta_i, \eta_i)_{\mu_k}$ of η_i with respect to the metric g_k decays to zero as $k\to\infty$. Indeed,

$$
\begin{aligned}
(\eta_i, \eta_i)_{\mu_k} &= \int_M \langle \eta_i, \eta_i \rangle_{g_k} d\mu_k \quad (\mu_k = dv_{g_k}/\mathrm{Vol}(M, g_k)) \\
&= \frac{1}{\mathrm{Vol}(M, g_k)} \int_U \langle \eta_i, \eta_i \rangle_{g_0} \phi_k^2 dv_{g_0} \to 0 \quad \text{as } k\to\infty .
\end{aligned}
$$

In particular, the norm of the harmonic part $\xi_{i,k}$ of η_i, $(\xi_{i,k}, \xi_{i,k})_{\mu_k}$, tends to zero as $k\to\infty$. Therefore the dual torus of $\mathscr{A}(M, g_k)$ collapses to a point as $k\to\infty$. In other words, the Albanese torus $\mathscr{A}(M, g_k)$ itself diverges (or converges to r-dimensional Euclidean space $\boldsymbol{R}^r$ with respect to the pointed Hausdorff distance) and in fact

$$\operatorname{diam}\mathscr{A}(M, g_k) \to +\infty \quad \text{as } k \to \infty .$$

We note that the pointwise norm $\langle \xi_{i,k}, \xi_{i,k} \rangle_{g_k}$ converges to zero outside U, but its maximun value goes to infinity as $k \to \infty$, because $\xi_{i,k}$ belongs to $\mathscr{H}^1(M, g_k, \boldsymbol{Z})$.

9.3. Let us close with an observation suggested by Akutagawa [2]. Let $\mathscr{M}_1(n)$ be the set of isometry classes of compact connected Riemannian manifolds with dimension n and unit volume. Given positive constants, a, τ, Λ and integer $p > n/2$, we consider a subset $\mathscr{S}$ of $\mathscr{M}_1(n)$ such that for a manifold $M = (M, g) \in \mathscr{S}$, the heat kernel $p_M(t, x, y)$ of M satisfies

$$p_M(t, x, x) \leq \frac{a}{t^{n/2}}, \qquad 0 < t \leq \tau, \quad x \in M,$$

and further the curvature tensor R_M of M has a bound of the form

$$\int_M |R_M|^p dv_g \leq \Lambda .$$

To this class $\mathscr{S}$ of manifolds, we can apply some results by Anderson [3, Section 3] together with Lemma 2.5, and we obtain a proposition stated as follows: *the set $\mathscr{S}$ above is precompact in $C^\alpha \cap L^{2,p}$ topology, $\alpha = 2 - n/2$; to be precise, for a sequence $\{(M_i, g_i)\}$ in $\mathscr{S}$, there exist a subsequence $\{M_j\}$, a compact smooth n-manifold X equipped with $C^\alpha \cap L^{2,p}$-metric g_X, and a diffeomorphism $h_j \colon X \to M_j$ for large j such that $h_j^* g_j$ converges to g_X in the $C^{\alpha'}$ topology for $\alpha' < \alpha$ and weakly in the $L^{2,p}$ topology on X.* Thus on this set $\mathscr{S}$, the topologies of the spectral distance and the Gromov-Hausdorff distance coincide and in fact they are expressed in a finer manner as above. We refer the reader to [3], [2] and the references therein for details and releted topics.

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