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EXISTENCE, UNIQUENESS AND ASYMPTOTIC STABILITY OF PERIODIC SOLUTIONS OF PERIODIC FUNCTIONAL DIFFERENTIAL SYSTEMS

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Abstract. We consider here a general Lotka-Volterra type n-dimensional periodic functional differential system. Sufficient conditions for the existence, uniqueness and global asymptotic stability of periodic solutions are established by combining the theory of monotone flow generated by FDEs, Horn's asymptotic fixed point theorem and linearized stability analysis. These conditions improve and generalize the recent ones obtained by Freedman and Wu (1992) for scalar equations. We also present a nontrivial application of our results to a delayed nonautonomous predator-prey system.

1. Introduction. The n-dimensional Lotka-Volterra system takes the following form:

$$\dot{x}_i(t)=x_i(t)\left(b_i+\sum_{j=1}^{n} a_{ij}x_j(t)\right), \tag{1.1}$$

where the dot denotes the differentiation with respect to t and b_i, a_{ij} $(i,j=1,\dots,n)$ are constants. This system has long played an important role in mathematical population biology (cf. Hofbauer and Sigmund [15]). However, realistic models often require the inclusion of effects of time delays and the changing environment. This leads us to the study of the following more general nonautonomous Lotka-Volterra system with time delay:

$$\dot{x}_i(t)=x_i(t)\left(b_i(t)+\sum_{j=1}^{n} a_{ij}(t)x_j(t)+\sum_{j=1}^{n} c_{ij}(t)x_j(t-\tau_{ij}(t))\right), \tag{1.2}$$

where $i=1,\dots,n$. Biologically, due to the difference in species, $\tau_{ij}(t)$ are usually different. However, to minimize the technical complexity in the presentation of our results, we assume in the rest of this paper that $\tau_{ij}(t)=\tau(t)$. Nevertheless, we would like to men-

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tion here that our methods and results in this paper can be applied to the system (1.2) even if τ_{ij} are different. This can be seen from the proofs of the theorems to be presented (proper adjustments are necessary).

In this paper, we assume that the environment changes periodically (seasonal changes), which leads us to assume that $b_i(t)$, $a_{ij}(t)$ and $c_{ij}(t)$ $(i, j=1, \ldots, n)$ and $\tau(t)$ are continuously differentiable ω-periodic functions. In addition, we assume that $b_i(t)>0$, $a_{ii}(t)<0$, $\tau(t)\geq 0$ for $t\in \boldsymbol{R}=(-\infty, +\infty)$. In fact, we will consider the more general system

$$\dot{x}_i(t)=x_i(t)F_i(t, x_1(t), \ldots, x_n(t), x_1(t-\tau(t)), \ldots, x_n(t-\tau(t))) , \tag{1.3}$$

where F_i are periodic with respect to t for $i=1, \ldots, n$. Our objective is to establish conditions for the existence and uniqueness of nonconstant periodic solutions in (1.3).

Almost all of the existing works on the system (1.2) are concerned with the stability of its equilibria. Results for scalar equations can be found in Haddock and Kuang [7], Kuang [17] and the references cited therein. Results for systems can be found in Gopalsamy [6], Kuang [17], Kuang and Smith [18], Martin and Smith [19] and the references cited therein.

Motivated by the laboratory work of the group led by Halbach (see [2], [8]–[11], [20], [25] and the references cited therein), Freedman and Wu [5] studied the existence of periodic solutions of the scalar equation

$$\dot{x}(t)=x(t)[a(t)-b(t)x(t)+c(t)x(t-\tau(t))] , \tag{1.4}$$

where $a(t)$, $b(t)$, $c(t)$ and $\tau(t)$ are continuously differentiable ω-periodic functions, and $a(t)>0$, $b(t)>0$, $c(t)\geq 0$, $\tau(t)\geq 0$ for $t\in \boldsymbol{R}$. In this paper, we will greatly improve and generalize their results to the system (1.3) by combining the theory of monotone semiflows generated by functional differential equations (FDEs) and Horn's asymptotic fixed point theorem. In addition, by a careful local stability analysis of the linearized system, we are able to obtain sufficient conditions for the uniqueness and asymptotic stability of periodic solutions of the system (1.3).

It should be pointed out that both our methods and results are somewhat different from those existing ones (e.g., see [5], [1] and [14]). To some extent, our results indicate that the uniform persistence and uniform boundedness (see the condition (A4) in Section 3) imply the existence of periodic solutions in the system (1.3). For the definitions of persistence and related notions, see [3], [4], [13], [24] and the references cited therein.

The theory of monotone semiflows generated by functional differential equations has been developed by Smith [21], Smith and Thieme [22] and others. For details we refer to [21], [22] and the references cited therein.

This paper is organized as follows. In the next section, we present some preliminaries. The existence results for periodic solutions of (1.3) are given in Section 3. Section 4 is devoted to the uniqueness theorem for periodic solutions of (1.3). The final section contains applications of our main results to some well-known systems.

2. Preliminaries. In order to apply the theory of monotone semiflows generated by FDEs, we consider first the following auxiliary nonautonomous delay system

$$\dot{x}_i(t)=x_i(t)G_i(t, x_1(t), \dots, x_n(t), x_1(t-\tau(t)), \dots, x_n(t-\tau(t))), \tag{2.1}$$

where $i=1, \dots, n$, $x=(x_1, \dots, x_n)\in \boldsymbol{R}^n_+=\{x\in \boldsymbol{R}^n : x_i\geq 0\}$, and $\mathring{\boldsymbol{R}}^n_+=\{x\in \boldsymbol{R}^n_+ : x_i>0\}$. We assume in (2.1),

(H1) $\partial G_i(t, x_1, \dots, x_n, y_1, \dots, y_n)/\partial x_j>0$ for $j\neq i$ and $\partial G_i(t, x_1, \dots, x_n, y_1, \dots, y_n)/\partial y_k>0$, $i, j, k=1, \dots, n$.

(H2) There is a $p=(p_1, \dots, p_n)\in \mathring{\boldsymbol{R}}^n_+$ such that

$$G_i(t, p_1, \dots, p_n, p_1, \dots, p_n)<0 \quad \text{for } t\in \boldsymbol{R}, \qquad i=1, \dots, n.$$

It is easy to see that (H1) guarantees that the system (2.1) generates monotone semiflows while (H2) ensures some solutions are bounded by p (see Lemma 2.2 below).

Suppose $\tau(t)$ is continuously differentiable, nonnegative and bounded above by τ^*. We define $C^n=\{\phi(\theta)=(\phi_1(\theta), \dots, \phi_n(\theta)) : \phi_i(\theta)\in C([-\tau^*, 0], \boldsymbol{R}), i=1, \dots, n\}$. For $\phi\in C^n$, the norm of ϕ is defined by $\|\phi\|=\sum_{i=1}^n \max_{-\tau^*\leq\theta\leq 0}|\phi_i(\theta)|$. For $x, y\in \boldsymbol{R}^n$, $x\geq y$ means $x_i\geq y_i$ and $x>y$ means $x_i>y_i$ for $1\leq i\leq n$. If $\phi, \psi\in C^n$, we write $\phi\leq\psi$ ($\phi<\psi$) if the indicated inequality holds pointwise, with the above partial ordering on $\boldsymbol{R}^n$. *In the rest of this paper we always assume that* $\phi(\theta)\geq 0$, *for* $\theta\in[-\tau^*, 0]$, $\phi(0)>0$. It is a well-known fact that for any given $\phi\in C^n$ with $\phi(\theta)\geq 0$, for $\theta\in[-\tau^*, 0]$, $\phi(0)>0$, there exists $\alpha\in(0, \infty)$ and a unique solution $x(t)=x(t, \phi)$ of the equation (2.1) on $[-\tau^*, \alpha)$ satisfying $x(\theta)=\phi(\theta)$, $\theta\in[-\tau^*, 0]$, and $x(t)>0$ for $t\in[0, \alpha)$. In other words, such a solution is unique and stays positive.

In order to establish conditions for the existence of periodic solutions in the system (1.3), we need the following lemmas.

LEMMA 2.1 (Smith [21]). *Let Ω be an open subset of $\boldsymbol{R}\times C^n$, and let $f, g : \Omega\to\boldsymbol{R}^n$ be continuous. Suppose that the system*

$$\dot{x}(t)=g(t, x_t) \tag{2.2}$$

generates a monotone semiflow. Assume $f(t, \phi)\leq g(t, \phi)$ for all $(t, \phi)\in\Omega$. If (t_0, ϕ), $(t_0, \psi)\in\Omega$ with $\phi\leq\psi$, then

$$x(t, t_0, \phi, f)\leq x(t, t_0, \psi, g)$$

for all $t\geq t_0$ for which both are defined, where $x(t, t_0, \phi, f)$ is a solution of

$$\dot{x}=f(t, x_t) \tag{2.3}$$

and $x(t, t_0, \phi, g)$, a solution of (2.2).

LEMMA 2.2. *Suppose that* (H1) *and* (H2) *hold. Then for any solution $x(t, \phi)$ of* (2.1) *with* $x(t_0+\theta, \phi)\le p$ *for* $\theta\in[-\tau^*, 0]$ *and some* $t_0\ge 0$, *there is a* $T>0$ *such that for* $t\ge T$,

$$x(t, \phi)<p .$$

PROOF. First we notice that if $x_i(t_1, \phi)=p_i$, $x_i(t, \phi)\le p_i$ and $x_j(t, \phi)\le p_j$ for $t\in[t_1-\tau^*, t_1]$, then

$$\begin{aligned}\dot{x}_i(t_1, \phi)&=p_iG_i(t_1, x_1(t_1, \phi), \dots, x_i(t_1, \phi), \dots, x_n(t_1, \phi), x_1(t_1-\tau(t_1), \phi), \dots, x_n(t_1-\tau(t_1), \phi))\\ &\le p_iG_i(t_1, p_1, \dots, p_i, \dots, p_n, p_1, \dots, p_n)<0 ,\end{aligned}$$

and so there is an $\varepsilon_1>0$ such that $x_i(t, \phi)<p_i$ for $t\in(t_1, t_1+\varepsilon_1)$. Hence there is an $\varepsilon>0$ such that $x(t, \phi)<p$ for $t\in(t_1, t_1+\varepsilon)$, where $t_1\ge t_0$.

We claim that there is a $T>0$ such that $x(t, \phi)<p$ for $t\ge T$. If not, there must be a $t_2>t_0$ such that for some i,

$$\begin{aligned}&x_i(t_2, \phi)=p_i , \quad x_i(t, \phi)\le p_i , \qquad t\in(t_0-\tau^*, t_2) ,\\ &x_i(t, \phi)<p_i \qquad \text{for} \quad t\in(t_2-\delta, t_2) \quad \text{and}\\ &x_j(t, \phi)\le p_j \qquad \text{for} \quad t\in[t_0-\tau^*, t_2] ,\end{aligned}$$

where δ is sufficiently small, and

$$\dot{x}_i(t_2, \phi)\ge 0 . \tag{2.4}$$

But (H1) yields that

$$\dot{x}_i(t_2, \phi)\le p_iG_i(t, p_1, \dots, p_i, \dots, p_n, p_1, \dots, p_n)<0 ,$$

a contradiction to (2.4). □

We need the following two assumptions in our next lemma.

(H3) $G_i(t, \lambda x_1, \dots, \lambda x_n, \lambda y_1, \dots, \lambda y_n)\ge \lambda G_i(t, x_1, \dots, x_n, y_1, \dots, y_n)$ for $\lambda\in(0, 1]$, where $i=1, \dots, n$.

(H4) $G_i(t, x_1, \dots, x_n, y_1, \dots, y_n)$ is uniformly continuous with respect to $(x_1, \dots, x_n, y_1, \dots, y_n)$ and $G_i(t+\omega, x_1, \dots, x_n, y_1, \dots, y_n)=G_i(t, x_1, \dots, x_n, y_1, \dots, y_n)$ for some $\omega>0$ and $i=1, \dots, n$.

Note that (H3) is equivalent to

(H3)′ $\bar{\lambda}G_i(t, x_1, \dots, x_n, y_1, \dots, y_n)\ge G_i(t, \bar{\lambda}x_1, \dots, \bar{\lambda}x_n, \bar{\lambda}y_1, \dots, \bar{\lambda}y_n)$ for $\bar{\lambda}\in[1, +\infty)$, $i=1, \dots, n$.

(H3) implies that G is concave in its variables (except the time variable). It is easy to check that if G is linear with respect to its variables except the time variable, then (H3) and (H4) are satisfied.

LEMMA 2.3. *Suppose that the system* (2.1) *satisfies* (H1)–(H4). *Then*:

(i) *For any* $\eta\in\mathring{\boldsymbol{R}}^n_+$, *there exists and* $M(\eta)\in\mathring{\boldsymbol{R}}^n_+$ *such that for any* $\phi\in C^n$ *with* $0\le\phi\le\eta$ *on* $[-\tau^*, 0]$, *one has* $0\le x(t, \phi)\le M(\eta)$ *for all* $t\ge 0$.

(ii) *There exists a* $\Delta\in\mathring{\boldsymbol{R}}^n_+$ *such that for any* $\alpha\in\boldsymbol{R}^n_+$, *there is a constant* $T=T(\alpha)>0$

such that for any $\phi \in C^n$ *with* $0 \le \phi \le \alpha$ *on* $[-\tau^*, 0]$, *one has* $0 \le x(t, \phi) \le \Delta$ *for all* $t \ge T(\alpha)$.

PROOF. By (H2) and (H3), we have

$$G_i(t, \bar{\lambda} p_1, \dots, \bar{\lambda} p_n, \bar{\lambda} p_1, \dots, \bar{\lambda} p_n) \le \bar{\lambda} G_i(t, p_1, \dots, p_n, p_1, \dots, p_n) < 0 \tag{2.5}$$

for $i = 1, \dots, n$ and $\bar{\lambda} \in [1, +\infty)$.

(i) For any $\eta \in \mathring{\boldsymbol{R}}^n_+$, there exists a $\bar{\lambda}_0 \in [1, +\infty)$ such that $\eta \le \bar{\lambda}_0 p$ and the conclusion follows from Lemma 2.2.

(ii) Let $\Delta = p$. If $\alpha \le p$, then Lemma 2.2 implies the conclusion. Suppose that $\alpha > p$. Then there is $\bar{\lambda}_0 \in (1, +\infty)$ such that $\alpha \le \bar{\lambda}_0 p$. Let $\phi \in C^n$ with $0 \le \phi \le \alpha$ on $[-\tau^*, 0]$ and let $x(t, \phi)$ be a solution. Then Lemma 2.2 yields that there is a $T_1 > 0$ such that $T_1 - \tau^* > 0$ and $x(t, \phi) < \bar{\lambda}_0 p$ for $t \ge T_1 - \tau^*$. Let $y_i^1 = \max_{T_1 - \tau^* \le t \le T_1} x_i(t, \phi)$ and $y^1 = (y_1^1, \dots, y_n^1)$. Then $y^1 < \bar{\lambda}_0 p$. If $y^1 < p$, then we are done by Lemma 2.2. Suppose $y^1 < p$ is not true. Then we can find $\bar{\lambda}_1 \in (1, \bar{\lambda}_0)$ such that $y^1 \le \bar{\lambda}_1 p$. In this case, Lemma 2.2 implies that there is a $T_2 > 0$ such that $T_2 - \tau^* > T_1$, and $x(t, \phi) < \bar{\lambda}_1 p$ for $t \ge T_2 - \tau^*$. Let $y_i^2 = \max_{T_2 - \tau^* \le t \le T_2} x_i(t, \phi)$ and $y^2 = (y_1^2, \dots, y_n^2)$. Then $y^2 < \bar{\lambda}_1 p$. Repeating the above procedure, we have two possibilities:

(i) For some $T > 0$, we have $x(t, \phi) < p$ for all $t \ge T$;

(ii) There exist two sequences $\{T_m\}$ and $\{\bar{\lambda}_m\}$ such that $x(t, \phi) < \bar{\lambda}_m p$ for $t \ge T_m - \tau^*$, $\bar{\lambda}_m \in (1, \bar{\lambda}_0)$ and $\lim_{m \to \infty} T_m = +\infty$.

In the following, we assume that (ii) is true. Without loss of generality, we suppose that $\lim_{m \to +\infty} \bar{\lambda}_m = \beta$. By (H4), we can find a $\delta_0 > 0$ such that for $i = 1, \dots, n$,

$$\begin{aligned} &G_i(t, (\beta + \delta_0) p_1, \dots, \beta p_i, (\beta + \delta_0) p_1, \dots, (\beta + \delta_0) p_n) \\ &\qquad \le G_i(t, \beta p_1, \dots, \beta p_i, \dots, \beta p_n, \beta p_1, \dots, \beta p_n) + \varepsilon \mathbf{1} < 0, \end{aligned} \tag{2.6}$$

where $0 < \varepsilon < -\min_{1 \le i \le n} \min_{t \in [0, w]} G_i(t, \beta p_1, \dots, \beta p_i, \dots, \beta p_n, \beta p_1, \dots, \beta p_n)$ and $\mathbf{1} = (1, \dots, 1)$.

If there is a $T_0 > 0$ such that $x_i(t, \phi) \le \beta p_i$ for all $t \ge T_0$, then by Lemma 2.2, we know that there is another $\bar{T} > T_0 + \tau^*$ such that $x_i(t, \phi) < \beta p_i$ for all $t \ge \bar{T}$, which contradicts the definition of β. Hence, there is an $i \in \{1, \dots, n\}$ and a sequence $\{t_k\}$ with $\lim_{k \to +\infty} t_k = +\infty$ such that $x_i(t_k, \phi) = \beta p_i$ and $\dot{x}_i(t_k, \phi) \ge 0$. We choose an $m_0 > 0$ such that $\bar{\lambda} m_0 < (\beta + \delta_0)$ and $x(t, \phi) < \bar{\lambda}_{m_0} p$ for $t \ge T_{m_0} - \tau^*$. If we choose a $k_0 > 0$ such that $t_{k_0} > T_{m_0}$, then (H1) and (2.6) imply that

$$\begin{aligned} \dot{x}_i(t_{k_0}, \phi) &= \beta p_i G_i(t_{k_0}, x_1(t_{k_0}), \dots, x_i(t_{k_0}), \dots, x_n(t_{k_0}), x_1(t_{k_0} - \tau(t_{k_0})), \dots, x_n(t_{k_0} - \tau(t_{k_0}))) \\ &\le \beta p_i G_i(t_{k_0}, (\beta + \delta_0) p_1, \dots, \beta p_i, \dots, (\beta + \delta_0) p_n, (\beta + \delta_0) p_1, \dots, (\beta + \delta_0) p_n) < 0, \end{aligned}$$

a contradiction. This shows that (ii) cannot be true and hence (i) holds.

Let $\psi(\theta) = \alpha$. Then there is a $T(\alpha) > 0$ such that $x(t, \psi) < p$ for all $t \ge T(\alpha)$. The monotonicity of the solution flow of the system (2.1) implies the conclusion of the lemma. □

It is easy to see that Lemma 2.3 remains true if (H3) is replaced by the assumption

(H3)″ $G_i(t, f_1(\lambda)x_1, \ldots, f_n(\lambda)x_n, f_1(\lambda)y_1, \ldots, f_n(\lambda)y_n) \geq g_i(\lambda)G_i(t, x_1, \ldots, x_n, y_1, \ldots, y_n)$ for $\lambda \in [0, 1]$ and $i=1, \ldots, n$, where f_i, g_i: $[0, 1] \to [0, 1]$ satisfy that $f_i(0)=0$, $g_i(0)=0$, $f_i(1)=g_i(1)=1$ and f_i, g_i are nondecreasing $i=1, \ldots, n$.

We will use the following result from [16] to establish the existence of an ω-periodic solution in (2.1) in the following section.

LEMMA 2.4 (Horn's fixed-point theorem). *Let $S_0 \subset S_1 \subset S_2$ be convex subsets of a Banach space X, with S_0 and S_2 compact and S_1 open relative to S_2. Let P: $S_2 \to X$ be a continuous mapping such that, for some integer $m>0$,*

(a) $P^j(S_1) \subseteq S_2$, $1 \leq j \leq m-1$

and

(b) $P^j(S_1) \subseteq S_0$, $m \leq j \leq 2m-1$.

Then P has a fixed point in S_0.

3. Existence of periodic solutions. We now return to the discussion of the system (1.3), namely,

$$\dot{x}_i(t) = x_i(t)F_i(t, x(t), \ldots, x_n(t), x_1(t-\tau(t)), \ldots, x_n(t-\tau(t))) , \tag{3.1}$$

where $x=(x_1, \ldots, x_n) \in \boldsymbol{R}^n_+$. We make the following assumptions on the system (3.1):

(A1) For $1 \leq i \leq n$, $F_i(t, x_1, \ldots, x_n, y_1, \ldots, y_n)$ is continuously differentiable with respect to its variables;

(A2) There is an $\omega>0$ such that for $i=1, \ldots, n$,

$$F_i(t+\omega, x_1, \ldots, x_n, y_1, \ldots, y_n) = F_i(t, x_1, \ldots, x_n, y_1, \ldots, y_n) .$$

(A3) $\tau(t)$ is a continuously differentiable ω-periodic function and $\tau(t) \geq 0$ for $t \in \boldsymbol{R}$. We denote $\tau^* = \max_{0 \leq t \leq \omega} \tau(t)$.

(A4) (i) For any $\eta_1, \eta_2 \in \boldsymbol{R}^n_+$ with $0 \neq \eta_1 \leq \eta_2$, there exists $\gamma(\eta_1, \eta_2) \in \mathring{\boldsymbol{R}}^n_+$ such that for any $\phi \in C^n$ with $\eta_1 \leq \phi(\theta) \leq \eta_2$ on $[-\tau^*, 0]$, one has

$$x(t, \phi) \geq \gamma(\eta_1, \eta_2) \qquad \text{for all} \quad t \geq 0 ;$$

(ii) There exists $\tilde{\delta} \in \mathring{\boldsymbol{R}}^n_+$ such that if $\phi(\theta) \in C^n$, $\phi(\theta) \geq 0$, and $\phi(0)>0$, then $\liminf_{t \to +\infty} x(t, \phi) \geq \tilde{\delta}$, where $\liminf_{t \to +\infty} x(t, \phi) = (\liminf_{t \to +\infty} x_1(t, \phi), \ldots, \liminf_{t \to +\infty} x_n(t, \phi))$;

(A5) For any $K>0$, there exists an $L(K)>0$ such that if $\|\phi\| \leq K$, then $|F(t, \phi)| = \sum_{i=1}^n |F_i(t, \phi)| \leq L(K)$ for $t \in \boldsymbol{R}$;

(A6) There are G_i : $R \times R^{2n} \to R$ such that

$$\begin{aligned} &F_i(t, x_1(t), \ldots, x_n(t), x_1(t-\tau(t)), \ldots, x_n(t-\tau(t))) \\ &\quad \leq G_i(t, x_1(t), \ldots, x_n(t), x_1(t-\tau(t)), \ldots, x_n(t-\tau(t))) \qquad \text{for} \quad i=1, \ldots, n , \end{aligned}$$

where $G_i(t, x_1(t), \ldots, x_n(t), x_1(t-\tau(t)), \ldots, x_n(t-\tau(t)))$ $(i=1, \ldots, n)$ satisfies the assumptions (H1)–(H4).

We would like to point out that the assumptions (A1), (A2), (A3) and (A5) are almost always satisfied by those systems studied in the literature [6], [17], since most of them are Lotka-Volterra type systems. For Lotka-Volterra type systems, F_i are simply linear functions of x_t, so (A1), (A5) are satisfied as long as the coefficient functions are bounded. (A4) (ii) is in fact the definition of the so-called uniform persistence [13], [17]. For Lotka-Volterra type systems (A4) (ii) is often satisfied. (A4) (i) and (A6) are technical.

LEMMA 3.1. *Suppose that the system* (3.1) *satisfies* (A1)–(A6), *and* G_i $(i=1,\ldots,n)$ *satisfies* (H1)–(H4). *Then there are* $\bar{\delta}, \bar{\Delta} \in \mathring{\boldsymbol{R}}^n_+$ *such that for any* $\eta_1, \eta_2 \in \mathring{\boldsymbol{R}}^n_+$ *with* $\eta_1 \le \eta_2$ *and any compact subset* S *of* C^n *which satisfies that* $S \subset \{\phi(\theta) \in C^n : \eta_1 \le \phi(\theta) \le \eta_2\}$, *there exists a constant* $T = T(\eta_1, \eta_2, S) > 0$ *such that for any* $\phi \in S$, *one has*

$$\bar{\Delta} \ge x(t, \phi) \ge \bar{\delta} \qquad \text{for all} \quad t \ge T .$$

PROOF. Let $x(t, \phi)$ be the solution of (3.1) and $u(t, \psi)$ the solution of (2.1) with $\phi \le \psi$. Then Lemma 2.1 implies that $x(t, \phi) \le u(t, \psi)$ for all $t \ge 0$. By Lemma 2.3 and (A4) (i), we have:

(i)′ For any $\eta_1, \eta_2 \in \mathring{\boldsymbol{R}}^n_+$ with $\eta_1 \le \eta_2$, there exists $\gamma(\eta_1, \eta_2)$, $M(\eta_1, \eta_2) \in \mathring{\boldsymbol{R}}^n_+$ such that for any $\phi \in C^n$ with $\eta_1 \le \phi(\theta) \le \eta_2$ on $[-\tau^*, 0]$, one has $\gamma(\eta_1, \eta_2) \le x(t, \phi) \le M(\eta_1, \eta_2)$ for all $t \ge 0$;

(ii)′ There exists $\tilde{\Delta} \in \mathring{\boldsymbol{R}}^n_+$ such that for any $\eta_1, \eta_2 \in \mathring{\boldsymbol{R}}^n_+$ with $\eta_1 \le \eta_2$, there is a constant $T_1 = T_1(\eta_1, \eta_2) > 0$ such that for any $\phi \in C^n$ with $\eta_1 \le \phi(\theta) \le \eta_2$ on $[-\tau^*, 0]$, one has $x(t, \phi) \le \tilde{\Delta}$ for all $t \ge T_1$.

Let $\tilde{\delta}$ be given in (A4) (ii) and $\tilde{\Delta}$ be given in (ii)′. By (i)′, we can find $\bar{\delta}, \bar{\Delta} \in \mathring{\boldsymbol{R}}^n_+$ with $\bar{\delta} < \tilde{\delta}/2 < \tilde{\Delta} \le \bar{\Delta}$ such that for any $\phi \in C^n$ with $\tilde{\delta}/2 \le \phi(\theta) \le \tilde{\Delta}$ on $[-\tau^*, 0]$, one has $\bar{\delta} \le x(t, \phi) \le \bar{\Delta}$ for all $t \ge 0$.

Let $\eta_1, \eta_2 \in \mathring{\boldsymbol{R}}^n_+$ and S be a compact subset of C^n with $S \subset \{\phi(\theta) \in C^n : \eta_1 \le \phi(\theta) \le \eta_2\}$.

By (ii)′, there is $\tilde{T} > 0$ such that for any $\phi \in C^n$ with $\eta_1 \le \phi(\theta) \le \eta_2$ on $[-\tau^*, 0]$, one has $x(t, \phi) \le \tilde{\Delta}$ for $t \ge \tilde{T}$.

For $\phi \in S$, (A4) (ii) implies that there is an integer $m(\phi) > 0$ such that $m(\phi)\omega > \tilde{T}$ and $x(t, \phi) \ge 2\tilde{\delta}/3$ for $t \ge m(\phi)\omega$. The continuous dependence on the initial conditions yields that there is a $\delta(\phi) > 0$ such that if $\psi \in C^n$ and $\|\psi - \phi\| < \delta(\phi)$, then $x(t, \psi) \ge \tilde{\delta}/2$ for $t \in [0, m(\phi)\omega + \tau^*]$. Since $x_{t+m(\phi)\omega}(\psi) = x_t(x_{m(\phi)\omega}(\psi))$, we have $\bar{\delta} \le x(t, \psi) \le \bar{\Delta}$ for all $t \ge m(\phi)\omega$. Since $\bigcup_{\phi \in S} \{\psi \in C^n : \|\psi - \phi\| < \delta(\phi)\} \supset S$, the compactness of S implies that $\bigcup_{i=1}^k \{\psi \in C^n : \|\psi - \phi_i\| < \delta(\phi_i), \phi_i \in S\} \supset S$. Let $T(\eta_1, \eta_2, S) \ge \max_{1 \le i \le k} \{m(\phi_i)\omega\}$. Then for any $\psi \in S$, there is an $i \in \{1, \ldots, k\}$ such that $\|\psi - \phi_i\| < \delta(\phi_i)$ and so $\bar{\delta} \le x(t, \psi) \le \bar{\Delta}$ for all $t \ge m(\phi_i)\omega$. Clearly, $\bar{\delta} \le x(t, \psi) \le \bar{\Delta}$ for all $t \ge T(\eta_1, \eta_2, S)$. □

Now we are in a position to state and prove our main result of this section.

THEOREM 3.1. *Suppose that the system* (3.1) *satisfies* (A1)–(A6) *and* G_i $(i=1,\ldots,n)$ *satisfies* (H1)–(H4). *If the system* (3.1) *has no positive constant solution, then it has a* *nonconstant positive ω-periodic solution.*

PROOF. Let $x(t, \phi)$ be a solution of (3.1). Let $\bar{\delta}$, $\bar{\Delta}$ be given in Lemma 3.1. By (i)$'$ in the proof of Lemma 3.1, we can find $\gamma_i \in \mathring{\boldsymbol{R}}^n_+$, $\Delta_i \in \mathring{\boldsymbol{R}}^n_+$ $(i=1, 2, 3)$ with $\gamma_3 < \gamma_2 < \gamma_1 < \bar{\delta} < \bar{\Delta} + 2 \cdot \mathbf{1} < \Delta_1 + \mathbf{1} < \Delta_2 < \Delta_3$ such that

$$\bar{\delta} \le \phi(\theta) \le \bar{\Delta} + \mathbf{1} \text{ on } [-\tau^*, 0] \text{ implies that } \gamma_1 \le x(t, \phi) \le \Delta_1 \quad \text{for all } t \ge 0;$$

$$\gamma_1 \le \phi(\theta) \le \Delta_1 + \mathbf{1} \text{ on } [-\tau^*, 0] \text{ implies that } \gamma_2 \le x(t, \phi) \le \Delta_2 \quad \text{for } t \ge 0;$$

$$\gamma_2 \le \phi(\theta) \le \Delta_2 \text{ on } [-\tau^*, 0] \text{ implies that } \gamma_3 \le x(t, \phi) \le \Delta_3 \quad \text{for } t \ge 0.$$

For Δ_3, (A5) yields that there is an $L > 0$ such that for $\|\phi\| \le |\Delta_3| = \sum_{i=1}^n \delta_i$, where $\Delta_3 = (\delta_1, \ldots, \delta_n)$, one has $\sum_{i=1}^n |F_i(t, \phi)| \le L$.

Define

$$S = \left\{ \phi \in C^n : \gamma_1 \le \phi(\theta) \le \Delta_1 + \mathbf{1}, \sum_{i=1}^n |\phi_i(\theta) - \phi_i(\xi)| \le nL |\theta - \xi| \text{ for } \theta, \xi \in [-\tau^*, 0] \right\}.$$

By repeated application of Arzela-Ascoli's theorem, we conclude that S is compact. Lemma 3.1 implies that there is a constant $T = T(\gamma_1, \Delta_1 + \mathbf{1}, S) > 0$ such that for any $\phi \in S$, one has $\bar{\delta} \le x(t, \phi) \le \bar{\Delta}$ for $t \ge T$.

Define

$$S_0 = \left\{ \phi \in C^n : \bar{\delta} \le \phi(\theta) \le \bar{\Delta} + \mathbf{1}, \sum_{i=1}^n |\phi_i(\theta) - \phi_i(\xi)| \le nL |\theta - \xi| \text{ for } \theta, \xi \in [-\tau^*, 0] \right\},$$

$$S_1 = \left\{ \phi \in C^n : \gamma_1 < \phi(\theta) \le \Delta_1 + \mathbf{1}, \sum_{i=1}^n |\phi_i(\theta) - \phi_i(\xi)| \le nL |\theta - \xi| \text{ for } \theta, \xi \in [-\tau^*, 0] \right\},$$

and

$$S_2 = \left\{ \phi \in C^n : \gamma_2 \le \phi(\theta) \le \Delta_2, \sum_{i=1}^n |\phi_i(\theta) - \phi_i(\xi)| \le nL |\theta - \xi| \text{ for } \theta, \xi \in [-\tau^*, 0] \right\}.$$

Then $S_0 \subset S_1 \subset S_2$ are convex subsets of the Banach space C^n. Arzela-Ascoli's theorem implies that S_0 and S_2 are compact. Also S_1 is open relative to S_2.

Define the Poincaré map $P: S_2 \to C^n$ by

$$P(\phi) = x_\omega(\phi) \qquad \text{for} \quad \phi \in S_2.$$

Then P is continuous and $P^m(\phi) = x_{m\omega}(\phi)$ for $m = 1, 2, \ldots$.

For $\phi \in S_1$, we have $x(t, \phi) \le \Delta_2$ for $t \ge 0$ and so $\|x(t, \phi)\| \le |\Delta_3|$. Thus

$$\sum_{i=1}^n |F_i(t, x_1(t), \ldots, x_n(t), x_1(t - \tau(t)), \ldots, x_n(t - \tau(t)))| \le L$$

and so

$$|\dot{x}_i(t, \phi)| = |F_i(t, x_1(t), \dots, x_n(t), x_1(t-\tau(t)), \dots, x_n(t-\tau(t))| \le L .$$

Note that $\|P^m(\phi)\| = \|x_{m\omega}(\phi)\| \le |\Delta_3|$. Hence,

$$\text{(3.2)} \qquad \sum_{i=1}^{n} |P_i^m(\phi)(\xi_1) - P_i^m(\phi)(\xi_2)| \le \sum_{i=1}^{n} |x_i(m\omega + \xi_1, \phi) - x_i(m\omega + \xi_2, \phi)|$$

$$\le \sum_{i=1}^{n} L|\xi_1 - \xi_2| = nL|\xi_1 - \xi_2|$$

for $\xi_1, \xi_2 \in [-\tau^*, 0]$ and $m = 1, 2, \dots$. This shows that $P^m(\phi) \in S_2$. Therefore, $P^m(S_1) \subset S_2$ for $m = 1, 2, \dots$.

We choose an integer $m_0 > 0$ such that $m_0\omega > T$. Then the definition of T and (3.2) imply that $P^m(\phi) \in S_0$ for $m \ge m_0$. Thus $P^m(S_1) \subset S_0$ for $m \ge m_0$. By Lemma 2.4, P has a fixed point in S_0, which corresponds to an ω-periodic solution $x(t, \phi_0)$ of the system (3.1) with $x(t, \phi_0) \ge \tilde{\delta}$. Also the fact that the system (3.1) has no positive constant solution implies that $x(t, \phi_0)$ is nonconstant. □

The following assumption is clearly less restrictive than (H2).

(H2)′ There exist positive ω-periodic functions $B_1(t), \dots, B_n(t)$ and $p = (p_1, \dots, p_n) \in \mathring{\boldsymbol{R}}^n_+$ such that

$$G_i(t, p_1B_1(t), \dots, p_nB_n(t), p_1B_1(t-\tau(t)), \dots, p_nB_n(t-\tau(t))) + \frac{|\dot{B}_i(t)|}{B_i(t)} < 0$$

for $t \in \boldsymbol{R}$ and $i = 1, \dots, n$.

The following theorem is slightly more general than Theorem 3.1.

Theorem 3.2. *Suppose that the system* (3.1) *satisfies* (A1)–(A6) *and* $G_i(t, x_1(t), \dots, x_n(t), x_1(t-\tau(t)), \dots, x_n(t-\tau(t)))$ $(i = 1, \dots, n)$ *satisfy the assumptions* (H1), (H2)′, (H3)″ *and* (H4). *If* (3.1) *has no constant solution, then it has a nonconstant positive ω-periodic solution.*

Proof. For the system (2.1), we make the following transformation:

$$z_i = \frac{x_i}{B_i}, \qquad i = 1, \dots, n .$$

Then the system (2.1) becomes

$$\dot{z}_i = z_i\Big(G_i(t, B_1(t)z_1(t), \dots, B_n(t)z_n(t), B_1(t-\tau(t))z_1(t-\tau(t)), \dots,$$
$$B_n(t-\tau(t))z_n(t-\tau(t))) - \frac{\dot{B}_i(t)}{B_i(t)}\Big).$$

Let

$$\begin{aligned}&\tilde{G}_i(t, z_1(t), \ldots, z_n(t), z_1(t-\tau(t)), \ldots, z_n(t-\tau(t)))\\&\quad = G_i(t, B_1(t)z_1(t), \ldots, B_n(t)z_n(t), B_1(t-\tau(t))z_1(t-\tau(t)), \ldots,\\&\qquad\qquad B_n(t-\tau(t))z_n(t-\tau(t))) + \frac{|\dot{B}_i(t)|}{B_i(t)}\end{aligned}$$

and consider the system

$$\dot{z}_i = z_i \tilde{G}_i(t, z_1(t), \ldots, z_n(t), z_1(t-\tau(t)), \ldots, z_n(t-\tau(t))) . \tag{3.3}$$

Then Lemma 2.2 and Lemma 2.3 are true for (3.3) and so Lemma 2.1 implies that Lemma 2.2 and Lemma 2.3 are still true for (2.1). □

4. Uniqueness theorems. In this section we will establish the uniqueness and asymptotic stability of periodic solutions of the system (3.1).

Let $y(t, \psi) = (y_1(t), \ldots, y_n(t)) > 0$ be a solution of the system (3.1). We denote for $i, j = 1, 2, \ldots, n$,

$$a_{ij}(t) = \left.\frac{\partial F_i(t, x_1, \ldots, x_n, y_1, \ldots, y_n)}{\partial x_j}\right|_{(y_1(t), \ldots, y_n(t), y_1(t-\tau(t)), \ldots, y_n(t-\tau(t)))},$$

$$b_{ij}(t) = \left.\frac{\partial F_i(t, x_1, \ldots, x_n, y_1, \ldots, y_n)}{\partial y_j}\right|_{(y_1(t), \ldots, y_n(t), y_1(t-\tau(t)), \ldots, y_n(t-\tau(t)))}.$$

Let $x(t, \phi) = (x_1(t), \ldots, x_n(t))$ be another solution of the system (3.1), and

$$\begin{aligned}\tilde{F}_i &= F_i(t, x_1(t), \ldots, x_n(t), x_1(t-\tau(t)), \ldots, x_n(t-\tau(t)))\\&\quad - F_i(t, y_1(t), \ldots, y_n(t), y_1(t-\tau(t)), \ldots, y_n(t-\tau(t)))\\&\quad - \sum_{j=1}^n a_{ij}(t)(x_j(t) - y_j(t)) - \sum_{j=1}^n b_{ij}(t)(x_j(t-\tau(t)) - y_j(t-\tau(t))) .\end{aligned}$$

We make the following technical assumption:

$(\tilde{\mathrm{A}}_1)$ There exist positive constants $p_1, \ldots, p_n, q_1, \ldots, q_n$ such that for all t, $y(t, \psi)$,

$$\sum_{\substack{i=1\\ i\neq j}}^n p_i |a_{ij}(t)| + p_j a_{jj}(t) + q_j < 0 \qquad j = 1, \ldots, n ,$$

and

$$\sum_{i=1}^n p_i |b_{ij}(t)| - q_j(1 - \tau'(t)) \leq 0 , \qquad j = 1, \ldots, n .$$

LEMMA 4.1. *Suppose* $(\tilde{\mathrm{A}}_1)$ *holds. Let* $y(t, \psi)$ *be a solution of* (3.1) *with* $\psi(\theta) \geq 0$ *on* $[-\tau^*, 0]$, $\psi(0) > 0$ *and* $\psi \in C^n$. *If there is an* $M \in \mathring{\boldsymbol{R}}^n_+$ *such that* $y(t, \psi) \leq M$ *for* $t \geq 0$, *then there is a* $\delta(\psi) > 0$ *such that for any solution* $x(t, \phi)$ *of* (3.1) *with* $\phi \in C^n$, $\phi(\theta) \geq 0$ *and*

$\|\phi-\psi\|<\delta$, *we have*

$$\lim_{t\to+\infty}(x(t,\phi)-y(t,\psi))=0\,.$$

PROOF. Let $x(t,\phi)=(x_1(t),\dots,x_n(t))$ and $y(t,\psi)=(y_1(t),\dots,y_n(t))$. Let $v_i(t)=(x_i(t)/y_i(t))$. Then

$$\begin{aligned}\dot v_i(t)=v_i(t)(F_i(t,x_1(t),\dots,x_n(t),x_1(t-\tau(t)),\dots,x_n(t-\tau(t)))\\ -F_i(t,y_1(t),\dots,y_n(t),y_1(t-\tau(t)),\dots,y_n(t-\tau(t))))\,,\end{aligned}$$

that is,

$$\dot v_i(t)=v_i(t)\left[\sum_{j=1}^n a_{ij}(t)y_j(t)(v_j(t)-1)+\sum_{j=1}^n b_{ij}(t)y_j(t-\tau(t))(v_j(t-\tau(t))-1)+\tilde F_i\right],\qquad i=1,\dots,n\,.$$

Let $u_i(t)=v_i(t)-1$. Then

$$\dot u_i(t)=(u_i+1)\left[\sum_{j=1}^n a_{ij}(t)y_j(t)u_j(t)+\sum_{j=1}^n b_{ij}(t)y_j(t-\tau(t))u_j(t-\tau(t))+\tilde F_i\right],\qquad i=1,\dots,n\,. \tag{4.1}$$

Let $V(u_t)=\sum_{i=1}^n(p_i|u_i(t)|+\int_{t-\tau(t)}^t q_iy_i(s)|u_i(s)|ds)$. Denote by D^+ the upper right Dini derivative. Then

$$\begin{aligned}D^+V(u_t)\le\sum_{j=1}^n\left[\left(\sum_{\substack{i=1\\ i\ne j}}^n p_i|a_{ij}(t)|+p_ja_{jj}(t)+q_j\right)y_j(t)+|\bar F_j|\right]|u_j(t)|\\ +\sum_{j=1}^n\left(\sum_{i=1}^n p_i|b_{ij}(t)|-q_j(1-\tau'(t))\right)y_j(t-\tau(t))|u_j(t-\tau(t))|\,,\end{aligned} \tag{4.2}$$

where

$$\bar F_j=\sum_{k=1}^n a_{jk}(t)y_k(t)u_k(t)+\sum_{k=1}^n b_{jk}(t)y_k(t-\tau(t))u_k(t-\tau(t))+\tilde F_j\,,\qquad j=1,\dots,n\,. \tag{4.3}$$

Now $\bar F_j$ $(j=1,\dots,n)$ are continuous with respect to their arguments and $y_j(t)$ $(j=1,\dots,n)$ are bounded away from zero. Hence if $\|u_t\|$ is sufficiently small, then the assumption $(\tilde{\mathrm{A}}_1)$ implies that $D^+V(u_t)<0$. Therefore if $\|\phi_1\|$ is sufficiently small, then for the solution $u(t,\phi_1)$ of (4.1), one has $\lim_{t\to+\infty}u(t,\phi_1)=0$. But $u_i(t)=(x_i(t)-y_i(t))/y_i(t)$ and $y_i(t)\le M_i$ for $t\ge0$, where $M=(M_1,\dots,M_n)$. Thus there is a $\delta(\psi)>0$ such that if $\phi\in C^n$, $\phi(\theta)\ge0$ and $\|\phi-\psi\|\le\delta$, then $\lim_{t\to+\infty}(x(t,\phi)-y(t,\psi))=0$. □

The following technical assumption is needed in our next lemma:

$(\tilde{\mathrm{A}}_2)$ For any $\eta\in\mathring{\boldsymbol{R}}^n_+$, there is an $M(\eta)\in\mathring{\boldsymbol{R}}^n_+$ such that for any $\phi\in C^n$ with $0\le\phi(\theta)\le\eta$

on $[-\tau^*, 0]$, one has

$$x(t, \phi) \leq M(\eta) \qquad \text{for} \quad t \geq 0 .$$

LEMMA 4.2. *Assume that the system* (3.1) *satisfies* $(\tilde{A}_1)$ *and* $(\tilde{A}_2)$. *Let* S *be a compact connected subset of* C^n *such that if* $\phi \in S$, *then* $\eta_2 \geq \phi(\theta) \geq \eta_1$ *on* $[-\tau^*, 0]$, *where* $\eta_1, \eta_2 \in \mathring{R}^n_+$. *Then for any two solutions* $x(t, \phi)$ *and* $y(t, \psi)$ *with* $\phi, \psi \in S$, *one has*

$$\lim_{t \to +\infty} (x(t, \phi) - y(t, \psi)) = 0 .$$

PROOF. By $(\tilde{A}_2)$, we have $M(\eta_2) \in \mathring{R}^n_+$ such that $x(t, \phi) \leq M(\eta_2)$ for $\phi \in S$, and $t \geq 0$. For any $\psi \in S$, there exists a $\delta(\psi) > 0$ such that if $\| \phi - \psi \| < \delta(\psi)$ for $\phi \in C^n$, then Lemma 4.1 implies that $\lim_{t \to +\infty} (x(t, \phi) - y(t, \psi)) = 0$. Clearly

$$\bigcup_{\psi \in S} B(\psi, \delta(\psi)) = \bigcup_{\psi \in S} \{ \phi \in C^n : \| \phi - \psi \| < \delta(\psi) \} \supset S .$$

The compactness of S yields that there exist $\psi_1, \ldots, \psi_m \in S$ such that $\bigcup_{i=1}^m B(\psi_i, \delta(\psi_i)) \supset S$. Since S is connected, for ψ_{i_0} and ψ_{j_0}, $i_0, j_0 \in \{1, \ldots, m\}$, we can find $\phi_{i_1}, \ldots, \phi_{i_l} \in C^n$ such that $\eta_2 \geq \phi_{i_k}(\theta) \geq \eta_1$ on $[-\tau^*, 0]$, $\phi_{i_k}, \phi_{i_{k+1}} \in B(\psi_i, \delta(\psi_i))$ ($k = 0, 1, \ldots, l$ and $\phi_{i_0} = \psi_{i_0}$, $\phi_{i_{l+1}} = \psi_{j_0}$). Thus Lemma 4.1 and the definition of $\delta(\psi_i)$ yield that

$$\lim_{t \to +\infty} (x(t, \phi_{i_k}) - x(t, \phi_{i_{k+1}})) = 0 , \qquad k = 0, 1, \ldots, l$$

and so

$$\lim_{t \to +\infty} (x(t, \psi_{i_0}) - x(t, \psi_{j_0})) = 0 \qquad \text{for} \quad i, j \in \{1, \ldots, m\} . \tag{4.4}$$

For $\phi, \psi \in S$, there exist $j_1, j_2 \in \{1, \ldots, n\}$ such that $\phi \in B(\psi_{j_1}, \delta(\psi_{j_1}))$ and $\psi \in B(\psi_{j_2}, \delta(\psi_{j_2}))$. Then $\lim_{t \to +\infty} (x(t, \phi) - x(t, \psi_{j_1})) = 0$ and $\lim_{t \to +\infty} (y(t, \psi) - x(t, \psi_{j_2})) = 0$. The conclusion now follows from (4.4). □

THEOREM 4.1. *Suppose that the system* (3.1) *satisfies* $(\tilde{A}_1)$ *and* $(\tilde{A}_2)$. *Then there exists at most one positive* ω*-periodic solution* $x(t, \phi)$ *for* (3.1). *If* $x(t, \phi)$ *is the positive* ω*-periodic solution, then it is globally asymptotically stable for* $C^n_+ := \{ \phi(\theta) \in C^n : \phi(\theta) \geq 0, \phi(0) > 0 \}$.

PROOF. Suppose that the system (3.1) has two different positive ω-periodic solutions $x(t, \phi)$ and $y(t, \psi)$. Then there are $\eta_1, \eta_2 \in \mathring{R}^n_+$ such that $\eta_1 \leq \phi, \psi \leq \eta_2$.

Let $S = \{ (1 - \lambda)\phi + \lambda\psi : \lambda \in [0, 1] \}$. Then the fact that $\phi, \psi \in C^n$ implies that S is compact. Clearly, S is connected. Lemma 4.2 yields that $\lim_{t \to +\infty} (x(t, \phi) - y(t, \psi)) = 0$, a contradiction, which shows the uniqueness.

Let $x(t, \phi_0)$ be the unique positive ω-periodic solution of (3.1). For $\phi \in C^n_+$, there exists $T(\phi) > 0$ such that $x(t, \phi) > 0$ for $t \geq T(\phi)$. Hence, without loss of generality, we suppose that $\phi(\theta) > 0$ for $\theta \in [-\tau^*, 0]$. Then there exist $\eta_1, \eta_2 \in \mathring{R}^n_+$ such that $\eta_1 \leq \phi(\theta)$,

$\phi_0(\theta) \leq \eta_2$ on $[-\tau^*, 0]$.

The above arguments show that $\lim_{t\to+\infty}(x(t, \phi) - x(t, \phi_0)) = 0$. □

If we replace $(\tilde{A}_1)$ by the assumption:

$(\tilde{A}_1)'$ There exist positive constants $p_1, \ldots, p_n$, $q_1, \ldots, q_n$ and positive ω-periodic functions $B_1(t), \ldots, B_n(t)$ such that

$$\sum_{i=1}^{n} p_i B_j(t) | a_{ij}(t) | + p_j a_{jj}(t) B_j(t) + q_j < 0$$

and

$$\sum_{i=1}^{n} p_j B_j(t - \tau(t)) | b_{ij}(t) | - q_j(1 - \tau'(t)) \leq 0$$

for $j = 1, \ldots, n$,

then we have:

THEOREM 4.2. *Suppose that the system* (3.1) *satisfies* $(\tilde{A}_1)'$, $(\tilde{A}_2)$. *Then there exists at most one positive* ω*-periodic solution* $x(t, \phi)$ *for* (3.1). *If* $x(t, \phi)$ is the positive ω*-periodic solution of* (3.1), *then it is globally asymptotically stable in* C^n_+.

PROOF. For the system (3.1), we take the transformation: $z_i = x_i / B_i$, $i = 1, \ldots, n$. Then the system (3.1) becomes

$$\dot{z}_i = z_i \Big(F_i(t, B_1(t) z_1(t), \ldots, B_n(t) z_n(t), B_1(t - \tau(t)) z_1(t - \tau(t)), \ldots, B_n(t - \tau(t)) z_n(t - \tau(t))) - \frac{\dot{B}_i(t)}{B_i(t)} \Big). \tag{4.5}$$

It is easy to check that Lemma 4.1 and Lemma 4.2 are true for the system (4.5), and so the theorem follows from the proof of Theorem 4.1. □

5. Applications. Now we apply the above results to some well-known systems.

5.1. Lotka-Volterra system. We consider first the system (1.2), namely, the following n-dimensional delay Lotka-Volterra system:

$$\dot{x}_i(t) = x_i(t) \Big(c_i(t) - a_{ii}(t) x_i(t) + \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}(t) x_j(t) + \sum_{j=1}^{n} b_{ij}(t) x_j(t - \tau(t)) \Big). \tag{5.1}$$

For the system (5.1) we assume that

(i) $a_{ij}(t), c_i(t), b_{ij}(t)$, $(i, j = 1, \ldots, n)$ and $\tau(t)$ are continuously differentiable, ω-periodic functions, and $c_i(t) > 0$, $a_{ii}(t) > 0$, $a_{ij}(t) \geq 0$, $b_{ij}(t) \geq 0$ and $\tau(t) \geq 0$ for $t \in \boldsymbol{R}$.

(ii) There exist positive differentiable ω-periodic functions $B_1(t), \ldots, B_n(t)$ and $p = (p_1, \ldots, p_n) \in \mathring{\boldsymbol{R}}^n_+$ such that

$$p_i a_{ii}(t)B_i(t) > c_i(t) + \left|\frac{\dot{B}_i(t)}{B_i(t)}\right| + \sum_{\substack{j=1\\ j\neq i}}^{n} p_j a_{ij}(t)B_j(t) + \sum_{j=1}^{n} p_j b_{ij}(t)B_j(t-\tau(t))$$

for $t \in \boldsymbol{R}$ and $i = 1, \ldots, n$.

THEOREM 5.1. *Under the above assumptions, if the system* (5.1) *has no constant solution, then it has a nonconstant periodic solution.*

Clearly, the system (5.1) generates a monotone semiflow by the assumption (i). In order to prove Theorem 5.1, we need the following lemma.

LEMMA 5.1. *Under the assumption* (i), *there exists a* $\gamma > 0$ *such that for* $\phi \in C^n$ *with* $\phi \geq \gamma \cdot \mathbf{1}$, *one has*

$$x(t, \phi) \geq \gamma \cdot \mathbf{1} \qquad \textit{for} \quad t \geq 0 .$$

PROOF. By the assumption (i), the system (5.1) generates a monotone semiflow and for any $\phi \in C^n$ with $\phi(\theta) \geq 0$ on $[-\tau^*, 0]$ and $\phi(0) \neq 0$, one has $x(t, \phi) \geq 0$ for $t \geq 0$. Thus we have

$$\dot{x}_i(t) \geq x_i(t)[c_i(t) - a_{ii}(t)x_i(t)] , \qquad i = 1, \ldots, n . \tag{5.2}$$

Therefore, if $\phi(0) > 0$, then $\liminf_{t\to\infty} x_i(t, \phi) \geq \min\{c_i(t)/a_{ii}(t), t \in [0, \omega]\} \equiv \gamma_i$. Let $\gamma = \min\{\gamma_i, i = 1, \ldots, n\}$. Then the conclusion of the lemma follows. □

PROOF OF THEOREM 5.1. Let

$$\begin{aligned} F_i(t, x_1(t), \ldots, x_1(t), x_1(t-\tau(t)), \ldots, x_n(t-\tau(t))) \\ = G_i(t, x_1(t), \ldots, x_n(t), x_1(t-\tau(t)), \ldots, x_n(t-\tau(t))) \\ = c_i(t) - a_{ii}(t)x_i(t) + \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}(t)x_j(t) + \sum_{j=1}^{n} b_{ij}(t)x_j(t-\tau(t)) . \end{aligned}$$

The assumption (i) yields that (A1)–(A3) and (A5) hold. Lemma 5.1 guarantees that (A4) is true. Now (H1), (H3) and (H4) are satisfied by the assumption (i) and the linear form of functions G_i. (H2) follows from the assumption (ii). Theorem 5.1 now follows from Theorem 3.2. □

In order to state our next result, we assume further that

(iii) there exists $p' = (p'_1, \ldots, p'_n) \in \mathring{\boldsymbol{R}}^n_+$ and $q = (q_1, \ldots, q_n) \in \mathring{\boldsymbol{R}}^n_+$ such that for $j = 1, \ldots, n$,

$$\sum_{i=1}^{n} p'_i B_j(t) \,|\, a_{ij}(t) \,|\, - p'_j a_{jj}(t)B_j(t) + q_j < 0 ,$$

$$\sum_{i=1}^{n} p'_i B_j(t' - \tau(t)) \,|\, b_{ij}(t) \,|\, - q_j(1 - \tau'(t)) \leq 0 .$$

THEOREM 5.2. *Under the assumptions* (i), (ii) *and* (iii), *if the system* (5.1) *has no positive constant solution, then the system* (5.1) *has a unique and globally asymptotically stable nonconstant ω-periodic solution with respect to positive solutions of* (5.1).

In order to prove Theorem 5.2, we need the following lemmas.

LEMMA 5.2. *Let γ be defined as in Lemma* 5.1. *For any $\phi \in C^n$ with $\phi(\theta) \geq 0$ on* $[-\tau^*, 0]$ *and $\phi(0) \neq 0$, there exists $T(\phi) > 0$ such that for $t \geq T(\phi)$,*

$$x(t, \phi) \geq \gamma \cdot \mathbf{1}\,.$$

PROOF. It follows from the argument for Lemma 5.1. □

LEMMA 5.3. *For any $\phi \in C^n$ and $\psi \in C^n$ with $\phi(\theta) \geq \gamma \cdot \mathbf{1}$ and $\psi(\theta) \geq \gamma \cdot \mathbf{1}$ on* $[-\tau^*, 0]$, *one has*

$$\lim_{t \to +\infty} (x(t, \phi) - x(t, \psi)) = 0\,.$$

PROOF. Let

$$x_i^0 = \min\{\min_{-\tau^* \leq \theta \leq 0} \phi_i(\theta),\ \min_{-\tau^* \leq \theta \leq 0} \psi_i(\theta)\}\,, \quad y_i^0 = \max\{\max_{-\tau^* \leq \theta \leq 0} \phi_i(\theta),\ \max_{-\tau^* \leq \theta \leq 0} \psi_i(\theta)\}\,,$$

where $i = 1, \ldots, n$ and $x^0 = (x_1^0, \ldots, x_n^0)$, $y^0 = (y_1^0, \ldots, y_n^0)$. Then $\gamma \cdot \mathbf{1} \leq x^0 \leq \phi(\theta) \leq y^0$ and $\gamma \cdot \mathbf{1} \leq x^0 \leq \psi(\theta) \leq y^0$ on $[-\tau^*, 0]$. Thus we have

$$x(t, x^0) \leq x(t, \phi) \leq y(t, y^0) \qquad \text{for} \quad t \geq 0$$

and

$$x(t, x^0) \leq y(t, \psi) \leq y(t, y^0) \qquad \text{for} \quad t \geq 0\,.$$

An argument similar to that in Lemma 4.1 and Lemma 4.2 shows that $\lim_{t \to +\infty}(x(t, x^0) - y(t, y^0)) = 0$. Hence $\lim_{t \to +\infty}(x(t, \phi) - y(t, \psi)) = 0$. □

PROOF OF THEOREM 5.2. The existence and uniqueness parts follow from Theorems 5.1 and 4.2, and the global asymptotic stability follows from Lemma 5.2 and Lemma 5.3. □

The following corollary improves a similar result of Freedman and Wu [5] for a scalar equation.

COROLLARY 5.1. *Consider the equation*

$$\dot{x}(t) = x(t)[a(t) - b(t)x(t) + c(t)x(t - \tau(t))]\,, \tag{5.3}$$

where $a(t)$, $b(t)$, $c(t)$ and $\tau(t)$ are continuously differentiable, ω-periodic functions and $a(t) > 0$, $b(t) > 0$, $c(t) \geq 0$, $\tau(t) \geq 0$ for $t \in \boldsymbol{R}$. Suppose that

(i) *the equation $a(t) - b(t)K(t) + c(t)K(t - \tau(t)) = 0$ has a positive, ω-periodic, continuously differentiable solution $K(t)$.*

Then the equation (5.3) *has a positive ω-periodic solution $Q(t)$. If, in addition*

(ii) $\min_{0\le t\le\omega} b(t)K(t) > \max_{0\le t\le\omega}[c(t)K(t-\tau(t))/(1-\tau'(t))]$,

then $Q(t)$ is globally asymptotically stable with respect to positive solutions of (5.3).

PROOF. Let $B(t)=K(t)$. Then the condition (i) implies that for $t\in[0,\omega]$, $b(t)>c(t)K(t-\tau(t))/K(t)$. This in turn implies that there is a $p_1>0$ such that $p_1>(a(t)+|\dot K(t)|/K(t))/(b(t)K(t)-c(t)K(t-\tau(t)))$, i.e., the condition (ii) in Theorem 5.1 is satisfied. Hence, the existence of periodic solutions follows from Theorem 5.1. A simple calculation yields the uniqueness and global asymptotic stability of periodic solutions. □

For the sake of comparison, we state below the main result of Freedman and Wu [5].

THEOREM (Freedman and Wu [5]). *For the equation* (5.3), *if* (i) *is satisfied, then it has a positive ω-periodic solution $Q(t)$. Moreover, if for all $t\in[0,\omega]$,*

$$(*)\qquad b(t)>c(t)Q(t-\tau(t))/Q(t),$$

then $Q(t)$ is globally asymptotically stable with respect to positive solutions of (2.1).

Clearly, we have the same existence conditions. However, our condition (ii) for the global stability of $Q(t)$ is verifiable, while $(*)$ is not due to the dependence of $Q(t)$ which is unknown.

For example, if $a(t)=2+(3\sin t)/2$, $b(t)=1$, $c(t)=1/2$, and $\tau(t)=\pi$. Then $\omega=2\pi$, and $K(t)=4+\sin t$, $K(t-\tau(t))=4-\sin t$. Clearly

$$3=\min_{0\le t\le 2\pi} b(t)K(t)=\min_{0\le t\le 2\pi}(4+\sin t)$$
$$>\frac{5}{2}=\max_{0\le t\le 2\pi}[c(t)K(t-\tau(t))/(1-\tau'(t))]=\max_{0\le t\le 2\pi}\frac{1}{2}(4-\sin t).$$

Hence both (i) and (ii) of Corollary 5.1 are satisfied. By Corollary 5.1, we conclude that in this case, the equation (5.3) has a globally asymptotically stable positive periodic solution $Q(t)$ (with respect to positive solutions of (5.3)).

5.2. A delay nonautonomous predator-prey system. In the following, we would like to apply our main results to the delay nonautonomous predator-prey system

$$(5.4)\qquad \begin{aligned}\dot x(t)&=x(t)\left[a_1(t)-b_1(t)x(t)-\frac{c_1(t)y(t)}{c_2(t)x(t)+1}\right],\\ \dot y(t)&=y(t)\left[-a_2(t)-b_2(t)y(t)+\frac{d(t)x(t-1)}{c_2(t)x(t-1)+1}\right],\end{aligned}$$

where $a_i(t)$, $b_i(t)$, $c_i(t)$, $(i=1,2)$ and $d(t)$ are nonnegative continuously differentiable, ω-periodic functions. Moreover, $a_i(t)$, $b_i(t)$ $(i=1,2)$, $c_1(t)$ and $d(t)$ are positive. Let

$$F_1(t, x(t), y(t), x(t-1), y(t-1)) = a_1(t) - b_1(t)x(t) - \frac{c_1(t)y(t)}{c_2(t)x(t)+1},$$

$$F_2(t, x(t), y(t), x(t-1), y(t-1)) = -a_2(t) - b_2(t)y(t) + \frac{d(t)x(t-1)}{c_2(t)x(t-1)+1},$$

$$G_1(t, x(t), y(t), x(t-1), y(t-1)) = a_1(t) - b_1(t)x(t)$$

and

$$G_2(t, x(t), y(t), x(t-1), y(t-1)) = a_2(t) - b_2(t)y(t) + \frac{d(t)x(t-1)}{c_2(t)x(t-1)+1}.$$

It is easy to check that G_i ($i=1,2$) satisfies the assumptions (H1)–(H4). Clearly, $F_i \le G_i$ for $i=1,2$.

In the following, we define

$$\bar{a}_i = \max_{0 \le t \le \omega} a_i(t), \quad \underline{a}_i = \min_{0 \le t \le \omega} a_i(t), \qquad i=1,2,$$

and similarly $\bar{b}_i$, $\underline{b}_i$, $\bar{c}_i$, $\underline{c}_i$, $\bar{d}$ and $\underline{d}$. We assume that

$$\frac{\bar{d}\bar{a}_1\underline{b}_1^{-1}}{\underline{c}_2\bar{a}_1\underline{b}_1^{-1}+1} > \bar{a}_2. \tag{5.5}$$

Then we have:

LEMMA 5.4. *Assume that* (5.5) *holds. Then*

(i) *For any* $\eta \in \mathring{\boldsymbol{R}}^2_+$, *there exists* $M(\eta) \in \mathring{\boldsymbol{R}}^2_+$ *such that for any* $\phi \in C^2$ *with* $0 \le \phi \le \eta$ *on* $[-\tau^*, 0]$, *one has*

$$0 \le X(t, \phi) \le M(\eta) \qquad \textit{for all} \quad t \ge 0;$$

where $X(t, \phi)$ *is the solution of* (5.4) *with* $X_0(\phi) = \phi$.

(ii) *Let* $\Delta = (\bar{a}_1\underline{b}_1^{-1}, \underline{b}_2^{-1}((\bar{d}\bar{a}_1\underline{b}_1^{-1}/\underline{c}_2\bar{a}_1\underline{b}_1^{-1}+1)+\bar{a}_2))$. *Then for any* $\eta \in \boldsymbol{R}^2_+$, *there is a constant* $T = T(\eta) > 0$ *such that for any* $\phi \in C^2$ *with* $0 \le \phi \le \eta$ *on* $[-\tau^*, 0]$, *one has* $0 \le X(t, \phi) \le \Delta + \mathbf{1}$ *for all* $t \ge T(\eta)$.

PROOF. Let $Y(t, \psi) = (x(t, \psi), y(t, \psi))$ be the solution of the system

$$\dot{x}(t) = G_1(t, x(t), y(t), x(t-1), y(t-1)),$$

$$\dot{y}(t) = G_2(t, x(t), y(t), x(t-1), y(t-1)).$$

It is easy to check that $(\lim_{t\to+\infty} \sup x(t, \psi), \lim_{t\to+\infty} \sup y(t, \psi)) \le \Delta$. Now the conclusions follow from Lemma 2.3. $\square$

Observe that for $t \ge 0$, we have $\dot{x}(t) \le \bar{a}_1 x(t)$, hence $x(t) \le x(t_0)\exp[\bar{a}_1(t-t_0)]$ for $t \ge t_0 \ge 0$, which implies that $x(t-1) \ge x(t)e^{-\bar{a}_1}$ for $t \ge 1$.

LEMMA 5.5. *Suppose that the conditions in Lemma 5.4 hold. If*

$$\bar{a}_2 < \frac{\underline{d}e^{-\bar{a}_1}\underline{a}_1\bar{b}_1^{-1}}{\bar{c}_2 e^{-\bar{a}_1}\underline{a}_1\bar{b}_1^{-1}+1}, \tag{5.6}$$

then for any $\eta_1, \eta_2 \in \mathring{R}^2_+$ with $\eta_1 \le \eta_2$, there exists $\gamma(\eta_1, \eta_2) \in \mathring{R}^2_+$ such that for any $\phi \in C^2$ with $\eta_1 \le \phi(\theta) \le \eta_2$ on $[-\tau^*, 0]$, one has $X(t, \phi) \ge \gamma(\eta_1, \eta_2)$ for all $t \ge 0$.

PROOF. For $\eta_1, \eta_2 \in \mathring{R}^2_+$ with $\eta_1 \le \eta_2$, Lemma 5.4 implies that there exists $M(\eta_2) = (M_1(\eta_2), M_2(\eta_2)) \in \mathring{R}^2_+$ such that $M(\eta_2) \ge \eta_2$ and for any $\phi \in C^2$ with $\eta_1 \le \phi \le \eta_2$ on $[-\tau^*, 0]$, one has $0 \le X(t, \phi) < M(\eta_2)$ for all $t \ge 0$, i.e., $0 \le x(t, \phi) < M_1(\eta_2)$, $0 \le y(t, \phi) < M_2(\eta_2)$ for $t \ge 0$. Hence for $t \ge 0$, we have

$$\dot{x}(t) \ge x(t)[\underline{a}_1 - \bar{b}_1 M_1(\eta_2) - \bar{c}_1 M_2(\eta_2)] = -\beta x(t),$$

which implies that for $t \ge 1$, $x(t-1) \le x(t)e^{\beta}$, where $-\beta = \underline{a}_1 - \bar{b}_1 M_1(\eta_2) - \bar{c}_1 M_2(\eta_2) < 0$.

Now we would like to compare the solutions of (5.4) with those of the following two systems of ordinary differential equations

$$\dot{u}(t) = u(t)\left[\underline{a}_1 - \bar{b}_1 u(t) - \frac{\bar{c}_1 v(t)}{\underline{c}_2 u(t)+1}\right], \quad \dot{v}(t) = v(t)\left[-\underline{a}_2 + \frac{\bar{d}e^{\beta}u(t)}{\underline{c}_2 e^{\beta}u(t)+1}\right] \tag{5.7}$$

and

$$\dot{u}(t) = u(t)\left[\underline{a}_1 - \bar{b}_1 u(t) - \frac{\bar{c}_1 v(t)}{\underline{c}_2 u(t)+1}\right], \quad \dot{v}(t) = v(t)\left[-\bar{a}_2 - \bar{b}_2 v(t) + \frac{\underline{d}e^{-\bar{a}_1}u(t)}{\bar{c}_2 e^{-\bar{a}_1}u(t)+1}\right]. \tag{5.8}$$

We denote by $(\bar{u}(t), \bar{v}(t))$ and $(\underline{u}(t), \underline{v}(t))$ solutions of (5.7) and (5.8) with initial data $(u_0, v_0) \in R^2_+$, respectively. We denote

$$\hat{x} = \min\left\{\min_{0 \le t \le \omega} \frac{1}{2} d^{-1}(t) a_2(t), \underline{a}_2 e^{-\beta}(\bar{d} - \underline{a}_2 \underline{c}_2)^{-1}\right\}.$$

Without loss of generality, we suppose that $\eta_1, \eta_2 < (\hat{x}, \underline{a}_1 \bar{c}_1^{-1})$. Denote $\bar{u}_1(t) = \bar{u}(t, \eta_1, M_2(\eta_2))$, $\bar{v}_1(t) = \bar{v}(t, \eta_1, M_2(\eta_2))$. Then there is a $\tau_1 > 0$ such that

$$\dot{\bar{u}}_1(\tau_1) = 0 \quad \text{and} \quad \dot{\bar{u}}_1(t) < 0 \qquad \text{for } t \in [0, \tau_1).$$

Denote $\underline{u}_1(t) = \underline{u}(t, \bar{u}_1(\tau_1), \bar{v}_1(\tau_1))$, $\underline{v}_1(t) = \underline{v}(t, \bar{u}_1(\tau_1), \bar{v}_1(\tau_1))$. Then there exist $0 < \tau_2 < \tau_3 < \tau_4$ such that either

(a) $\underline{v}_1(\tau_2) = \eta_{12}$, $\underline{u}_1(\tau_3) = \eta_{11}$, $\dot{\underline{v}}_1(\tau_4) = 0$ and $(\underline{u}_1(t), \underline{v}_1(t)) < (\eta_{11}, \eta_{12})$ on (τ_2, τ_3),

or

(b) $\underline{u}_1(\tau_2) = \eta_{11}$, $\underline{v}_1(\tau_3) = \eta_{12}$, $\dot{\underline{v}}_1(\tau_4) = 0$ and $(\underline{u}_1(t), \underline{v}_1(t)) > (\eta_{11}, \eta_{12})$ on (τ_2, τ_3), where $\eta_1 = (\eta_{11}, \eta_{12})$.

We first assume (a). We define

$$\Gamma_1 = \{(\bar{u}_1(t), \bar{v}_1(t)) : 0 \le t \le \tau_1\}, \quad \Gamma_2 = \{(\underline{u}_1(t), \underline{v}_1(t)) : 0 \le t \le \tau_4\},$$

$\Gamma_3=\{(x, \underline{v}_1(\tau_4)): \underline{u}_1(\tau_4)\leq x\leq M_1(\eta_2)\}$, $\Gamma_4=\{(M_1(\eta_2), y): \underline{v}_1(\tau_4)\leq y\leq M_2(\eta_2)\}$,
$\Gamma_5=\{(x, M_2(\eta_2)): \eta_1\leq x\leq M_1(\eta_2)\}$.

Then $\bigcup_{i=1}^{5}\Gamma_i$ constitutes the boundary of a closed bounded region Ω in the x-y plane.

CLAIM. *For $t\geq 0$, we have $X(t, \phi)\in\Omega$ with $\eta_1\leq\phi\leq\eta_2$ on $[-\tau^*, 0]$.*

First we observe that for $t\geq 0$, $x(t, \phi)=(x(t), y(t))$ can never leave Ω from Γ_1. Since if $(x(t), y(t))=(\bar{u}(t^*), \bar{v}(t^*))\in\Gamma_1$, then $\dot{x}(t)\geq\dot{\bar{u}}(t^*)$, $\dot{y}(t)\leq\dot{\bar{v}}(t^*)$. Similarly, we see that $(x(t), y(t))$ cannot leave Ω through Γ_2. It is obvious that it cannot cross Γ_3, Γ_4 and Γ_5. This proves the claim.

Now we assume (b). In this case, we first choose one point (x_0, y_0) such that $(x_0, y_0)<\eta_1$. Denote

$$\underline{u}_2(t)=\underline{u}(t, x_0, y_0), \quad \underline{v}_2(t)=\underline{v}(t, x_0, y_0).$$

There exist $\tau_4>\tau_3>0>\tau_2>\tau_1$ such that $\dot{\underline{v}}_2(\tau_4)=0$, $\underline{u}_2(\tau_3)=\eta_{11}$, $\underline{v}_2(\tau_2)=\eta_{12}$ and

$$(\underline{u}_2(\tau_1), \underline{v}_2(\tau_1)) \text{ satisfies } \underline{a}_1-\bar{b}_1\underline{u}_2(\tau_1)=\frac{\bar{c}_1\underline{v}_2(\tau_1)}{\underline{c}_2\underline{u}_2(\tau_1)+1}.$$

Denote

$$\bar{u}_2(t)=\bar{u}_2(t, \underline{u}_2(\tau_1), \underline{v}_2(\tau_1)), \quad \bar{v}_2(t)=\bar{v}_2(t, \underline{u}_2(\tau_1), \underline{v}_2(\tau_2)).$$

Then there is a $t_1<0$ such that $\bar{v}_2(t_1)=M_2(\eta_2)$. Clearly, $\bar{u}_2(t_1)<\eta_{11}$. We define

$\Gamma'_1=\{(\bar{u}_2(t), \bar{v}_2(t)): t_1\leq t\leq 0\}$, $\Gamma'_2=\{(\bar{u}_2(t), \bar{v}_2(t)): \tau_1\leq t\leq\tau_4\}$,
$\Gamma'_3=\{(x, \underline{v}_2(\tau_4)): \underline{u}_2(\tau_4)\leq x\leq M_1(\eta_2)\}$, $\Gamma'_4=\{(M_1(\eta_2), y): \underline{v}_2(\tau_4)\leq y\leq M_2(\eta_2)\}$,
$\Gamma'_5=\{(x, M_1(\eta_2)): \bar{u}_2(t_1)\leq x\leq M_1(\eta_2)\}$.

Then $\bigcup_{i=1}^{5}\Gamma'_i$ constitutes the boundary of a closed bounded region Ω' in the x-y plane. Similarly, the above claim is true for Ω' and the proof follows. $\square$

The accompanying Figure should be helpful in understanding the above arguments. In the following, we need the notation:

$$\Delta^1=\bar{a}_1\underline{b}_1^{-1}+1, \quad \Delta^2=\underline{b}_2^{-1}\left(\frac{\bar{d}\bar{a}_1\underline{b}_1^{-1}}{\underline{c}_2\bar{a}_1\underline{b}_1^{-1}+1}+\bar{a}_2\right)+1.$$

LEMMA 5.6. *Assume that the conditions in Lemma 5.5 hold. Then there exists $\tilde{\delta}\in\mathring{\boldsymbol{R}}^2_+$ such that for any $\phi(\theta)\in C^2$ with $\phi(\theta)\geq 0$, $\phi(0)>0$,*

$$\liminf_{t\to+\infty} X(t, \phi)\geq\tilde{\delta}.$$

PROOF. From Lemma 5.4 (ii), we know that there is a $T_0>0$ such that for $t>T_0$, $X(t, \phi)\leq\Delta+\mathbf{1}$. Therefore, we may assume that $X(t, \phi)\leq\Delta+\mathbf{1}$ for $t\geq -1$. Then we have

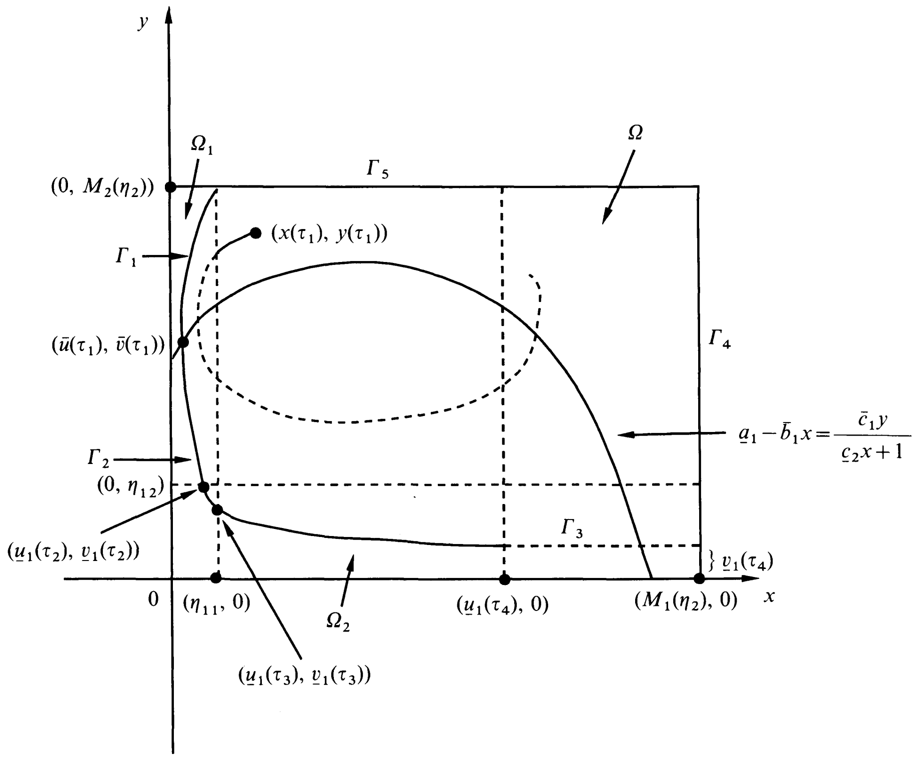

FIGURE. Illustration of the proof of Lemma 5.5.

$\dot{x}(t) \geq x(t)[\underline{a}_1 - \bar{b}_1 \Delta_b^1 - \bar{c}_1 \Delta^2] = -\hat{\beta} x(t)$, which implies that $x(t-1) \leq x(t) e^{\hat{\beta}}$ for $t \geq 1$.

Now we would like to compare the solutions of (5.4) with those of the system (5.8) and the following system of ordinary differential equations:

$$(5.9) \qquad \dot{u}(t) = u(t)\left[\underline{a}_1 - \bar{b}_1 u(t) - \frac{\bar{c}_1 v(t)}{\underline{c}_2 u(t) + 1}\right], \quad \dot{v}(t) = v(t)\left[-\underline{a}_2 + \frac{\bar{d} e^{\hat{\beta}} u(t)}{\underline{c}_2 e^{\hat{\beta}} u(t) + 1}\right].$$

Let $\sigma = \max\{\Delta^1 + 1, \Delta^2 + 1\}$ and $\tilde{x} = \min\{\min_{0 \leq t \leq \omega} (1/2) d^{-1}(t) a_2(t), \underline{a}_2 e^{-\hat{\beta}} (\bar{d} - \underline{a}_2 \underline{c}_2)^{-1}\}$. We denote by $(\bar{u}(t), \bar{v}(t))$ and $(\underline{u}(t), \underline{v}(t))$ solutions of (5.9) and (5.8) with initial data $(u_0, v_0) \in \boldsymbol{R}_+^2$, respectively. We denote $\bar{u}(t) = \bar{u}(t, \tilde{x}, \sigma)$, $\bar{v}(t) = \bar{v}(t, \tilde{x}, \sigma)$. Then there exists a $\tau_1 > 0$, such that $\dot{\bar{u}}(\tau_1) = 0$ and $\dot{\bar{u}}(t) < 0$ for $t \in [0, \tau_1)$. Denote

$$\underline{u}(t) = \underline{u}(t, \bar{u}(\tau_1), \bar{v}(\tau_1)), \quad \bar{v}(t) = \underline{v}(t, \bar{u}(\tau_1), \bar{v}(\tau_1)).$$

Then there is a $\tau_2 > 0$ such that $\dot{\underline{v}}(\tau_2) = 0$. Define

$$\Gamma_1 = \{(\bar{u}(t), \bar{v}(t)) : 0 \leq t \leq \tau_1\}, \quad \Gamma_2 = \{(\underline{u}(t), \underline{v}(t)) : 0 \leq t \leq \tau_2\},$$

$$\Gamma_3 = \{(x, \underline{v}(\tau_2)) : \underline{u}(\tau_2) \leq x \leq \sigma\}, \quad \Gamma_4 = \{(\sigma, y) : \underline{v}(\tau_2) \leq y \leq \sigma\},$$

$$\Gamma_5 = \{(x, \sigma) : \tilde{x} \le x \le \sigma\} .$$

Then $\bigcup_{i=1}^{5} \Gamma_i$ constitutes the boundary of a closed bounded region Ω in the x-y plane. Note that Ω is independent of $(x(t), y(t))$.

(a) If $(x(t_1), y(t_1)) \in \Omega$ for some $t_1 > 0$, then an argument similar to that in Lemma 5.5 shows that for $t \ge t_1$, $X(t, \phi)$ stays in Ω.

(b) If there exists some $t_1 > 0$ such that $(x(t), y(t)) \notin \Omega$ for $t > t_1$, then we have two cases to consider:

(I) There exists a $T > 0$ such that $(x(t), y(t)) \in \Omega_1$ for $t > T$, where $\Omega_1 = \{(x, y) \in \boldsymbol{R}_+^2 : x < \tilde{x}, y < \sigma$ and $(x, y) \notin \Omega\}$. Then $\dot{y}(t) < 0$ and $\lim_{t \to +\infty} y(t) = 0$, which leads to $\limsup_{t \to +\infty} x(t) \ge \underline{a}_1 \bar{b}_1^{-1}$. However, from (5.6), we would then have $\dot{y}(t) > 0$ for some large t, a contradiction. Hence $(x(t), y(t))$ must leave Ω_1 and enter Ω_2 through $x = \tilde{x}$, where $\Omega_2 = \{(x, y) \in \boldsymbol{R}_+^2 : \sigma > x > \tilde{x}, 0 < y < \sigma$ and $(x, y) \notin \Omega\}$.

(II) It is clear that $(x(t), y(t))$ cannot enter Ω_1 from Ω_2 since $\bar{\Omega}_1 \cap \bar{\Omega}_2 = \{(\tilde{x}, y) \in \boldsymbol{R}_+^2 : 0 \le y \le \underline{v}(\tilde{\tau})$ where $\underline{u}(\tilde{\tau}) = \tilde{x}\}$ and on $\bar{\Omega}_1 \cap \bar{\Omega}_2$, $\dot{x}(t) > 0$. Suppose that there exist a $T > 0$ and a $\sigma^* > 0$ such that for $t > T$,

$$\underline{a}_1 - \bar{b}_1 x(t) - \frac{\bar{c}_1 y(t)}{\underline{c}_2 x(t) + 1} > \delta^* .$$

Then $\dot{x}(t) > 0$, which leads to $\lim_{t \to +\infty} x(t) = +\infty$, a contradiction. Hence there exists a $t_2 > 0$ such that $(x(t_2), y(t_2)) \in \Omega$ and by (a), we have that $(x(t), y(t)) \in \Omega$ for $t \ge t_2$. □

THEOREM 5.3. *Suppose that the conditions in Lemma* 5.5 *are true. If* (5.4) *has no positive constant solution, then the system* (5.4) *has a nonconstant positive* ω*-periodic solution.*

PROOF. By Lemma 5.4, Lemma 5.5 and Lemma 5.6, the condition (A4) is true. It is easy to check that the other conditions in Theorem 3.1 are satisfied. □

The above theorem implies that (5.4) has a nonconstant positive ω-periodic solution if $a_2(t)$ is small enough.

The application of Theorem 4.1 yields:

THEOREM 5.4. *Suppose that the conditions in Lemma* 5.5 *are true. Assume that there are positive numbers* p_1, q_1, q_2 *such that*

$$\begin{cases} p_1(-b_1(t) + c_1(t)c_2(t)\Delta^2) + q_1 < 0 , \\ -b_2(t) + p_1 c_1(t) + q_2 < 0 , \\ d(t) - q_1 \le 0 . \end{cases}$$

Then the system (5.4) *has a unique and globally asymptotically stable* ω*-periodic solution.*

PROOF. By Lemma 5.4 and the property that $X_{t+m\omega}(\phi)=X_t(X_{m\omega}(\phi))$, we confine ourselves to the initial space

$$\tilde{C}^n=\{\phi(\theta)\in C^n:\ 0\le\phi(\theta)\le\Delta\}\,.$$

Then the theorem follows from Theorem 4.1. □

The assumption of $b_2(t)>0$ is important but not essential. Without such an assumption, the proof will become even more technical and complicated. We leave this case to our future work.

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