Tôhoku Math. J.
49 (1997), 379–413

COMPACTIFICATIONS OF THE CONFIGURATION SPACE OF SIX POINTS OF THE PROJECTIVE PLANE AND FUNDAMENTAL SOLUTIONS OF THE HYPERGEOMETRIC SYSTEM OF TYPE (3, 6)

JIRO SEKIGUCHI* AND NOBUKI TAKAYAMA†

(Received February 13, 1996, revised September 24, 1996)

Abstract. We discuss two kinds of compactifications of the configuration space of six points in the complex projective plane. One is Naruki's cross ratio variety and the other is a toric variety obtained from the regular triangulations of the product of two copies of the 2-simplex. The former admits a biregular action of the Weyl group of type E_6. The latter admits a biregular action of $S_3 \times S_3$. The complement of the complex torus of the toric variety consists of normal crossing divisors. The action of $S_3 \times S_3$ leaves the set of normal crossing points invariant and decomposes this set into five orbits.

We explicitly show that the natural birational map between the two varieties is locally biregular around the normal crossing points of the toric variety and the corresponding points of the cross ratio variety. Utilizing this map, we study fundamental systems of solutions of the hypergeometric system $E(3, 6)$ on the cross ratio variety which is a natural domain of definition of the hypergeometric functions of type (3, 6).

1. Introduction. We consider the integral

$$f(y_1, \dots, y_n) = \int_{\text{a cycle}} \prod_i \Big(\sum_j y_{ij} t_j \Big)^{\beta_i} \Big(\sum_s (-1)^s t_s dt_1 \wedge \cdots dt_{s-1} \wedge dt_{s+1} \cdots \wedge dt_k \Big),$$

where $y_1 = (y_{11} : \cdots : y_{1k}), \dots, y_n = (y_{n1} : \cdots : y_{nk})$ are points on the projective space $\boldsymbol{P}^{k-1}$ and $(\beta_1, \dots, \beta_n)$ is a parameter with $\sum \beta_i = n-k$. We can naturally regard $f(y_1, \dots, y_n)$ as a function on the configuration space $P(k, n)$ of n points on the projective space $\boldsymbol{P}^{k-1}$:

$$P(k, n) = GL(k, \boldsymbol{C}) \backslash \{k \times n \text{ matrices of which all } k \times k \text{ minors are not } 0\} / (\boldsymbol{C}^*)^n .$$

The projective space $\boldsymbol{P}^{(k-1)(n-k-1)}$ is a compactification of the configuration space $P(k, n)$. The function $f(y_1, \dots, y_n)$ satisfies a holonomic system of differential equations on $\boldsymbol{P}^{(k-1)(n-k-1)}$ which we denote by $E(k, n)$ and has been intensively studied by a lot of people (see, e.g., [1], [12] and their bibliography).

We are interested in the following problems:

---

* Partly supported by the Grants-in-Aid for Scientific Research, the Ministry of Education, Science, Sports and Culture, Japan.

† Partly supported by the Grant-in-Aid for Encouragement of Young Scientists, the Ministry of Education, Science, Sports and Culture, Japan.

1991 *Mathematics Subject Classification*. Primary 33C70; Secondary 14M15, 14M25.

(A1) Construct a compactification X of the configuration space $P(k, n)$ so that $X-P(k, n)$ is the union of non-singular hypersurfaces with normal crossings.

(A2) Construct fundamental series solutions of the holonomic system on X around the normal crossing points.

We explain results concerning the problems (A1), (A2) for the Appell-Lauricella hypergeometric function $F_D(a, b_1, \ldots, b_n, c; x_1, \ldots, x_n)$ (cf. [2]) which is interpreted as a solution to $E(2, n+3)$ on $P(2, n+3)$ in a natural manner. In this case, Terada [37] constructed a compactification of $P(2, n+3)$ which is called the n-dimensional *Terada model* in [24]. We denote it by $\mathscr{M}_n$ for a moment. The Terada model $\mathscr{M}_n$ has some nice properties:

(B1) $\mathscr{M}_n$ is non-singular.

(B2) The configuration space $P(2, n+3)$ is regarded as a Zariski open subset of $\mathscr{M}_n$ and its complement S is the union of divisors with normal crossings.

(B3) S coincides with the pull-back to $\mathscr{M}_n$ of the singular locus of the holonomic system for F_D.

(B4) Permutations among $n+3$ points of the projective line naturally induce a biregular action of the symmetric group on $n+3$ letters on $\mathscr{M}_n$.

The properties (B1)–(B4) of the Terada model give an answer to the problem (A1) for the case of $P(2, n+3)$. As to the problem (A2) for the Appell-Lauricella case, it is possible to solve it for small n. For example, an answer for $E(2, 5)$ (the Appell function F_1) is given in [33], but it is not solved for general n. One reason for that is the difficulty in classifying the normal crossing points of $\mathscr{M}_n$ with respect to the natural action of the group S_{n+3}.

Let us return to the configuration space $P(k, n)$ and the system $E(k, n)$ on it $(k \geq 3, n \geq 2k)$. No one has yet tried to solve the problems (A1) and (A2). The toric variety constructed in [11] is a compactification of $P(k, n)$ related to holonomic systems $E(k, n)$, but it does not satisfy the properties corresponding to (B1) and (B2). For this reason, it is worthwhile to solve the problems above.

Compared with the general case, the three spaces $P(3, n)$ $(n=6, 7, 8)$ have fruitful geometric background related with classical topics on del Pezzo surfaces (cf. [6], [18]). For example, since a non-singular cubic surface in $\boldsymbol{P}^3$ is obtained as a six-point blowing up of $\boldsymbol{P}^2$, $P(3, 6)$ is regarded as a moduli space of cubic surfaces. In this case, from a purely geometric motivation to study a moduli space of marked cubic surfaces, Naruki [21] succeeded in constructing a compactification $\mathscr{C}$ of $P(3, 6)$ having properties similar to (B1) and (B2). In this paper, we call $\mathscr{C}$ *Naruki's cross ratio variety* following [13]. Since $\mathscr{C}$ enjoys the properties analogous to the properties (B1)–(B4), $\mathscr{C}$ can be regarded as a solution to the problem (A1) for $P(3, 6)$.

Noting these in mind, we focus our attention to the holonomic system $E(3, 6)$ in this paper. Our interest therefore lies in the following:

(C1) Study the compactification $\chi(N'(\Sigma(\Delta_2 \times \Delta_2)))$ (for definition, see §4) which is a modification of the toric variety constructed by the method in [11] in this case.

(C2) Clarify the relationship between $\mathscr{C}$ and the compactification $\chi(N'(\Sigma(\Delta_2 \times \Delta_2)))$.

(C3) Determine the normal crossing points of $\mathscr{C}$ and those of $\chi(N'(\Sigma(\Delta_2 \times \Delta_2)))$ and study the correspondence among them.

(C4) Construct fundamental solutions to the pull-back to $\mathscr{C}$ of $E(3, 6)$ around normal crossing points of $\mathscr{C}$ which correspond to those of $\chi(N'(\Sigma(\Delta_2 \times \Delta_2)))$.

We can say little about (C1) and (C2) and leave them for further study. As to (C3), we obtain a rather satisfactory results; we determine all the regular triangulation of the product of the 2-simplices $\Delta_2 \times \Delta_2$ and, roughly speaking, these correspond to normal crossing points of $\chi(N'(\Sigma(\Delta_2 \times \Delta_2)))$. Moreover, we clarify the correspondence in a concrete manner among these points and normal crossing points of $\mathscr{C}$ which we call normal crossing points of $\mathscr{C}$ attached to triangulations in this introduction. As a preparation for treating (C4), we construct three kinds of power series in four variables denoted by

$$(1) \qquad F_{(3,6),A}(\lambda; X_1, X_2, X_3, X_4)\,, \quad F_{(3,6),B}(\lambda; X_1, X_2, X_3, X_4)\,,$$
$$F_{(3,6),C}(\lambda; X_1, X_2, X_3, X_4)$$

depending on parameters

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) \quad (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = 0)\,.$$

Then our main result is that each fundamental solution of $E(3, 6)$ around each normal crossing point of $\mathscr{C}$ attached to a triangulation is expressed in terms of one of $F_{(3,6),Z}$ $(Z = A, B, C)$, only changing variables and parameters in a certain manner. This result is partly contained in a general theory of hypergeometric functions on $P(k, n)$ (cf. [11]), but it is stressed here that it is sufficient to use three kind of power series introduced above. There are normal crossing points of $\mathscr{C}$ which are not attached to triangulations. We do not enter into the construction of fundamental solutions around such points in this article and only give a remark at the end of §5.

We are going to briefly explain the contents of this article. Section 2 is devoted to the construction of a 4-dimensional toric variety associated to the triangulations of the product of two copies of the 2-simplex Δ_2. The product $\Delta_2 \times \Delta_2$ admits an $S_3 \times S_3$-action induced from the natural S_3-action on Δ_2. Our study starts with showing a result of Postnikov [12] on the $S_3 \times S_3$-orbital structure on the set of triangulations of $\Delta_2 \times \Delta_2$. By a general theory of [11], we construct a 4-dimensional toric variety associated to the triangulations and finally introduce a non-singular model of the toric variety and a power series to each triangulation. They are solutions of the holonomic system. In Section 3, we first review the definition of the configuration space of six points in P^2 and its compactification $\mathscr{C}$ due to Naruki. Naruki's cross ratio variety $\mathscr{C}$ admits a biregular $W(E_6)$-action, where $W(E_6)$ is the Weyl group of type E_6. It naturally contains S_6. Noting that the hypergeometric system of type (3, 6) is preserved by the S_6-action, we see S_6-orbits of normal crossing points of $\mathscr{C}$. By studying the intersections of hypersurfaces of $\mathscr{C}$, we determine that there are nine S_6-orbits denoted by (NC.i)

($i=1, 2, \ldots, 9$). In Section 4, we study the correspondence between the totality of $S_3 \times S_3$-orbits of triangulations investigated in Section 2 and those of normal crossing points of $\mathscr{C}$. The types of such points are (NC.i) ($i=2, 4, 5, 6, 7$). Noting this, we are going to construct fundamental solutions of the holonomic system around each normal crossing point whose type is one of (NC.i) ($i=2, 4, 5, 6, 7$). We show that each fundamental solution of the holonomic system around each normal crossing point whose type is one of (NC.i) ($i=2, 4, 5, 6, 7$) is expressed in terms of one of the functions above by a suitable choice of variables and parameters. Finally, we discuss fundamental solutions at other normal crossing points.

2. Triangulation of the product of two copies of 2-simplices. This section is devoted to a brief introduction of a toric variety associated to triangulations of the product of 2-simplices. Let Δ_2 be a 2-simplex. We first state a theorem due to Postnikov on the enumeration of all triangulations of $\Delta_2 \times \Delta_2$. Secondly, under the guidance of the general theory due to [11] and [14], we will construct a 4-dimensional toric variety associated to these triangulations and study properties of it. Finally, we will construct series solutions of the hypergeometric system of type (3, 6) on the toric variety based on the general method developed in [11].

We begin this section with giving an embedding of $\Delta_2 \times \Delta_2$ into the 6-dimensional Euclidean space: $\Delta_2 \times \Delta_2$ is the convex hull of the nine column vectors of the following matrix A regarded as points in $\boldsymbol{R}^6$:

$$A := \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

We denote by $\{i, j\}$ the $(3(i-1)+j)$-th column vector of the matrix A. By a triangulation of $\Delta_2 \times \Delta_2$, we mean a triangulation of the product of the simplices of which each vertex is one of the nine vectors $\{i, j\}$. We can show that any triangulation of $\Delta_2 \times \Delta_2$ consists of six 4-simplices. So each triangulation is given by six sets of five points.

Let $\mathscr{T}$ be the set of all triangulations of $\Delta_2 \times \Delta_2$. Since the 2-simplex Δ_2 admits an action of the group S_3 of the permutations on three letters, $\Delta_2 \times \Delta_2$ naturally admits an action of the product $S_3 \times S_3$. The set $\mathscr{T}$ is decomposed into $S_3 \times S_3$-orbits. The representatives of such orbits are given in the following theorem.

THEOREM 1 (cf. Postnikov [12, p. 249]). *The set $\mathscr{T}$ is decomposed into five $S_3 \times S_3$-orbits whose representatives are the triangulations T_i ($i=a, \ldots, e$) below*:

$$T_a = \begin{matrix} \{1,1\}, & \{1,2\}, & \{1,3\}, & \{2,3\}, & \{3,3\} \\ \{1,1\}, & \{1,2\}, & \{2,2\}, & \{2,3\}, & \{3,1\} \\ \{1,1\}, & \{2,1\}, & \{2,2\}, & \{2,3\}, & \{3,1\} \\ \{1,2\}, & \{2,2\}, & \{2,3\}, & \{3,1\}, & \{3,3\} \\ \{1,2\}, & \{2,2\}, & \{3,1\}, & \{3,2\}, & \{3,3\} \\ \{1,1\}, & \{1,2\}, & \{2,3\}, & \{3,1\}, & \{3,3\} \end{matrix}$$

$$T_b = \begin{matrix} \{1,1\}, & \{1,2\}, & \{1,3\}, & \{2,3\}, & \{3,2\} \\ \{1,1\}, & \{1,2\}, & \{2,2\}, & \{2,3\}, & \{3,2\} \\ \{1,1\}, & \{2,1\}, & \{2,2\}, & \{2,3\}, & \{3,2\} \\ \{1,1\}, & \{2,1\}, & \{2,3\}, & \{3,1\}, & \{3,2\} \\ \{1,1\}, & \{2,3\}, & \{3,1\}, & \{3,2\}, & \{3,3\} \\ \{1,1\}, & \{1,3\}, & \{2,3\}, & \{3,2\}, & \{3,3\} \end{matrix}$$

$$T_c = \begin{matrix} \{1,1\}, & \{1,2\}, & \{1,3\}, & \{2,3\}, & \{3,1\} \\ \{1,1\}, & \{1,2\}, & \{2,2\}, & \{2,3\}, & \{3,1\} \\ \{1,1\}, & \{2,1\}, & \{2,2\}, & \{2,3\}, & \{3,1\} \\ \{1,2\}, & \{2,2\}, & \{2,3\}, & \{3,1\}, & \{3,2\} \\ \{1,3\}, & \{2,3\}, & \{3,1\}, & \{3,2\}, & \{3,3\} \\ \{1,2\}, & \{1,3\}, & \{2,3\}, & \{3,1\}, & \{3,2\} \end{matrix}$$

$$T_d = \begin{matrix} \{1,1\}, & \{1,2\}, & \{1,3\}, & \{2,3\}, & \{3,1\} \\ \{1,1\}, & \{1,2\}, & \{2,2\}, & \{2,3\}, & \{3,1\} \\ \{1,1\}, & \{2,1\}, & \{2,2\}, & \{2,3\}, & \{3,1\} \\ \{1,2\}, & \{2,2\}, & \{2,3\}, & \{3,1\}, & \{3,2\} \\ \{1,2\}, & \{2,3\}, & \{3,1\}, & \{3,2\}, & \{3,3\} \\ \{1,2\}, & \{1,3\}, & \{2,3\}, & \{3,1\}, & \{3,3\} \end{matrix}$$

$$T_e = \begin{matrix} \{1,1\}, & \{1,2\}, & \{1,3\}, & \{2,3\}, & \{3,1\} \\ \{1,1\}, & \{1,2\}, & \{2,2\}, & \{2,3\}, & \{3,1\} \\ \{1,1\}, & \{2,1\}, & \{2,2\}, & \{2,3\}, & \{3,1\} \\ \{1,2\}, & \{2,2\}, & \{2,3\}, & \{3,1\}, & \{3,3\} \\ \{1,2\}, & \{2,2\}, & \{3,1\}, & \{3,2\}, & \{3,3\} \\ \{1,2\}, & \{1,3\}, & \{2,3\}, & \{3,1\}, & \{3,3\} \end{matrix} .$$

The lengths of the orbits of T_a, T_b *and* T_c *are* 12, 6 *and* 18, *respectively. The lengths of the orbits of* T_d *and* T_e *are* 36. *The product* $\Delta_2 \times \Delta_2$ *admits* 108 *triangulations. Furthermore, these triangulations are regular (coherent) triangulations in the sense of Definition* 2.3 *in* [12, p. 228].

REMARK 1. Theorem 1 was firstly obtained by Postnikov around the end of the 1980's. Though there is no reference on how he derived it, it is possible to check his result in the following manner. We first enumerate and triangulations of the boundary of $\Delta_2 \times \Delta_2$. Next we check the possibility of extensions of the triangulations of the boundary obtained above to those of $\Delta_2 \times \Delta_2$. (We are deeply indebted to *Mathematica* in carrying out this idea.)

Although no algorithmic method to obtain all triangulations is known, we have a systematic method to get all regular triangulations; computer programs are available to enumerate all regular triangulations (cf. [17] and [19]). Actually the 108 regular triangulations of $\Delta_2 \times \Delta_2$ can be obtained in a few minutes by means of these programs. The readers who are interested in the algorithm may consult [4] and [12, pp. 231–233]. Here, we only note that the enumeration is done by utilizing *the circuits* of the nine points. *A* subset Z of the nine points is called a circuit if any proper subset of Z is linearly independent but Z itself is linearly dependent. Let us denote by the 3×3 matrix (c_{ij}) a circuit of the nine points; the set of $\{i, j\}$ for which $c_{ij} \neq 0$ is the circuit and moreover (c_{ij}) corresponds to the relation

$$\sum_{ij} c_{ij}\{i, j\} = 0 .$$

In the case of $\Delta_2 \times \Delta_2$, the $\boldsymbol{S}_3 \times \boldsymbol{S}_3$-orbits of

$$c_1 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

are all the circuits. The enumeration of the regular tiangulations is done by modifying a given triangulation along a suitable circuit. Modifications along circuits and flops of hypergeometric functions have a close relationship. As to this topic, see [36].

Let T be a regular triangulation of $\Delta_2 \times \Delta_2$. We define a vector $\phi_T = ((\phi_T)_{ij})$, $(1 \leq i \leq 3, 1 \leq j \leq 3)$ in $\boldsymbol{Z}^{3 \times 3}$ by letting $(\phi_T)_{ij}$ to be the number of the appearances of the vertex $\{i, j\}$ in the triangulation T. For example

$$\phi_{T_b} = \begin{pmatrix} 6 & 2 & 2 \\ 2 & 2 & 6 \\ 2 & 6 & 2 \end{pmatrix},$$

where the triangulation T_b is as given in Theorem 1.

The secondary polytope $\Sigma(\Delta_2 \times \Delta_2)$ is the convex hull of the 108 vectors $\{\phi_T \mid T \in \mathscr{T}\}$ in $\boldsymbol{R}^{3 \times 3}$. The following theorem is shown by an implementation of the algorithm obtaining the convex hull of a given set of points by Edelsbrunner (cf. [7, Chap. 8]).

THEOREM 2. (i) *The secondary polytope $\Sigma(\Delta_2 \times \Delta_2)$ is a* 4-*dimensional polytope in $\boldsymbol{R}^9$. It has* 108 *vertices corresponding to the* 108 *regular triangulations. The numbers of* 1-, 2-, 3-*faces are* 222, 144, 30, *respectively.*

(ii) *The secondary polytope $\Sigma(\Delta_2 \times \Delta_2)$ has two types of vertices corresponding* *to the properties* (1) *and* (2) *below*:

(1) *The numbers of the adjacent* 1-, 2- *and* 3-*faces are* 6, 9 *and* 5, *respectively.*

(2) *The numbers of the adjacent* 1-, 2- *and* 3-*faces are* 4, 6 *and* 4, *respectively.*

ϕ_{T_b} *is of the type* (1) *and* ϕ_{T_i}, $(i=a, c, d, e)$ *are of the type* (2).

(iii) *The triangulation* T_a *has support on the circuit* c_2 *in the sense of* [12, p. 232, Definition 2.9].

(iv) *The facets of the secondary polytope decompose into three* $S_3 \times S_3$ *orbits. Let* f_1, f_2 *and* f_3 *be representatives of the orbits respectively. The* 3-*polytope* f_1 *has twelve facets consisting of four* 4-*gons, four* 5-*gons and four* 6-*gons. The* 3-*polytope* f_2 *has twelve facets consisting of eight* 4-*gons and four* 6-*gons. The* 3-*polytope* f_3 *has six facets consisting of six* 4-*gons.*

REMARK 2. The statement (iii) can be understood as a combinatorial counterpart to the fact that the power series $F_{(3,6),B}(X_1, X_2, X_3, X_4)$ introduced later (cf. §5) is reduced to the generalized hypergeometric function ${}_3F_2(X_4)$ when $X_1 = X_2 = X_3 = 0$. Details on this subject will be discussed elsewhere (see also [36]).

We consider the normal fan $N(\Sigma(\Delta_2 \times \Delta_2))$ of the secondary polytope $\Sigma(\Delta_2 \times \Delta_2)$; the normal fan is the collection of the normal cones at the faces f:

$$N(\Sigma(\Delta_2 \times \Delta_2), f) = \{v \mid \langle v, p-q \rangle \geq 0 \text{ for all } p \in \Sigma(\Delta_2 \times \Delta_2) \text{ and all } q \in f\} .$$

We are going to consider the 4-dimensional toric variety $\chi(N(\Sigma(\Delta_2 \times \Delta_2)))$ defined by the normal fan. Let C be a cone of the fan $N(\Sigma(\Delta_2 \times \Delta_2))$. We can get the semi-group ring defined by the integral points of the dual cone $\boldsymbol{C}[C^\vee \cap \boldsymbol{Z}^{3\times 3}]$. The toric variety $\chi(N(\Sigma(\Delta_2 \times \Delta_2)))$ is obtained by gluing the spectra of the semi-group rings corresponding to the cones of the fan by the incidence relations among the cones; the semi-group rings are the coordinate rings of the affine charts of the toric variety (see [23, §2], or [9, §§1.3, 1.4 and 1.5] for the definitions on toric varieties). Noting the definition in mind, we are going to look at the semi-group ring corresponding to each of the normal cones at the vertices ϕ_{T_i}. We put

$$\boldsymbol{v} = {}^t(1 \quad 1 \quad 1) .$$

Let τ be a simplex of a regular triangulation T in $\mathscr{T}$. Then there exist four vectors $b_\tau^{(ij)}$ ($\{i, j\} \notin \tau$) in $\boldsymbol{Z}^{3\times 3}$ such that

(D1) $(b_\tau^{(ij)})_{ij} = 1$

(D2) $(b_\tau^{(ij)})_{kl} = 0$ $(\{k, l\} \notin \tau,\ \{k, l\} \neq \{i, j\})$

(D3) $b_\tau^{(ij)} \in \ker(A : \boldsymbol{Z}^{3\times 3} \to \boldsymbol{Z}^6)$, i.e., $b_\tau^{(ij)}\boldsymbol{v} = {}^t b_\tau^{(ij)}\boldsymbol{v} = 0$.

We can show that the conditions (D1), (D2) and (D3) uniquely determine the vector $b_\tau^{(ij)}$. For example, if τ is given by the five points

$$\{1, 1\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 3\} ,$$

then the four vectors $b_\tau^{(21)}$, $b_\tau^{(22)}$, $b_\tau^{(31)}$, $b_\tau^{(33)}$ are as follows:

$$\begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

(The explicit forms of $b_\tau^{(ij)}$ corresponding to the triangulations T_a, T_b, T_c, T_d, T_e will be given in Section 6.)

PROPOSITION 1. *The normal cone at the vertex ϕ_{T_i} is given by*

$$N(\Sigma(\Delta_2 \times \Delta_2), \phi_{T_i}) = \{x \in \mathbf{R}^{3\times 3} \mid \langle x, b_\tau^{(pq)} \rangle \geq 0, \tau \in T_i, \{p, q\} \notin \tau\}.$$

PROOF. The normal cone to the secondary polytope at the point ϕ_{T_i} agrees with the cone of the weight vectors for the regular triangulation T_i (cf. [12, p. 228, Theorem 2.4]). The cone of the weight vectors are characterized by the right-hand side of the proposition by [4, Lemma 4.2]. q.e.d.

Consider 3×3 indeterminates u_{ij} $(1 \leq i \leq 3, 1 \leq j \leq 3)$. For each $b=(b_{ij}) \in \mathbf{Z}^{3\times 3}$, we put $u^b = \prod_{ij} u_{ij}^{b_{ij}}$. The following proposition is a consequence of Proposition 1; what we have only to do is to show that $\{b_\tau^{pq} \mid \tau \in T_i, \{p, q\} \notin \tau\}$ generates the semi-group $N(\Sigma(\Delta_2 \times \Delta_2), \phi_{T_i})^\vee \cap \mathbf{Z}^{3\times 3}$ which follows from case-by-case computations.

PROPOSITION 2.

$$\mathbf{C}[N(\Sigma(\Delta_2 \times \Delta_2), \phi_{T_i})^\vee \cap \mathbf{Z}^{3\times 3}] \simeq \mathbf{C}[u^{b_\tau^{(pq)}} \mid \tau \in T_i, \{p, q\} \notin \tau].$$

The ring given in the proposition is the coordinate ring of the affine toric variety defined by the normal cone at the vertex ϕ_{T_i}. The coordinate ring is isomorphic to $\mathbf{C}[x_1, x_2, x_3, x_4]$ in the case of $T_i \neq T_b$, but not isomorphic in the case of $T_i = T_b$ from the following theorem and a general argument in the theory of toric varieties.

THEOREM 3. *There exist two types of maximal cones;*

(1) *$N(\Sigma(\Delta_2 \times \Delta_2), \phi_{T_i})$ is the direct sum of a linear space and a 4-dimensional unimodular cone where $i = a, c, d, e$.*

(2) *$N(\Sigma(\Delta_2 \times \Delta_2), \phi_{T_b})$ is not unimodular.*

The theorem can be shown by explicit presentation of the secondary polytope $\Sigma(\Delta_2 \times \Delta_2)$ as a convex hull of 108 vectors $\{\phi_T\}$.

It is well-known in the theory of toric varieties that the toric variety defined by a given fan is non-singular if and only if all the cones are unimodular (cf. [23, Theorem 1.10] or [9, p. 29]). In our case, since the cone at the point ϕ_{T_b} is not unimodular, the toric variety $\chi(N(\Sigma(\Delta_2 \times \Delta_2)))$ is singular. We look at the coordinate ring for the cone at ϕ_{T_b} and refine the cone to get a unimodular fan as follows.

THEOREM 4. (i) *The semi-group*

$$N(\Sigma(\Delta_2 \times \Delta_2), \phi_{T_b})^\vee \cap \mathbf{Z}^{3\times 3} =: S^\vee$$

is generated by the following six vectors:

$$\tilde{x}:=\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \tilde{y}:=\begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix},\ \tilde{p}:=\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix},\ \tilde{q}:=\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix},$$

$$\tilde{x}+\tilde{y}-\tilde{p}=\begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix},\quad \tilde{p}+\tilde{q}-\tilde{x}=\begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\begin{aligned} \boldsymbol{C}[S^\vee] &\overset{f_1}{\underset{\sim}{\longrightarrow}} \boldsymbol{C}[x, y, p, q, xyp^{-1}, pqx^{-1}] \\ &\overset{f_2}{\underset{\sim}{\longrightarrow}} \boldsymbol{C}[x_1, x_2, x_3, y_1, y_2, y_3]/(x_1y_2-x_2y_1, x_1y_3-x_3y_1, x_2y_3-x_3y_2), \end{aligned}$$

where f_1 is defined by

$$u^{\tilde{x}} \mapsto x,\quad u^{\tilde{y}} \mapsto y,\quad u^{\tilde{p}} \mapsto p,\quad u^{\tilde{q}} \mapsto q$$

and f_2 is defined by

$$x \mapsto x_1,\quad y \mapsto y_3,\quad p \mapsto y_1,\quad q \mapsto x_2,\quad xyp^{-1} \mapsto x_3,\quad pqx^{-1} \mapsto y_2\,.$$

(ii) *Put*

$$p_1=(0, 0, 0, 1),\ p_2=(1, 0, 0, 1),\ p_3=(0, 1, 0, 0),$$
$$p_4=(1, 0, 1, 0),\ p_5=(0, 1, 1, 0)$$

and

$$q_1=(1, 0, 0, 0),\ q_2=(0, 1, 0, 0),\ q_3=(0, 0, 1, 0),$$
$$q_4=(0, 0, 0, 1),\ q_5=(1, 1, -1, 0),\ q_6=(-1, 0, 1, 1)\,.$$

Then

$$\boldsymbol{C}\left[\left(\sum_{i=1}^{5} \boldsymbol{R}_{\geq 0}p_i\right)^\vee \cap \boldsymbol{Z}^4\right]=\boldsymbol{C}\left[\sum_{j=1}^{6} \boldsymbol{Z}_{\geq 0}q_j\right] \simeq \boldsymbol{C}[x, y, p, q, xyp^{-1}, pqx^{-1}]$$

where

$$q_1 \leftrightarrow x,\quad q_2 \leftrightarrow y,\quad q_3 \leftrightarrow p,\quad q_4 \leftrightarrow q,\quad q_5 \leftrightarrow xyp^{-1},\quad q_6 \leftrightarrow pqx^{-1}\,.$$

(iii) *Put*

$$C_1=(\boldsymbol{R}_{\geq 0}p_1+\boldsymbol{R}_{\geq 0}p_3+\boldsymbol{R}_{\geq 0}p_4+\boldsymbol{R}_{\geq 0}p_5),\quad C_2=(\boldsymbol{R}_{\geq 0}p_1+\boldsymbol{R}_{\geq 0}p_2+\boldsymbol{R}_{\geq 0}p_3+\boldsymbol{R}_{\geq 0}p_4)\,.$$

The cones C_1 and C_2 are unimodular and $C_1 \cup C_2=\sum_{i=1}^{5} \boldsymbol{R}_{\geq 0}p_i$.

(iv)

$$C[C_1^\vee \cap Z^4] \simeq C[x^{-1}p, x, xyp^{-1}, q],$$
$$C[C_2^\vee \cap Z^4] \simeq C[xp^{-1}, y, p, x^{-1}, pq]$$

where the correspondence between the monomials and lattice points is that of (ii).

PROOF. The statement (i) follows from Proposition 2.

Let us show (ii). We first note that q_j, $(j=1,\dots,6)$ are exponent vectors of monomials $x, y, p, q, xyp^{-1}, pqx^{-1}$. Taking the dual cone of $\sum_{j=1}^{6} \boldsymbol{R}_{\geq 0} q_j$, we obtain vectors p_i. The isomorphisms of the rings in (ii) can be easily checked.

We have (iii), because $|\det(p_1, p_3, p_4, p_5)| = |\det(p_1, p_2, p_3, p_4)| = 1$.

The statement (iv) is easy to prove. q.e.d.

The cone C_i in Theorem 4 (iii) above defines the corresponding cone contained in the cone of the fan $N(\Sigma(\Delta_2 \times \Delta_2))$ that we also call C_i. The orbit of the cone C_1 by the action of $\boldsymbol{S}_3 \times \boldsymbol{S}_3$ consists of twelve elements which contains C_2 (we checked this fact by *Mathematica*). The other cones in the orbit is outside of $C_1 \cup C_2$, which means that the action of $\boldsymbol{S}_3 \times \boldsymbol{S}_3$ is compatible with the refinement by C_1 and C_2. Thus, we obtain a refined fan $N'(\Sigma(\Delta_2 \times \Delta_2))$ by taking the $\boldsymbol{S}_3 \times \boldsymbol{S}_3$ orbit of the cone C_1 in the fan $N(\Sigma(\Delta_2 \times \Delta_2))$. This fan consists of 114 maximal cones and admits the action of $\boldsymbol{S}_3 \times \boldsymbol{S}_3$. The toric variety $\chi(N'(\Sigma(\Delta_2 \times \Delta_2)))$ is non-singular. The proper regular map from $\chi(N'(\Sigma(\Delta_2 \times \Delta_2)))$ to $\chi(N(\Sigma(\Delta_2 \times \Delta_2)))$ is denoted by r.

Before closing this section, we review the construction due to [11] of series solutions of the hypergeometric system of type (3, 6) which is denoted by $E(3, 6)$ from now on. We regard the series as functions on the non-singular toric variety that has been constructed. In the sequel, we take parameters $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ with the condition

$$\alpha_1 + \alpha_2 + \alpha_3 = \beta_1 + \beta_2 + \beta_3 . \tag{2}$$

We put

$$\alpha = {}^t(\alpha_1 \ \ \alpha_2 \ \ \alpha_3), \quad \beta = {}^t(\beta_1 \ \ \beta_2 \ \ \beta_3).$$

We take a regular triangulation T of $\mathscr{T}$ and its simplex τ. Then there exist four vectors $b_\tau^{(ij)}$ $(\{i,j\} \notin \tau)$ with conditions (D1), (D2), (D3). Associated to the four vectors, we introduce a semi-lattice $L(\tau)$ defined by

$$L(\tau) = \sum_{\{i,j\} \notin \tau} \boldsymbol{Z}_{\geq 0} b_\tau^{(ij)},$$

which is on a four-dimensional subspace of $\boldsymbol{R}^{3\times 3}$.

On the other hand, we take a 3×3 matrix $\gamma = (\gamma_{ij})$ such that

$$\gamma \boldsymbol{v} = \alpha, \quad {}^t\gamma \boldsymbol{v} = \beta, \tag{3}$$

$$\gamma_{ij} = 0 \quad \text{if } \{i,j\} \notin \tau . \tag{4}$$

We now consider a 3×3 matrix

$$u=\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix} \tag{5}$$

and by using the semi-lattice $L(\tau)$ and the 3×3 matrix γ introduced above, we define a formal power series in u

$$F_{\tau,T}=\sum_{k\in L(\tau)} u^{k+\gamma}/\Gamma(1+k+\gamma)\,, \tag{6}$$

where

$$u^{k+\gamma}=\prod_{i,j} u_{ij}^{k_{ij}+\gamma_{ij}}\,, \tag{7}$$

$$\Gamma(1+k+\gamma)=\prod_{i,j}\Gamma(1+k_{ij}+\gamma_{ij})\,. \tag{8}$$

Let $\tau_1=\tau$, $\tau_2\dots,\tau_6$ be the six simplices of T. Then

$$F_{\tau_1,T}\,,\quad F_{\tau_2,T}\,,\quad F_{\tau_3,T}\,,\quad F_{\tau_4,T}\,,\quad F_{\tau_5,T}\,,\quad F_{\tau_6,T}$$

are linearly independent over $\boldsymbol{C}$ for generic choices of the parameters α_i and β_j. These functions are naturally regarded as solutions of the hypergeometric system $E(3, 6)$.

Later, we shall compute explicit forms of the functions defined by the series of the form (6).

3. The configuration space of six points in $\boldsymbol{P}^2$. In this section, we will first reviw the configuration space $P(3, 6)$ of six points in $\boldsymbol{P}^2$ and its compactification $\mathscr{C}$ due to Naruki. For details on this subject and related topics, see [21], [30], [31]. There exist seventy-five non-singular hypersurfaces whose union coincides with the complement of $P(3, 6)$ in $\mathscr{C}$. It is better to consider one more hypersurface denoted by Y_r of $\mathscr{C}$ when we treat $\mathscr{C}$ as a variety with $W(E_6)$-action. As a preparation for our purpose, we will study normal crossing points of the $76(=75+1)$ hypersurfaces of $\mathscr{C}$. In particular, we will determine the $\boldsymbol{S}_6$-orbit decomposition of the set of such points, regarding $\boldsymbol{S}_6$ as a subgroup of $W(E_6)$ in the standard manner.

We begin with defining the configuration space of six points in $\boldsymbol{P}^2$. For this purpose, we first introduce the linear space $M_{3,6}$ of 3×6 matrices:

$$M_{3,6}=\left\{\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & x_{36} \end{pmatrix};\ x_{ij}\in\boldsymbol{C}(1\le i\le 3,\ 1\le j\le 6)\right\}.$$

Clearly $M_{3,6}$ admits a left $GL(3, \boldsymbol{C})$-action and a right $GL(6, \boldsymbol{C})$-action in a natural way. For a moment, we identify $(\boldsymbol{C}^*)^6$ with the maximal torus of $GL(6, \boldsymbol{C})$ consisting of diagonal matrices and consider the action of $GL(3, \boldsymbol{C})\times(\boldsymbol{C}^*)^6$ on $M_{3,6}$ instead of that of $GL(3, \boldsymbol{C})\times GL(6, \boldsymbol{C})$.

Let $M'_{3,6}$ be the open subset of $M_{3,6}$ defined by

$$M'_{3,6}=\left\{\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & x_{36} \end{pmatrix}\in W;\ D(i_1, i_2, i_3)\neq 0\ (1\leq i_1<i_2<i_3\leq 6)\right\},$$

where

$$D(i_1, i_2, i_3)=\det\begin{pmatrix} x_{1i_1} & x_{1i_2} & x_{1i_3} \\ x_{2i_1} & x_{2i_2} & x_{2i_3} \\ x_{3i_1} & x_{3i_2} & x_{3i_3} \end{pmatrix}.$$

Then for any element $X\in M'_{3,6}$, there exist $(g, h)\in GL(3, \boldsymbol{C})\times(\boldsymbol{C}^*)^6$ and $(x_1, x_2, y_1, y_2)\in \boldsymbol{C}^4$ such that

$$gXh=\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & x_1 & x_2 \\ 0 & 0 & 1 & 1 & y_1 & y_2 \end{pmatrix}.$$

In this sense, $P(3, 6)=GL(3, \boldsymbol{C})\backslash M'_{3,6}/(\boldsymbol{C}^*)^6$ is identified with an open subset of $\boldsymbol{C}^4$. In this article, $P(3, 6)$ is called the configuration space of six points in $\boldsymbol{P}^2$. Transpositions of column vectors of $X\in M'_{3,6}$ induce birational transformations on $\boldsymbol{C}^4$ with coordinate system (x_1, x_2, y_1, y_2). Let $\tilde{s}_j$ $(1\leq j\leq 5)$ be the birational transformation on $\boldsymbol{C}^4$ corresponding to the transposition of the j-th column vector and $(j+1)$-column vector of $X\in M'_{3,6}$. Then, by an easy computation, we obtain

$$\tilde{s}_1:(x_1, x_2, y_1, y_2)\longrightarrow\left(\frac{1}{x_1}, \frac{1}{x_2}, \frac{y_1}{x_1}, \frac{y_2}{x_2}\right),$$

$$\tilde{s}_2:(x_1, x_2, y_1, y_2)\longrightarrow(y_1, y_2, x_1, x_2),$$

$$\tilde{s}_3:(x_1, x_2, y_1, y_2)\longrightarrow\left(\frac{x_1-y_1}{1-y_1}, \frac{x_2-y_2}{1-y_2}, \frac{y_1}{y_1-1}, \frac{y_2}{y_2-1}\right),$$

$$\tilde{s}_4:(x_1, x_2, y_1, y_2)\longrightarrow\left(\frac{1}{x_1}, \frac{x_2}{x_1}, \frac{1}{y_1}, \frac{y_2}{y_1}\right),$$

$$\tilde{s}_5:(x_1, x_2, y_1, y_2)\longrightarrow(x_2, x_1, y_2, y_1).$$

Let $\boldsymbol{S}_6$ be the symmetric group on six letters. If s_j is the transposition of j and $j+1$, $\boldsymbol{S}_6$ is generated by $s_1, \dots, s_5$. Then, from the construction, it is clear that the correspondence $s_j\mapsto\tilde{s}_j$ $(1\leq j\leq 5)$ induces a birational action of $\boldsymbol{S}_6$ on $\boldsymbol{C}^4$. In the sequel, we frequently identify $\boldsymbol{S}_6$ with the group generated by $\tilde{s}_j$ $(1\leq j\leq 5)$ and we frequently use s_j and $\tilde{s}_j$ interchangeably. The birational transformations s_j $(j=1, \dots, 5)$ are nonsingular outside the union of the fourteen hypersurfaces $R_j=\{p_j=0\}$ $(1\leq j\leq 14)$, where

$$p_1=x_1y_2-x_2y_1-x_1+x_2+y_1-y_2\,,\quad p_2=y_1-1\,,\quad p_3=x_1-1\,,\quad p_4=y_2-1\,,$$
$$p_5=x_2-1\,,\quad p_6=y_1-y_2\,,\quad p_7=x_1-x_2\,,\quad p_8=x_1-y_1\,,\quad p_9=x_2-y_2\,,$$
$$p_{10}=x_1y_2-x_2y_1\,,\quad p_{11}=x_2\,,\quad p_{12}=x_1\,,\quad p_{13}=y_2\,,\quad p_{14}=y_1\,.$$

Let s_0 be the birational transformation on $\boldsymbol{C}^4$ defined by

$$s_0: (x_1, x_2, y_1, y_2) \longrightarrow (1/x_1, 1/x_2, 1/y_1, 1/y_2)\,.$$

Then the group $\tilde{G}$ generated by $s_1, \dots, s_5$ and s_0 is isomorphic to the Weyl group of type E_6 as will be seen soon. We define the hypersurface $R_{15}=\{p_{15}=0\}$, where

$$p_{15}=x_1y_2(1-y_1)(1-x_2)-x_2y_1(1-x_1)(1-y_2)\,.$$

It follows from the definition that $s_1, \dots, s_5$, s_0 and therefore all the elements of $\tilde{G}$ are biregular outside the union R of the hypersurfaces R_j $(1\leq j\leq 15)$.

We are going to introduce the root system Δ of type E_6. For this purpose, we consider the 8-dimensional Euclidean space $\tilde{E}$ with a standard basis $\varepsilon_1, \dots, \varepsilon_8$. Let $\langle\cdot,\cdot\rangle$ be the inner product on $\tilde{E}$ defined by

$$\langle\varepsilon_j, \varepsilon_k\rangle=\delta_{jk}$$

and let E be the linear subspace of $\tilde{E}$ spanned by the six vectors

$$\varepsilon_1, \dots, \varepsilon_5\,,\quad \tilde{\varepsilon}=\varepsilon_6-\varepsilon_7-\varepsilon_8\,.$$

We introduce the thirty-six vectors

$$r=-\frac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5+\tilde{\varepsilon})\,,$$
$$r_{1j}=-\varepsilon_{j-1}+r_0\,,\quad 1<j<7$$
$$r_{jk}=\varepsilon_{j-1}-\varepsilon_{k-1}\,,\quad 1<j<k<7$$
$$r_{1jk}=-\varepsilon_{j-1}-\varepsilon_{k-1}\,,\quad 1<j<k<7$$
$$r_{ijk}=-\varepsilon_{i-1}-\varepsilon_{j-1}-\varepsilon_{k-1}+r_0\,,\quad 1<i<j<k<7$$

following [13], where

$$r_0=\frac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5-\tilde{\varepsilon})\,.$$

It is possible to take

$$\alpha_1=r_{12}\,,\quad \alpha_2=r_{123}\,,\quad \alpha_3=r_{23}\,,\quad \alpha_4=r_{34}\,,\quad \alpha_5=r_{45}\,,\quad \alpha_6=r_{56}$$

as a set of positive simple roots. Then the Dynkin diagram is as in Figure.

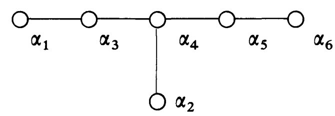

FIGURE.

Moreover, r, r_{jk}, r_{ijk} defined above are the totality of positive roots of Δ and r is the longest root.

Let g_j be the reflection on E with respect to α_j ($j=1,\dots,6$) and let g_0 be the reflection on E with respect to r. Then the Weyl group $W(E_6)$ of type E_6 is generated by g_j ($j=1,\dots,6$). The relation between $W(E_6)$ and the group generated by the birational transformations $\tilde{s}_j$ ($j=1,\dots,5$) and s_0 is given by the following lemma.

LEMMA 1. (i) *The correspondence*

$$g_1 \longmapsto \tilde{s}_1, \quad g_2 \longmapsto s_0, \quad g_3 \longmapsto \tilde{s}_2, \quad g_4 \longmapsto \tilde{s}_3, \quad g_5 \longmapsto \tilde{s}_4, \quad g_6 \longmapsto \tilde{s}_5$$

induces a group isomorphism Φ of $W(E_6)$ to $\tilde{G}$.

(ii) *If $c_0=\Phi(g_0)$, then*

$$c_0: \begin{cases} x_1 \longmapsto (x_1y_2-x_2y_1)(y_2-1)/((x_2-y_2)(y_1-y_2)) \\ x_2 \longmapsto (x_1y_2-x_2y_1)(y_1-1)/((x_1-y_1)(y_1-y_2)) \\ y_1 \longmapsto (x_1y_2-x_2y_1)(x_2-1)/((x_1-x_2)(x_2-y_2)) \\ y_2 \longmapsto (x_1y_2-x_2y_1)(x_1-1)/((x_1-x_2)(x_1-y_1)) . \end{cases}$$

Let G_0 be the group generated by $\boldsymbol{S}_6$ and c_0. Since $c_0^2=\mathrm{id}$ by definition and since c_0 centralizes $\boldsymbol{S}_6$, we find that $G_0 \simeq \boldsymbol{S}_6 \times \boldsymbol{Z}_2$.

REMARK 3. We now introduce an involution c on $\boldsymbol{C}^4$ defined by

$$c: (x_1, x_2, y_1, y_2) \longmapsto (y_2, x_2, y_1, x_1) .$$

Then it is easy to show that $c_0=c\circ(14)(26)(35)$, where (ij) means the transposition of i and j. In particular, G_0 is also generated by $\boldsymbol{S}_6$ and c.

We write

$$t_j=\langle \varepsilon_j, t\rangle , \quad j=1,\dots,5 , \quad t_6=\langle \tilde{\varepsilon}, t\rangle$$

for any $t\in E$. Then the linear forms h, h_{jk}, h_{ijk} given in [30] correspond to positive roots r, r_{jk}, r_{ijk}, that is,

$$h=\langle r, t\rangle , \quad h_{jk}=\langle r_{jk}, t\rangle , \quad h_{ijk}=\langle r_{ijk}, t\rangle .$$

We are going to define an embedding of $P(3, 6)$ into Naruki's cross ratio variety along the line in [30] and [31]. Let $Z(\Delta)$ be the Zariski open subset of $\boldsymbol{P}^5$ defined by

$$h\cdot \prod_{j<k} h_{jk}\cdot \prod_{i<j<k} h_{ijk}\neq 0 .$$

We now recall the definition of the D_4-cross ratio maps (cf. [31]). A D_4-cross ratio map of $Z(\Delta)$ to $\mathrm{CR}(\boldsymbol{P})$ is given by

$$t \longmapsto (h_{35}h_{345}h_{26}h_{246} : -h_{25}h_{245}h_{36}h_{346} : h_{23}h_{234}h_{56}h_{456}),$$

where $\mathrm{CR}(\boldsymbol{P})$ is the hyperplane of $\boldsymbol{P}^2$ with homogeneous coordinate $\xi=(\xi_1:\xi_2:\xi_3)$ defined by $\xi_1+\xi_2+\xi_3=0$. By permutations of indices among 1, 2, 3, 4, 5, 6, we obtain thirty maps of the form above. There is another D_4-cross ratio map defined by

$$t \longmapsto (h_{135}h_{245}h_{236}h_{146} : -h_{235}h_{145}h_{136}h_{246} : h_{12}h_{34}h_{56}h).$$

In this case, by permutations of indices among 1, 2, 3, 4, 5, 6, we obtain fifteen maps of the form above. As a result, we obtain 45 $(=30+15)$ D_4-cross ratio maps of $Z(\Delta)$ to $\mathrm{CR}(\boldsymbol{P})$.

By taking the product of these maps, we define a map cr_{Δ,D_4} of $Z(\Delta)$ to $\mathrm{CR}(\boldsymbol{P})^{45}$ which is actually $W(E_6)$-equivariant. Let $\mathscr{C}'=\mathrm{cr}_{\Delta,D_4}(Z(\Delta))$ and let $\mathscr{C}$ be its Zariski closure in $\mathrm{CR}(\boldsymbol{P})^{45}$.

THEOREM 5 (cf. [21]). (i) *$\mathscr{C}$ is 4-dimensional and non-singular.*

(ii) *The $W(E_6)$-action on $\mathscr{C}$ is biregular.*

(iii) *$\mathscr{C}-\mathscr{C}'$ is a divisor with normal crossings. There exist seventy-six irreducible components of $\mathscr{C}-\mathscr{C}'$ each of which is smooth.*

Following [13], we call $\mathscr{C}$ Naruki's cross ratio variety.

We define a map F of $Z(\Delta)$ to $\boldsymbol{C}^4$ by

$$F(t)=(x_1(t), x_2(t), y_1(t), y_2(t)),$$

where

$$\begin{aligned} x_1(t)&=\frac{h_{24}\cdot h_{234}\cdot h_{15}\cdot h_{135}}{h_{14}\cdot h_{134}\cdot h_{25}\cdot h_{235}}, \quad x_2(t)=\frac{h_{24}\cdot h_{234}\cdot h_{16}\cdot h_{136}}{h_{14}\cdot h_{134}\cdot h_{26}\cdot h_{236}}, \\ y_1(t)&=\frac{h_{34}\cdot h_{234}\cdot h_{15}\cdot h_{125}}{h_{14}\cdot h_{124}\cdot h_{35}\cdot h_{235}}, \quad y_2(t)=\frac{h_{34}\cdot h_{234}\cdot h_{16}\cdot h_{126}}{h_{14}\cdot h_{124}\cdot h_{36}\cdot h_{236}} \end{aligned} \tag{9}$$

as in [30]. Then it follows from [30, Theorem 4.4] that F is $W(E_6)$-equivariant and its image $F(Z(\Delta))$ coincides with

$$P_0(3, 6)=\left\{(x_1, x_2, y_1, y_2);\ \prod_{j=1}^{15} p_j\neq 0\right\}$$

which is an open dense subset of $P(3, 6)$.

We now put

$$Z(\Delta)_h=\left\{t\in E_{\boldsymbol{C}};\ \prod_{j<k} h_{jk}\cdot \prod_{i<j<k} h_{ijk}\neq 0\right\}.$$

Clearly $Z(\Delta)_h$ contains $Z(\Delta)$ and both of the maps cr_{Δ,D_4} and F are extended to $Z(\Delta)_h$. Then it is easy to show that $F(Z(\Delta)_h)$ coincides with $P(3, 6)$. On the other hand, $\mathscr{C}'$ is naturally identified with $F(Z(\Delta))$. Indeed, the identification is established by the correspondence

$$\mathrm{cr}_{\Delta,D_4}(t) \longmapsto F(t) \qquad (\forall t \in Z(\Delta)) .$$

On the other hand, the hypersurface $p_{15}=0$ is non-singular outside the hypersurface $p_1 \cdots p_{14}=0$. Therefore $P(3, 6)$ is regarded as a Zariski open subset of $\mathscr{C}$. This embedding of $P(3, 6)$ in $\mathscr{C}$ is G_0-equivariant, where $G_0 = \boldsymbol{S}_6 \times \boldsymbol{Z}_2$. This follows from the fact that the G_0-action on $\mathscr{C}$ preserves $p_{15}=0$ outside $p_1 \cdots p_{14}=0$.

We are going to write down the seventy-six irreducible components of $\mathscr{C} - \mathscr{C}'$. Each component is described in terms of a subroot system of Δ. Noting this, we put

$$Y_{ij} = Y_{\Delta,D_4}(\{\pm r_{ij}\})$$
$$Y_{ijk} = Y_{\Delta,D_4}(\{\pm r_{ijk}\})$$
$$Y_r = Y_{\Delta,D_4}(\{\pm r\})$$

following the notation in [31]. Then Y_{ij} and Y_{ijk} are hypersurfaces in $\mathscr{C}$. Roughly speaking, the subvariety Y_{ij} is the image of $h_{ij}=0$ by the map cr_{Δ,D_4}.

We now take three subsets Δ_1, Δ_2, Δ_3 of Δ with the following condition:

CONDITION 1. (i) *Each of Δ_1, Δ_2, Δ_3 is a root system of type A_2.*
(ii) *Δ_1, Δ_2, Δ_3 are mutually orthogonal.*
(iii) *The vectors of $\Delta_1 \cup \Delta_2 \cup \Delta_3$ span E.*

Let $Y_{\Delta,D_4}(\Delta_j)$ $(j=1, 2, 3)$ be the subvarieties of $\mathscr{C}$ defined in [31]. Then as is shown in [31, Lemma 3.5],

$$Y_{\Delta,D_4}(\Delta_1) = Y_{\Delta,D_4}(\Delta_2) = Y_{\Delta,D_4}(\Delta_3) .$$

We determine the triples $\{\Delta_1, \Delta_2, \Delta_3\}$ satisfying Condition 1. It is easy to see that there are two kinds of such sets. The first one is of the form

$$\Delta_1 = \{\pm r_{i_1 i_2}, \pm r_{i_2 i_3}, \pm r_{i_1 i_3}\} ,$$
$$\Delta_2 = \{\pm r_{i_4 i_5}, \pm r_{i_5 i_6}, \pm r_{i_4 i_6}\} ,$$
$$\Delta_3 = \{\pm r, \pm r_{i_1 i_2 i_3}, \pm r_{i_4 i_5 i_6}\} .$$

We denote by $Z_{i_1 i_2 i_3, i_4 i_5 i_6}$ the hypersurface $Y_{\Delta,D_4}(\Delta_1)$ in this case. The second one is of the form

$$\Delta_1 = \{\pm r_{i_1 i_2}, \pm r_{i_2 i_3 i_4}, \pm r_{i_1 i_3 i_4}\} ,$$
$$\Delta_2 = \{\pm r_{i_3 i_4}, \pm r_{i_3 i_5 i_6}, \pm r_{i_4 i_5 i_6}\} ,$$
$$\Delta_3 = \{\pm r_{i_5 i_6}, \pm r_{i_1 i_2 i_5}, \pm r_{i_1 i_2 i_6}\} .$$

We denote by $Z_{i_1i_2,i_3i_4,i_5i_6}$ the hypersurface $Y_{\Delta,D_4}(\Delta_1)$ in this case.

REMARK 4. From the definition, we have

(a) $Z_{i_1i_2i_3,i_4i_5i_6}=Z_{i_4i_5i_6,i_1i_2i_3}$,

(b) $Z_{i_1i_2,i_3i_4,i_5i_6}=Z_{i_5i_6,i_1i_2,i_3i_4}\neq Z_{i_1i_2,i_5i_6,i_3i_4}$.

In the sequel, we denote by Ω the totality of the seventy-six divisors in $\mathscr{C}-\mathscr{C}'$. Then Ω is decomposed into the following five G_0-orbits:

$$\Omega_1=\{Y_r\}\ ,\quad \Omega_2=\{Y_{ij};\,1\leq i<j\leq 6\}\ ,\quad \Omega_3=\{Y_{ijk};\,1\leq i<j<k\leq 6\}\ ,$$
$$\Omega_4=\{Z_{i_1i_2i_3,j_1j_2j_3};\,\{i_1,i_2,i_3,j_1,j_2,j_3\}=\{1,2,3,4,5,6\}\}\ ,$$
$$\Omega_5=\{Z_{i_1i_2,i_3i_4,i_5i_6};\,\{i_1,i_2,i_3,i_4,i_5,i_6\}=\{1,2,3,4,5,6\}\}\ .$$

The hypersurfaces contained in $\Omega_1\cup\Omega_2\cup\Omega_3$ (resp. $\Omega_4\cup\Omega_5$) are called hypersurfaces of the first kind (resp. of the second kind) (cf. [30]). Then we have the following.

PROPOSITION 3 (cf. [21], [31]). *Hypersurfaces of the first kind (resp. of the second kind) are isomorphic to the* 3*-dimensional Terada model* $\mathscr{M}_3$ *(resp.* $(\boldsymbol{P}^1)^3$*).*

The Terada model was constructed in Terada [37] (see also [24]).

We are going to describe the intersection relations among the seventy-six divisors above shown in [21] (see also [31], Theorem 3.6]). Let Y be one of the seventy-six hypersurfaces above.

(i) If Y intersects Y_r, then Y is isomorphic to one of the hypersurfaces

$$Y_{ij}\quad(i\neq j)\ ,\quad Z_{i_1i_2i_3,j_1j_2j_3}\quad(\{i_1,i_2,i_3,j_1,j_2,j_3\}=\{1,2,3,4,5,6\})\ .$$

(ii) If Y intersects Y_{12}, then Y is isomorphic to one of the hypersurfaces

$$Y_r,Y_{34},Y_{35},Y_{36},Y_{45},Y_{46},Y_{56},Y_{123},Y_{124},Y_{125},Y_{126},Y_{345},Y_{346},Y_{356},Y_{456},$$
$$Z_{123,456},Z_{124,356},Z_{125,346},Z_{126,345},Z_{12,34,56},Z_{12,35,46},Z_{12,36,45},$$
$$Z_{12,56,34},Z_{12,46,35},Z_{12,45,36}.$$

(iii) If Y intersects Y_{123}, then Y is isomorphic to one of the hypersurfaces

$$Y_{12},Y_{23},Y_{13},Y_{45},Y_{46},Y_{56},Y_{145},Y_{156},Y_{146},Y_{245},Y_{256},Y_{246},Y_{345},Y_{356},Y_{346},$$
$$Z_{123,456},Z_{12,56,34},Z_{12,46,35},Z_{12,45,36},Z_{13,56,24},Z_{13,46,25},Z_{13,45,26},$$
$$Z_{23,56,14},Z_{23,46,15},Z_{23,45,16}\ .$$

(iv) If Y intersects $Z_{123,456}$, then Y is isomorphic to one of the hypersurfaces

$$Y_r,Y_{12},Y_{23},Y_{13},Y_{45},Y_{46},Y_{56},Y_{123},Y_{456}.$$

(v) If Y intersects $Z_{12,34,56}$, then Y is isomorphic to one of the hypersurfaces

$$Y_{12},Y_{34},Y_{56},Y_{134},Y_{234},Y_{356},Y_{456},Y_{125},Y_{126}\ .$$

The action of $\boldsymbol{S}_6$ on the set Ω is same as that of $\boldsymbol{S}_6$ on the indices of Y_{ij}, Y_{ijk},

$Z_{i_1i_2i_3,j_1j_2j_3}$, $Z_{i_1i_2,i_3i_4,i_5i_6}$ as a permutation group. In particular, Y_r is left invariant by S_6. On the other hand, the action of g_0 on Ω is given as follows. The hypersurfaces in Ω_j ($j=1, 2, 4$) are fixed by g_0. Moreover, if $\{i_i, i_2, i_3, i_4, i_5, i_6\}=\{1, 2, 3, 4, 5, 6\}$, then

$$g_0: Y_{i_1i_2i_3} \longmapsto Y_{i_4i_5i_6}, \quad Z_{i_1i_2,i_3i_4,i_5i_6} \longmapsto Z_{i_1i_2,i_5i_6,i_3i_4}.$$

The property of $\mathscr{C}$ given in the following proposition might be of some interest, although we do not use it in our later discussion.

PROPOSITION 4. *Let φ be a biregular transformation on $\mathscr{C}$ such that $\mathscr{C}-\mathscr{C}'$ is left invariant by φ. Then there exists $g\in W(E_6)$ such that $\varphi(Y)=g(Y)$ for any $Y\in\Omega$.*

PROOF. We consider the action of φ on the hypersurface Y_r. Since $\varphi(Y_r)$ is contained in Ω, in virtue of Proposition 3 we find that $\varphi(Y_r)$ is contained in the union of Ω_j ($j=1, 2, 3$). Since $W(E_6)$ acts on the set of root hyperplanes in E transitively, there exists $k\in W(E_6)$ such that $k(\varphi(Y_r))=Y_r$. Noting this, we may assume from the beginning that $\varphi(Y_r)=Y_r$.

We put

$$\mathrm{Div}(Y_r)=\bigcup_{Y\in\Omega_2\cup\Omega_4} Y_r\cap Y.$$

Then we find that any $g\in S_6$ acts on Y_r as a biregular transformation and $g(\mathrm{Div}(Y_r))=\mathrm{Div}(Y_r)$. Moreover the action of S_6 on Y_r is faithful. This combined with the results in [37] implies that there exists $g\in S_6$ such that $g\circ\varphi(y)=y$ for any $y\in Y_r$. Therefore we may assume from the beginning that φ fixes Y_r pointwise. As a consequence,

$$\varphi(Y_{ij})\cap Y_r=\varphi(Y_{ij})\cap\varphi(Y_r)=\varphi(Y_{ij}\cap Y_r)=Y_{ij}\cap Y_r.$$

This shows that $\varphi(Y_{ij})=Y_{ij}$. Similary, we find that $\varphi(Z_{i_1i_2i_3,j_1j_2j_3})=Z_{i_1i_2i_3,j_1j_2j_3}$.

We now consider the image of Y_{123} by φ. We first note that $\varphi(Y_{123})$ is contained in Ω_3. Since Y_{123} intersects Y_{12}, Y_{13}, Y_{23}, so does $\varphi(Y_{123})$. These combined with the intersection relations imply that $\varphi(Y_{123})$ coincides with Y_{123} or Y_{456}. We may assume that $\varphi(Y_{123})=Y_{123}$. Indeed, suppose $\varphi(Y_{123})=Y_{456}$. Since g_0 permutes Y_{123} and Y_{456}, it follows that $g_0\circ\varphi(Y_{123})=Y_{123}$. Noting that g_0 fixes Y_r pointwise, we may take $g_0\circ\varphi$ instead of φ in this case. Then $\varphi(Y_{456})=Y_{456}$.

We next treat Y_{124}. Since Y_{124} intersects Y_{12}, Y_{14}, Y_{24} and Y_{456}, so does $\varphi(Y_{124})$. Then we conclude that $\varphi(Y_{124})=Y_{124}$. For the same reason, we find that $\varphi(Y_{ijk})=Y_{ijk}$. We finally treat $Z_{i_1i_2,i_3i_4,i_5i_6}$. Since $Z_{i_1i_2,i_3i_4,i_5i_6}$ intersects all of $Y_{i_1i_2}$, $Y_{i_1i_3}$, $Y_{i_2i_3}$, $Y_{i_1i_2i_5}$, so does $\varphi(Z_{i_1i_2,i_3i_4,i_5i_6})$. Noting that $\varphi(Z_{i_1i_2,i_3i_4,i_5i_6})\in\Omega_5$, we conclude that $\varphi(Z_{i_1i_2,i_3i_4,i_5i_6})=Z_{i_1i_2,i_3i_4,i_5i_6}$.

We have thus proved the proposition. q.e.d.

REMARK 5. As an easy consequence of Propositions 4 and 3, we find that if φ is a biregular transformation on $\mathscr{C}$ such that φ leaves the set Ω invariant, then $\varphi(y)=y$ for all $y\in\mathscr{C}-\mathscr{C}'$. It is conjectured that such a biregular transformation φ on $\mathscr{C}$ is the identity transformation on $\mathscr{C}$. If this is the case, then $W(E_6)$ coincides with the group of biregular transformations on $\mathscr{C}$ leaving $\mathscr{C}'$ invariant.

For later purpose, we are going to determine normal crossing points of four hypersurfaces of Ω and their isotropy subgroups in $W(E_6)$.

Let H_1, H_2, H_3, $H_4 \in \Omega$ be mutually distinct four hypersurfaces such that $H_1 \cap H_2 \cap H_3 \cap H_4$ is not empty. Then $H_1 \cap H_2 \cap H_3 \cap H_4$ consists of a unique point, say p, and H_1, H_2, H_3, H_4 have normal crossing at p and there is no other hypersurface of Ω containing p. Moreover, under the action of G_0, the quadruple (H_1, H_2, H_3, H_4) is transformed to one of the points (nc.1)–(nc.9) given in Table 1.

We are going to explain the notation in Table 1 briefly. Let p be the normal crossing point which is the intersection of the hypersurfaces given in (nc.j) ($j=1, \dots, 9$). The determination of the isotropy subgroup of p in S_6 and the cardinality $|G_0 \cdot p|$ are easy exercises and are left to the reader. In Table 1, Dh(8) and $W(B_3)$ mean the dihedral group of order 8 and the Weyl group of type B_3, respectively. Moreover, noting that S_3 is regarded as the quotient of $W(B_3)$, we denote by $W(B_3)_{\mathrm{alt}}$ the pull-back of the alternating group (S_3, S_3) in $W(B_3)$. In the sequel, a normal crossing point that is conjugate to the point (nc.i) by the S_6-action is called a normal crossing point of type (NC.i).

PROPOSITION 5. *Local coordinates in the neighborhoods of normal crossing points* (nc.i), $i=2, 4, 5, 6, 7$ *are given in Table* 2.

TABLE 1. Types of normal crossing points.

	$p=H_1 \cap H_2 \cap H_3 \cap H_4$	The isotropy of p in S_6	$\lvert G_0 \cdot p \rvert$
(nc.1)	$Y_{123} \cap Y_{145} \cap Y_{246} \cap Y_{356}$	S_4	30
(nc.2)	$Y_{234} \cap Y_{15} \cap Y_{34} \cap Y_{125}$	Dh(8)	90
(nc.3)	$Y_{12} \cap Y_{34} \cap Y_{56} \cap Y_r$	$W(B_3)$	15
(nc.4)	$Y_{234} \cap Z_{16,25,34} \cap Y_{136} \cap Y_{125}$	Z_3	240
(nc.5)	$Y_{234} \cap Z_{16,25,34} \cap Y_{34} \cap Y_{125}$	Z_2	360
(nc.6)	$Z_{15,34,26} \cap Y_{15} \cap Y_{34} \cap Y_{125}$	$Z_2 \times Z_2$	180
(nc.7)	$Z_{15,34,26} \cap Y_{15} \cap Y_{34} \cap Y_{26}$	$W(B_3)_{\mathrm{alt}}$	30
(nc.8)	$Y_r \cap Y_{12} \cap Y_{56} \cap Z_{123,456}$	Dh(8)	90
(nc.9)	$Y_{12} \cap Y_{45} \cap Y_{123} \cap Z_{123,456}$	$Z_2 \times Z_2$	180

TABLE 2. Local coordinates at normal crossing points.

(nc.2)	(x_2,	x_1/x_2,	y_2/x_2,	$x_2 y_1/x_1 y_2$)
(nc.4)	(x_1,	y_2/x_1,	x_2/y_2,	y_1/y_2)
(nc.5)	(x_1,	x_2/x_1,	y_2/x_2,	y_1/y_2)
(nc.6)	($1/x_2$,	x_1,	y_2,	$x_2 y_1/x_1 y_2$)
(nc.7)	($y_1/x_1 y_2$,	x_1,	y_2,	$x_1 y_2/x_2 y_1$)

PROOF. It is clear that in the (x_1, x_2, y_1, y_2)-space, the origin is not a normal crossing point of the union R of the fifteen hypersurfaces introduced before in this section. We are going to blow up R in the following manner:

(10) $$x_1 = z_1 z_2, \quad x_2 = z_1, \quad y_1 = z_1 z_2 z_3 z_4, \quad y_2 = z_1 z_3.$$

Let $R_{z\text{-space}}$ be the pull-back of R by the map

$$(z_1, z_2, z_3, z_4) \longmapsto (x_1, x_2, y_1, y_2) = (z_1 z_2, z_1, z_1 z_2 z_3 z_4, z_1 z_3).$$

Then it is easy to show that in the z-space, the origin is a normal crossing point of $R_{z\text{-space}}$. On the other hand, by direct computation, we have (cf. (9), (10))

$$z_1 = x_2 = \frac{h_{24} h_{234} h_{16} h_{136}}{h_{14} h_{134} h_{26} h_{236}}, \quad z_2 = \frac{x_1}{x_2} = \frac{h_{26} h_{236} h_{15} h_{135}}{h_{16} h_{136} h_{25} h_{235}},$$

$$z_3 = \frac{y_2}{x_2} = \frac{h_{34} h_{134} h_{26} h_{126}}{h_{24} h_{124} h_{36} h_{136}}, \quad z_4 = \frac{x_2 y_1}{x_1 y_2} = \frac{h_{25} h_{125} h_{36} h_{136}}{h_{35} h_{135} h_{26} h_{126}}.$$

It is shown that the hypersurfaces $z_1 = 0$, $z_2 = 0$, $z_3 = 0$, $z_4 = 0$ are local defining equations of Y_{234}, Y_{15}, Y_{34}, Y_{125}, respectively. Indeed, this is proved as follows. We treat the case $z_1 = 0$. By definition, $z_1 = 0$ is equivalent to $h_{24} h_{234} h_{16} h_{136} = 0$. Therefore there exist four possibilities

$$h_{24} = 0, \quad h_{234} = 0, \quad h_{16} = 0, \quad h_{136} = 0.$$

In the three cases except $h_{234} = 0$, at least one of z_2, z_3, z_4 becomes infinity. This implies that $z_1 = 0$ is a local defining equation of Y_{234}. Similarly, we show that $z_2 = 0$, $z_3 = 0$, $z_4 = 0$ are local defining equations of Y_{15}, Y_{34}, Y_{125}. Therefore we conclude that $z = (z_1, z_2, z_3, z_4)$ is regarded as a local coordinate system of $\mathscr{C}$ whose origin is $Y_{234} \cap Y_{15} \cap Y_{34} \cap Y_{125}$.

By an argument similar to the one above, we can determine local coordinates of Table 2 in neighborhoods of the normal crossing points (nc.i), $i = 4, 5, 6, 7$. q.e.d.

It is clear from the definition that there exist hypersurfaces $\tilde{R}_j$ $(j = 1, \dots, 15)$ on $\mathscr{C}$ corresponding to the hypersurfaces R_j $(j = 1, \dots, 15)$. Then the following proposition is easy to show.

PROPOSITION 6. *The following relations hold*:

$$[\tilde{R}_1] = [Y_{456}], \quad [\tilde{R}_2] = [Y_{245}], \quad [\tilde{R}_3] = [Y_{345}], \quad [\tilde{R}_4] = [Y_{246}], \quad [\tilde{R}_5] = [Y_{346}],$$

$$[\tilde{R}_6] = [Y_{256}], \quad [\tilde{R}_7] = [Y_{356}], \quad [\tilde{R}_8] = [Y_{145}],$$

$$[\tilde{R}_9] = [Y_{146}], \quad [\tilde{R}_{10}] = [Y_{156}], \quad [\tilde{R}_{11}] = [Y_{136}], \quad [\tilde{R}_{12}] = [Y_{135}],$$

$$[\tilde{R}_{13}] = [Y_{126}], \quad [\tilde{R}_{14}] = [Y_{125}], \quad [\tilde{R}_{15}] = [Y_r].$$

4. Triangulations and normal crossing points. The purpose of this section is to study the relationship between the toric variety $\chi(N(\Delta_2 \times \Delta_2))$ introduced in Section 2 and the normal crossing points of Naruki's cross ratio variety $\mathscr{C}$.

As was pointed out in Section 3, the set of normal crossing points of $\mathscr{C}$ is decomposed into nine S_6-orbits. Among these nine orbits, we focus our attention on five orbits which are denoted by (NC.2), (NC.4), (NC.5), (NC.6), (NC.7) in Section 3. We take representatives of such orbits by giving the relations among local coordinates of the points in question and (x_1, x_2, y_1, y_2).

Let T be a regular triangulation of $\Delta_2 \times \Delta_2$ and let τ be a simplex of T. Then there exist four vectors $b_\tau^{(ij)}$ $(\{i,j\} \notin \tau)$ satisfying the conditions (D1), (D2), (D3). Then $u^{b_\tau^{(ij)}}$ $(\{i,j\} \notin \tau)$ are monomials in the matrix entries of $u=(u_{ij})$. We now pay our attention to the restriction of u to the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & x_1 & x_2 \\ 1 & y_1 & y_2 \end{pmatrix}.$$

Then $u^{b_\tau^{(ij)}}$ $(\{i,j\} \notin \tau)$ turn out to be rational functions of x_1, x_2, y_1, y_2 which were introduced in the previous section. We are going to compute the functions thus defined for simplices of the triangulations T_a, T_b, T_c, T_d, T_e.

Here is the result:

PROPOSITION 7. *The relation between the simplices of the triangulations T_a, T_b, T_c, T_d, T_e and the variables x_1, x_2, y_1, y_2 introduced in the previous section are given as follows*:

$T_a=$	$\{1,1\}, \{1,2\}, \{1,3\}, \{2,3\}, \{3,3\}$	$1/x_2$	x_1/x_2	$1/y_2$	y_1/y_2
	$\{1,1\}, \{1,2\}, \{2,2\}, \{2,3\}, \{3,1\}$	x_1/x_2	$1/x_1$	y_1	x_1y_2/x_2
	$\{1,1\}, \{2,1\}, \{2,2\}, \{2,3\}, \{3,1\}$	$1/x_1$	$1/x_2$	y_1/x_1	y_2/x_2
	$\{1,2\}, \{2,2\}, \{2,3\}, \{3,1\}, \{3,3\}$	x_1y_2/x_2	x_1/x_2	y_2/x_2	x_2y_1/x_1y_2
	$\{1,2\}, \{2,2\}, \{3,1\}, \{3,2\}, \{3,3\}$	y_1	y_1/y_2	y_1/x_1	x_2y_1/x_1y_2
	$\{1,1\}, \{1,2\}, \{2,3\}, \{3,1\}, \{3,3\}$	$1/y_2$	y_2/x_2	x_1y_2/x_2	y_1
$T_b=$	$\{1,1\}, \{1,2\}, \{1,3\}, \{2,3\}, \{3,2\}$	$1/x_2$	x_1/x_2	$1/y_1$	y_2/y_1
	$\{1,1\}, \{1,2\}, \{2,2\}, \{2,3\}, \{3,2\}$	x_1/x_2	$1/x_1$	$1/y_1$	x_1y_2/x_2y_1
	$\{1,1\}, \{2,1\}, \{2,2\}, \{2,3\}, \{3,2\}$	$1/x_1$	$1/x_2$	x_1/y_1	x_1y_2/x_2y_1
	$\{1,1\}, \{2,1\}, \{2,3\}, \{3,1\}, \{3,2\}$	$1/y_1$	$1/x_2$	x_1/y_1	y_2/x_2
	$\{1,1\}, \{2,3\}, \{3,1\}, \{3,2\}, \{3,3\}$	$1/y_1$	$1/y_2$	y_2/x_2	x_1y_2/x_2y_1
	$\{1,1\}, \{1,3\}, \{2,3\}, \{3,2\}, \{3,3\}$	y_2/y_1	$1/x_2$	x_1y_2/x_2y_1	$1/y_2$

$$T_c=\begin{array}{lcccc}
\{1,1\},\ \{1,2\},\ \{1,3\},\ \{2,3\},\ \{3,1\} & 1/x_2 & x_1/x_2 & y_1 & y_2 \\
\{1,1\},\ \{1,2\},\ \{2,2\},\ \{2,3\},\ \{3,1\} & x_1/x_2 & 1/x_1 & y_1 & x_1y_2/x_2 \\
\{1,1\},\ \{2,1\},\ \{2,2\},\ \{2,3\},\ \{3,1\} & 1/x_1 & x_1/x_2 & y_1/x_1 & y_2/x_2 \\
\{1,2\},\ \{2,2\},\ \{2,3\},\ \{3,1\},\ \{3,2\} & y_1 & x_1/x_2 & y_1/x_1 & x_1y_2/x_2y_1 \\
\{1,3\},\ \{2,3\},\ \{3,1\},\ \{3,2\},\ \{3,3\} & y_2 & y_2/y_1 & y_2/x_2 & x_1y_2/x_2y_1 \\
\{1,2\},\ \{1,3\},\ \{2,3\},\ \{3,1\},\ \{3,2\} & y_1 & y_1/x_2 & x_1/x_2 & y_2/y_1
\end{array}$$

$$T_d=\begin{array}{lcccc}
\{1,1\},\ \{1,2\},\ \{1,3\},\ \{2,3\},\ \{3,1\} & 1/x_2 & x_1/x_2 & y_1 & y_2 \\
\{1,1\},\ \{1,2\},\ \{2,2\},\ \{2,3\},\ \{3,1\} & x_1/x_2 & 1/x_1 & y_1 & x_1y_2/x_2 \\
\{1,1\},\ \{2,1\},\ \{2,2\},\ \{2,3\},\ \{3,1\} & 1/x_1 & 1/x_2 & y_1/x_1 & y_2/x_2 \\
\{1,2\},\ \{2,2\},\ \{2,3\},\ \{3,1\},\ \{3,2\} & y_1 & x_1/x_2 & y_1/x_1 & x_1y_2/x_2y_1 \\
\{1,2\},\ \{2,3\},\ \{3,1\},\ \{3,2\},\ \{3,3\} & y_1 & y_1/y_2 & 1/x_2 & x_1y_2/x_2y_1 \\
\{1,2\},\ \{1,3\},\ \{2,3\},\ \{3,1\},\ \{3,3\} & y_2 & y_2/x_2 & x_1/x_2 & y_1/y_2
\end{array}$$

$$T_e=\begin{array}{lcccc}
\{1,1\},\ \{1,2\},\ \{1,3\},\ \{2,3\},\ \{3,1\} & 1/x_2 & x_1/x_2 & y_1 & y_2 \\
\{1,1\},\ \{1,2\},\ \{2,2\},\ \{2,3\},\ \{3,1\} & x_1/x_2 & 1/x_1 & y_1 & x_1y_2/x_2 \\
\{1,1\},\ \{2,1\},\ \{2,2\},\ \{2,3\},\ \{3,1\} & 1/x_1 & 1/x_2 & y_1/x_1 & y_2/x_2 \\
\{1,2\},\ \{2,2\},\ \{2,3\},\ \{3,1\},\ \{3,3\} & x_1y_2/x_2 & x_1/x_2 & y_2/x_2 & x_2y_1/x_1y_2 \\
\{1,2\},\ \{2,2\},\ \{3,1\},\ \{3,2\},\ \{3,3\} & y_1 & y_1/y_2 & y_1/x_1 & x_2y_1/x_1y_2 \\
\{1,2\},\ \{1,3\},\ \{2,3\},\ \{3,1\},\ \{3,3\} & y_2 & y_2/x_2 & x_1/x_2 & y_1/y_2
\end{array}$$

PROPOSITION 8. *We put* $\mathcal{S}_k^\vee = N(\Sigma(\Delta_2\times\Delta_2), \phi_{T_k})^\vee \cap \boldsymbol{Z}^{3\times 3}$ $(k=a, b, c, d, e)$. *Then,*

$$\boldsymbol{C}[S_a^\vee] \simeq \boldsymbol{C}[1/y_2, x_1y_2/x_2, x_2y_1/x_1y_2, 1/x_1]\,,$$

$$\boldsymbol{C}[S_b^\vee] \simeq \boldsymbol{C}[x_1/y_1, 1/y_2, x_1/x_2, y_2/y_1, 1/x_1, y_2/x_2]\,,$$

$$\boldsymbol{C}[S_c^\vee] \simeq \boldsymbol{C}[1/x_1, x_1/x_2, y_1, y_2/y_1]\,,$$

$$\boldsymbol{C}[S_d^\vee] \simeq \boldsymbol{C}[y_1/y_2, y_2, x_1y_2/x_2y_1, 1/x_1]\,,$$

$$\boldsymbol{C}[S_e^\vee] \simeq \boldsymbol{C}[1/x_1, x_1/x_2, y_2, x_2y_1/x_1y_2]\,.$$

The two propositions above are direct consequences of the table on $b_\tau^{(ij)}$ (see Section 6).

By using the system of coordinates (x_1, x_2, y_1, y_2), we find that the complex torus

$$(\boldsymbol{C}^*)^4 = \{(x_1, x_2, y_1, y_2) \mid x_i \in \boldsymbol{C}^*, y_j \in \boldsymbol{C}^*\}$$

is embedded into the toric variety $\chi(N(\Sigma(\Delta_2\times\Delta_2)))$:

$$f'\colon (\boldsymbol{C}^*)^4 \longrightarrow \chi(N(\Sigma(\Delta_2\times\Delta_2)))\,.$$

Regarded as a Zariski open subset of $(\boldsymbol{C}^*)^4$, the configuration space $P(3, 6)$ is naturally embedded into $\chi(N(\Sigma(\Delta_2\times\Delta_2)))$ by the composite of the natural inclusion $P(3, 6)\to(\boldsymbol{C}^*)^4$ and f':

$$f\colon P(3,6) \longrightarrow (C^*)^4 \xrightarrow{f'} \chi(N(\Sigma(\Delta_2 \times \Delta_2)))\,.$$

By definition, the map f is birational. There exist birational actions of the elements of S_6 on the configuration space $P(3, 6)$. Among them, the actions of s_1, s_2, s_4 and s_5 can be extended to biregular actions on the toric variety $\chi(N(\Sigma(\Delta_2 \times \Delta_2)))$; they act on each of the coordinate rings of the toric variety as follows:

$$s_1 : u^b \longmapsto u^{s_{12}b}\,, \quad s_2 : u^b \longmapsto u^{s_{23}b}\,, \quad s_4 : u^b \longmapsto u^{bs_{12}}\,, \quad s_5 : u^b \longmapsto u^{bs_{23}}$$

where

$$s_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad s_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

On the other hand, we defined in Section 3 a birational map from the configuration space $P(3, 6)$ into the cross ratio variety $\mathscr{C}$:

$$g\colon P(3,6) \longrightarrow \mathscr{C}\,.$$

Therefore the composite

$$f \circ g^{-1} \colon \mathscr{C} \longrightarrow \chi(N(\Sigma(\Delta_2 \times \Delta_2)))$$

is also birational.

THEOREM 6. *The birational map $f \circ g^{-1}$ is locally isomorphic at the normal crossing points which are contained in the $S_3 \times S_3$-orbits of the four points*

$$Y_{234} \cap Y_{15} \cap Y_{34} \cap Y_{125}\,, \quad Y_{234} \cap Z_{16,25,34} \cap Y_{136} \cap Y_{125}\,,$$

$$Y_{234} \cap Z_{16,25,34} \cap Y_{34} \cap Y_{125}\,, \quad Z_{15,34,26} \cap Y_{15} \cap Y_{34} \cap Y_{125}\,,$$

whose types are respectively, (NC.2), (NC.4), (NC.5), (NC.6) (cf. Table 2).

PROOF. Each of the local coordinate rings given in Table 2 is isomorphic to the corresponding ring given in Table 3 by the following actions of $S_3 \times S_3$:

TABLE 3.

		(w_1,	w_2,	w_3,	w_4)
T_a	(NC.4)	($1/y_2$,	x_1y_2/x_2,	x_2y_1/x_1y_2,	$1/x_1$)
T_b	(NC.7)	(y_2/x_1,	x_1/y_1,	$1/y_2$,	x_1/x_2)
		(x_1/y_2,	$1/x_1$,	y_2/y_1,	y_2/x_1)
T_c	(NC.2)	($1/x_1$,	x_1/x_2,	y_1,	y_2/y_1)
T_d	(NC.6)	(y_1/y_2,	y_2,	x_1y_2/x_2y_1,	$1/x_1$)
T_e	(NC.5)	($1/x_1$,	x_1/x_2,	y_2,	x_2y_1/x_1y_2)

$$(\mathrm{NC.2}): s_5 s_1\,, \quad (\mathrm{NC.4}): s_4 s_5 s_2\,, \quad (\mathrm{NC.5}): s_1\,, \quad (\mathrm{NC.6}): s_2 s_4 s_5 (s_1 s_2 s_1)\,.$$

q.e.d.

The birational map $f \circ g^{-1}$ is not locally isomorphic at the normal crossing points contained in the $\boldsymbol{S}_3 \times \boldsymbol{S}_3$-orbit of $Z_{15,34,26} \cap Y_{15} \cap Y_{34} \cap Y_{26}$ of type (NC.7) (cf. Table 2).

THEOREM 7. *The birational map*

$$r^{-1} \circ f \circ g^{-1} : \mathscr{C} \longrightarrow \chi(N(\Sigma(\Delta_2 \times \Delta_2))) \xrightarrow{r^{-1}} \chi(N'(\Sigma(\Delta_2 \times \Delta_2)))$$

is locally isomorphic in a neighborhood of the point $Z_{15,34,26} \cap Y_{15} \cap Y_{34} \cap Y_{26}$ *of type* (NC.7).

PROOF. Indeed, applying $s_1 s_2 s_1$ (resp. s_4) to the local coordinate rings of Table 2 of type (NC.7), we get the local coordinate ring $\boldsymbol{C}[C_1^\vee \cap \boldsymbol{Z}^4]$ (resp. $\boldsymbol{C}[C_2^\vee \cap \boldsymbol{Z}^4]$) given in Theorem 4 (iii). q.e.d.

REMARK 6. The correspondence among the variables s, y, p, q in Theorem 4 (iv) and the variables w_1, w_2, w_3, w_4 for the triangulation T_b in Table 3 are as follows:

$$y_2/x_1 = x^{-1}p \quad x_1/y_1 = x \quad 1/y_2 = xyp^{-1} \quad x_1/x_2 = q$$

$$x_1/y_2 = xp^{-1} \quad 1/x_1 = y \quad y_2/y_1 = p \quad y_2/x_2 = x^{-1}pq\,.$$

Noting that $\chi(N'(\Sigma(\Delta_2 \times \Delta_2)))$ admits an $\boldsymbol{S}_3 \times \boldsymbol{S}_3$-action, we now pose a problem concerning the relationship between $\mathscr{C}$ and $\chi(N'(\Sigma(\Delta_2 \times \Delta_2)))$.

PROBLEM 1. 1. *Does there exist an* $\boldsymbol{S}_3 \times \boldsymbol{S}_3$*-equivariant surjective map of* $\mathscr{C}$ *to* $\chi(N'(\Sigma(\Delta_2 \times \Delta_2)))$?

2. *Study the correspondence of hypersurfaces on* $\mathscr{C}$ *and* $\chi(N'(\Sigma(\Delta_2 \times \Delta_2)))$.

REMARK 7. Kapranov [14] constructed compactifications of the configurations spaces called the Chow quotients. Then it is interesting to clarify the relationship between the Chow quotient of the Grassmann variety $G(3, 6)$ of the 3-dimensional linear subspaces in $\boldsymbol{C}^6$ and Naruki's cross ratio variety $\mathscr{C}$.

5. Construction of fundamental solutions. The system $E(3, 6)$ of linear differential equations on $\boldsymbol{C}^4$ with coordinates (X_1, X_2, X_3, X_4) is given in [27] and plays an essential role in the study of the period map of a family of *K3* surfaces (cf. [20]).

The system of differential equations $E(3, 6)$ does not have singularities on $P(3, 6) \subseteq \boldsymbol{C}^4$. Any local holomorphic solution on $P(3, 6)$ can be analytically continued to a multivalued holomorphic function on $P(3, 6)$. We regard the local solutions as holomorphic functions defined on domains in the cross ratio variety $\mathscr{C}$ by the embedding $g: P(3, 6) \to \mathscr{C}$. These functions naturally define a holonomic system on $\mathscr{C}$; there exists a holonomic system $\tilde{E}(3, 6)$ defined on $\mathscr{C}$ of which spaces of local holomorphic solutions on the image of g agree with the spaces of local holomorphic functions obtained from those of $E(3, 6)$ by the embedding above. The group S_6 acts on the space of solutions of $\tilde{E}(3, 6)$ and the singular locus of $\tilde{E}(3, 6)$ is the union of the hypersurfaces belonging to Ω_j ($j=2, 3, 4, 5$). In particular, the singular locus of $\tilde{E}(3, 6)$ does not contain Y_r. Noting these, we discuss the problem of constructing fundamental solutions around the normal crossing points (NC.i) ($i=1, 2, 4, 5, 6, 7, 9$) which Y_r does not pass through. Among these points, (NC.i) ($i=2, 4, 5, 6, 7$) correspond to triangulations of the toric variety $\chi(N'(\Sigma(\Delta_2 \times \Delta_2)))$ whereas (NC.i) ($i=1, 9$) do not. We explained how to construct power series solutions on the toric variety $\chi(N'(\Sigma(\Delta_2 \times \Delta_2)))$ in Section 2 and proved that the normal crossing points (NC, i) ($i=2, 4, 5, 6, 7$) are locally isomorphic to the corresponding points on the toric variety in Theorems 6 and 7. By virtue of these results, it is possible to construct power series solutions of $\tilde{E}(3, 6)$ at the normal crossing points (NC, i) ($i=2, 4, 5, 6, 7$). First, we give power series solutions explicitly around these points. Next, we shall discuss fundamental solutions around the remaining normal crossing points.

We are going to introduce three kinds of functions defined by power series:

$$F_{(3,6),A}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6; X_1, X_2, X_3, X_4)$$
$$= \sum_{m_1,m_2,m_3,m_4=0}^{\infty} \gamma_{(3,6),A}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6; m_1, m_2, m_3, m_4) X_1^{m_1} X_2^{m_2} X_3^{m_3} X_4^{m_4},$$

$$\gamma_{(3,6),A}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6; m_1, m_2, m_3, m_4)$$
$$= \frac{\Gamma\left(\lambda_2+m_{12}+\frac{1}{2}\right)\Gamma\left(\lambda_3+m_{34}+\frac{1}{2}\right)\Gamma\left(-\lambda_5+m_{24}+\frac{1}{2}\right)\Gamma\left(-\lambda_6+m_{13}+\frac{1}{2}\right)}{m_1!m_2!m_3!m_4!\Gamma\left(\lambda_{234}+m_{1234}+\frac{3}{2}\right)},$$

$$F_{(3,6),B}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6; X_1, X_2, X_3, X_4)$$
$$= \sum_{m_1,m_2,m_3,m_4=0}^{\infty} \gamma_{(3,6),B}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6; m_1, m_2, m_3, m_4) X_1^{m_1} X_2^{m_2} X_3^{m_3} X_4^{m_4},$$

$$\gamma_{(3,6),B}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6; m_1, m_2, m_3, m_4)$$
$$= \frac{\Gamma\left(\lambda_3+m_{34}+\frac{1}{2}\right)\Gamma\left(-\lambda_5+m_{24}+\frac{1}{2}\right)\Gamma\left(-\lambda_{34}+m_1-m_{34}\right)\Gamma(\lambda_{15}+m_1-m_{24})}{m_1!m_2!m_3!m_4!\Gamma\left(\lambda_{156}+m_1-m_{234}+\frac{1}{2}\right)},$$

$$F_{(3,6),C}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6; X_1, X_2, X_3, X_4)$$
$$= \sum_{m_1,m_2,m_3,m_4=0}^{\infty} \gamma_{(3,6),C}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6; m_1, m_2, m_3, m_4) X_1^{m_1} X_2^{m_2} X_3^{m_3} X_4^{m_4},$$

$$\gamma_{(3,6),C}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6; m_1, m_2, m_3, m_4)$$
$$=\frac{(-1)^{m_{14}}\Gamma\left(\lambda_1+m_{12}+\frac{1}{2}\right)\Gamma\left(\lambda_3+m_{34}+\frac{1}{2}\right)\Gamma\left(-\lambda_6+m_{23}+\frac{1}{2}\right)}{m_1!m_2!m_3!m_4!\Gamma(\lambda_{15}+m_{12}-m_4+1)\Gamma(\lambda_{34}-m_1+m_{34}+1)},$$

where $m_{ij}=m_i+m_j$, $m_{ijk}=m_i+m_j+m_k$, $\lambda_{ij}=\lambda_i+\lambda_j$, $\lambda_{ijk}=\lambda_i+\lambda_j+\lambda_k$, etc. In the sequel, we write

$$F_{(3,6),Z}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6; X_1, X_2, X_3, X_4)=F_{(3,6),Z}\begin{pmatrix} \lambda_1, & \lambda_2, & \lambda_3 & \\ \lambda_4, & \lambda_5, & \lambda_6; & \\ X_1, & X_2, & X_3, & X_4 \end{pmatrix}$$

$(Z=A, B, C)$ for simplicity.

It is stressed here that each of the functions of the form (6) is reduced to one of $F_{(3,6),Z}$ $(Z=A, B, C)$. Noting this, we are going to construct a set of fundamental solutions around normal crossing points of types (NC.i) $i=2, 4, 5, 6, 7$. The result is given below where the variables $(w_1, \dots, w_4)$ are as given in Table 3 (we use those in the upper row in the case of T_b).

(I) A set of fundamental solutions around the normal crossing point of type (NC.4) corresponding to T_a.

$$w_1^{-\alpha_2-\alpha_3}w_2^{-\alpha_2}w_4^{-\alpha_2}F_{(3,6),A}\begin{pmatrix} -\alpha_1-1/2, & -\alpha_3-1/2, & -\alpha_2-1/2 & \\ \beta_3+1/2, & \beta_1+1/2, & \beta_2+1/2; & \\ w_1w_2w_3, & w_1, & w_1w_2, & w_1w_2w_4 \end{pmatrix},$$

$$w_1^{-\beta_3}(w_2w_3)^{-\alpha_1-\alpha_2+\beta_2}w_4^{-\alpha_2}F_{(3,6),A}\begin{pmatrix} -\alpha_3-1/2, & -\alpha_1-1/2, & -\alpha_2-1/2 & \\ \beta_2+1/2, & \beta_1+1/2, & \beta_3+1/2; & \\ w_1w_2w_3, & w_2w_3, & w_3, & w_2w_3w_4 \end{pmatrix},$$

$$w_1^{-\beta_3}w_2^{-\beta_3}w_4^{-\beta_2-\beta_3}F_{(3,6),A}\begin{pmatrix} -\alpha_2-1/2, & -\alpha_1-1/2, & -\alpha_3-1/2 & \\ \beta_1+1/2, & \beta_3+1/2, & \beta_2+1/2; & \\ w_4, & w_1w_2w_4, & w_2w_3w_4, & w_2w_4 \end{pmatrix},$$

$$w_1^{-\beta_3}w_2^{-\alpha_2}w_4^{-\alpha_2}F_{(3,6),B}\begin{pmatrix} -\alpha_1-1/2, & -\alpha_3-1/2, & -\alpha_2-1/2 & \\ \beta_3+1/2, & \beta_2+1/2, & \beta_1+1/2; & \\ w_1, & w_2w_3, & w_2w_4, & w_2 \end{pmatrix},$$

$$w_1^{-\beta_3}w_2^{-\alpha_1-\alpha_2+\beta_2}w_4^{-\alpha_2}F_{(3,6),B}\begin{pmatrix} -\alpha_3-1/2, & -\alpha_2-1/2, & -\alpha_1-1/2 & \\ \beta_2+1/2, & \beta_1+1/2, & \beta_3+1/2; & \\ w_3, & w_2w_4, & w_1w_2, & w_2 \end{pmatrix},$$

$$w_1^{-\beta_3}w_2^{-\beta_3}w_4^{-\alpha_2}F_{(3,6),B}\begin{pmatrix} -\alpha_2-1/2, & -\alpha_1-1/2, & -\alpha_3-1/2 & \\ \beta_1+1/2, & \beta_3+1/2, & \beta_2+1/2; & \\ w_4, & w_1w_2, & w_2w_3, & w_2 \end{pmatrix}.$$

(II) A set of fundamental solutions around the normal crossing point of type (NC.7) corresponding to T_b.

$$w_1^{-\alpha_2-\alpha_3}w_2^{-\alpha_3}w_3^{-\alpha_2-\alpha_3}w_4^{-\alpha_2}F_{(3,6),C}\begin{pmatrix} -\alpha_3-1/2, & -\alpha_1-1/2, & -\alpha_2-1/2 & \\ \beta_3+1/2, & \beta_2+1/2, & \beta_1+1/2; & \\ w_1w_2, & w_1w_2w_3, & w_1w_3w_4, & w_4 \end{pmatrix},$$

$$w_1^{-\alpha_2-\beta_2}w_2^{-\beta_2}w_3^{-\beta_2-\beta_3}w_4^{-\alpha_2}F_{(3,6),C}\begin{pmatrix} -\alpha_1-1/2, & -\alpha_3-1/2, & -\alpha_2-1/2 & \\ \beta_3+1/2, & \beta_1+1/2, & \beta_2+1/2; & \\ w_3, & w_1w_2w_3, & w_1w_2w_4, & w_1w_4 \end{pmatrix},$$

$$w_1^{-\beta_2-\beta_3}w_2^{-\beta_2}w_3^{-\beta_2-\beta_3}w_4^{-\beta_3}F_{(3,6),C}\begin{pmatrix} -\beta_3-1/2, & -\beta_1-1/2, & -\beta_2-1/2 & \\ \alpha_3+1/2, & \alpha_2+1/2, & \alpha_1+1/2; & \\ w_1w_4, & w_1w_3w_4, & w_1w_3, & w_2 \end{pmatrix},$$

$$w_1^{-\beta_2-\beta_3}w_2^{-\alpha_3}w_3^{-\beta_2-\beta_3}w_4^{-\beta_3}F_{(3,6),C}\begin{pmatrix} -\alpha_3-1/2, & -\alpha_2-1/2, & -\alpha_1-1/2 & \\ \beta_1+1/2, & \beta_2+1/2, & \beta_3+1/2; & \\ w_2, & w_1w_2, & w_1w_3w_4, & w_1w_3 \end{pmatrix},$$

$$w_1^{-\alpha_2-\alpha_3}w_2^{-\alpha_3}w_3^{-\alpha_2-\alpha_3}w_4^{-\beta_3}F_{(3,6),C}\begin{pmatrix} -\beta_1-1/2, & -\beta_2-1/2, & -\beta_3-1/2 & \\ \alpha_2+1/2, & \alpha_1+1/2, & \alpha_3+1/2; & \\ w_1w_3, & w_1w_2w_3, & w_1w_2w_4, & w_4 \end{pmatrix},$$

$$w_1^{-\alpha_2-\beta_2}w_2^{-\beta_2}w_3^{-\alpha_2-\alpha_3}w_4^{-\alpha_2}F_{(3,6),C}\begin{pmatrix} -\beta_2-1/2, & -\beta_3-1/2, & -\beta_1-1/2 & \\ \alpha_1+1/2, & \alpha_3+1/2, & \alpha_2+1/2; & \\ w_1w_2, & w_1w_2w_4, & w_1w_3w_4, & w_3 \end{pmatrix}.$$

(III) A set of fundamental solutions around the normal crossing point of type (NC.2) corresponding to T_c.

$$w_1^{-\alpha_2}w_2^{-\alpha_2}F_{(3,6),C}\begin{pmatrix} -\alpha_2-1/2, & -\alpha_1-1/2, & -\alpha_3-1/2 & \\ \beta_1+1/2, & \beta_3+1/2, & \beta_2+1/2; & \\ w_1w_2, & w_2, & w_3, & w_3w_4 \end{pmatrix},$$

$$w_1^{-\beta_2-\beta_3}w_2^{-\beta_3}F_{(3,6),A}\begin{pmatrix} -\alpha_2-1/2, & -\alpha_1-1/2, & -\alpha_3-1/2 & \\ \beta_1+1/2, & \beta_3+1/2, & \beta_2+1/2; & \\ w_1, & w_1w_2, & w_1w_3, & w_1w_2w_3w_4 \end{pmatrix},$$

$$w_1^{-\alpha_2}w_2^{-\alpha_2}w_3^{\alpha_3-\beta_1}w_4^{-\alpha_1-\alpha_2+\beta_3}F_{(3,6),A}\begin{pmatrix} -\alpha_3-1/2, & -\alpha_2-1/2, & -\alpha_1-1/2 & \\ \beta_3+1/2, & \beta_2+1/2, & \beta_1+1/2; & \\ w_1w_2w_3w_4, & w_2w_4, & w_3w_4, & w_4 \end{pmatrix},$$

$$w_1^{-\alpha_2}w_2^{-\beta_3}F_{(3,6),B}\begin{pmatrix} -\alpha_2-1/2, & -\alpha_1-1/2, & -\alpha_3-1/2 & \\ \beta_1+1/2, & \beta_3+1/2, & \beta_2+1/2; & \\ w_1, & w_2, & w_3, & w_2w_3w_4 \end{pmatrix},$$

$$w_1^{-\alpha_2}w_2^{-\alpha_2}w_3^{\alpha_3-\beta_1}F_{(3,6),B}\begin{pmatrix} -\alpha_3-1/2, & -\alpha_1-1/2, & -\alpha_2-1/2 & \\ \beta_3+1/2, & \beta_1+1/2, & \beta_2+1/2; & \\ w_4, & w_3, & w_2, & w_1w_2w_3 \end{pmatrix},$$

$$w_1^{-\beta_2-\beta_3}w_2^{-\beta_3}w_3^{\alpha_3-\beta_1}F_{(3,6),C}\begin{pmatrix} -\beta_1-1/2, & -\beta_2-1/2, & -\beta_3-1/2 & \\ \alpha_2+1/2, & \alpha_3+1/2, & \alpha_1+1/2; & \\ w_1w_3, & w_3, & w_2, & w_2w_4 \end{pmatrix}.$$

(IV) A set of fundamental solutions around the normal crossing point of type (NC.6) corresponding to T_d.

$$w_1^{-\alpha_2}w_3^{-\alpha_2}w_4^{-\alpha_2}F_{(3,6),C}\begin{pmatrix} -\alpha_3-1/2, & -\alpha_1-1/2, & -\alpha_2-1/2 & \\ \beta_3+1/2, & \beta_1+1/2, & \beta_2+1/2; & \\ w_2, & w_1w_2, & w_1w_3, & w_1w_3w_4 \end{pmatrix},$$

$$w_1^{-\beta_3}w_3^{-\beta_3}w_4^{-\beta_2-\beta_3}F_{(3,6),A}\begin{pmatrix} -\alpha_2-1/2, & -\alpha_3-1/2, & -\alpha_1-1/2 & \\ \beta_1+1/2, & \beta_2+1/2, & \beta_3+1/2; & \\ w_1w_2w_3w_4, & w_1w_2w_4, & w_1w_3w_4, & w_4 \end{pmatrix},$$

$$w_1^{-\beta_3}w_3^{-\beta_3}w_4^{-\alpha_2}F_{(3,6),B}\begin{pmatrix} -\alpha_2-1/2, & -\alpha_1-1/2, & -\alpha_3-1/2 & \\ \beta_1+1/2, & \beta_3+1/2, & \beta_2+1/2; & \\ w_4, & w_1w_3, & w_1w_2, & w_1w_2w_3 \end{pmatrix},$$

$$w_1^{-\alpha_1-\alpha_2+\beta_2}w_2^{\alpha_3-\beta_1}w_3^{-\alpha_2}w_4^{-\alpha_2}F_{(3,6),C}\begin{pmatrix} -\alpha_1-1/2, & -\alpha_3-1/2, & -\alpha_2-1/2 & \\ \beta_3+1/2, & \beta_2+1/2, & \beta_1+1/2; & \\ w_1, & w_1w_2, & w_1w_2w_3w_4, & w_3 \end{pmatrix},$$

$$w_1^{-\alpha_2}w_2^{\alpha_3-\beta_1}w_3^{-\alpha_2}w_4^{-\alpha_2}F_{(3,6),C}\begin{pmatrix} -\beta_2-1/2, & -\beta_3-1/2, & -\beta_1-1/2 & \\ \alpha_3+1/2, & \alpha_1+1/2, & \alpha_2+1/2; & \\ w_1, & w_1w_3, & w_1w_2w_3w_4, & w_2 \end{pmatrix},$$

$$w_1^{-\alpha_1-\alpha_2+\beta_2}w_2^{\alpha_3-\beta_1}w_3^{-\beta_3}w_4^{-\alpha_2}F_{(3,6),C}\begin{pmatrix} -\beta_3-1/2, & -\beta_2-1/2, & -\beta_1-1/2 & \\ \alpha_3+1/2, & \alpha_2+1/2, & \alpha_1+1/2; & \\ w_3, & w_1w_3, & w_1w_2, & w_1w_2w_4 \end{pmatrix}.$$

(V) A set of fundamental solutions around the normal crossing point of type (NC.5) corresponding to T_e.

$$w_1^{-\alpha_2}w_2^{-\alpha_2}F_{(3,6),C}\begin{pmatrix} -\alpha_2-1/2, & -\alpha_1-1/2, & -\alpha_3-1/2 & \\ \beta_1+1/2, & \beta_3+1/2, & \beta_2+1/2; & \\ w_1w_2, & w_2, & w_2w_3w_4, & w_3 \end{pmatrix},$$

$$w_1^{-\beta_2-\beta_3}w_2^{-\beta_3}F_{(3,6),A}\begin{pmatrix} -\alpha_2-1/2, & -\alpha_1-1/2, & -\alpha_3-1/2 & \\ \beta_1+1/2, & \beta_2+1/2, & \beta_3+1/2; & \\ w_1w_2, & w_1, & w_1w_2w_3, & w_1w_2w_3w_4 \end{pmatrix},$$

$$w_1^{-\alpha_2}w_2^{\alpha_3-\beta_1-\beta_3}w_3^{\alpha_3-\beta_1}w_4^{-\alpha_1-\alpha_2+\beta_2}F_{(3,6),A}\left(\begin{matrix} -\alpha_3-1/2, & -\alpha_2-1/2, & -\alpha_1-1/2 & \\ \beta_2+1/2, & \beta_3+1/2, & \beta_1+1/2; & \\ w_1w_2w_3w_4, & w_4, & w_2w_3w_4, & w_2w_4 \end{matrix}\right),$$

$$w_1^{-\alpha_2}w_2^{-\beta_2}F_{(3,6),B}\left(\begin{matrix} -\alpha_2-1/2, & -\alpha_1-1/2, & -\alpha_3-1/2 & \\ \beta_1+1/2, & \beta_3+1/2, & \beta_2+1/2; & \\ w_1, & w_2, & w_2w_3w_4, & w_2w_3 \end{matrix}\right),$$

$$w_1^{-\alpha_2}w_2^{\alpha_3-\beta_1-\beta_3}w_3^{\alpha_3-\beta_1}F_{(3,6),B}\left(\begin{matrix} -\alpha_3-1/2, & -\alpha_2-1/2, & -\alpha_1-1/2 & \\ \beta_2+1/2, & \beta_1+1/2, & \beta_3+1/2; & \\ w_4, & w_1w_2w_3, & w_2, & w_2w_3 \end{matrix}\right),$$

$$w_1^{-\alpha_2}w_2^{-\alpha_2}w_3^{\alpha_3-\beta_1}F_{(3,6),C}\left(\begin{matrix} -\beta_1-1/2, & -\beta_3-1/2, & -\beta_2-1/2 & \\ \alpha_1+1/2, & \alpha_3+1/2, & \alpha_2+1/2; & \\ w_3, & w_1w_2w_3, & w_2, & w_2w_4 \end{matrix}\right).$$

Before going into discussion on Problem (A2) in the Introduction for the case $E(3,6)$, we explain the relationship between our point of view and Horn's study on analytic continuations of the Appell hypergeometric functions (cf. [8]). We first recall the case of the Gaussian hypergeometric functions. The differential equation for $F(a, b, c; x)$ has singularities at $x=0, 1, \infty$. As is well-known, all the fundamental solutions around $x=0, 1, \infty$ are expressed in terms of such functions as $x^{e_1}(1-x)^{e_2}F(a', b', c'; x')$, where a', b', c', e_1, e_2 are linear with respect to a, b, c and x' is obtained by a linear fractional transformation of x.

In the case of the Appell hypergeometric functions F_1, F_2, F_3, F_4, the situation becomes slightly different. Taking F_2 as an example, we consider fundamental solutions of the holonomic system $\mathscr{S}_{F_2}$ for F_2. In this case, we take the 2-dimensional Terada model $\mathscr{M}_2$ as the blowing up of $\boldsymbol{P}^2$ where $\mathscr{S}_{F_2}$ is defined. Then the singular locus of the pull-back of $\mathscr{S}_{F_2}$ to $\mathscr{M}_2$ is the union of ten lines and there exist fifteen normal crossing points. As fundamental solutions around normal crossing points of $\mathscr{M}_2$, we obtain F_2, F_3 and one of Horn's functions denoted by H_2 (cf. [8]). To construct fundamental solutions around all normal crossing points, we need three other functions; two are, roughly speaking, two-variable versions of the generalized hypergeometric function ${}_3F_2(a_1, a_2, a_3; b_1, b_2; x)$ and the remaining one is complicated to describe, since we need the special values of ${}_3F_2$ at $x=1$ to write coefficients of its power series expression. (For details on this subject, see [29], [32] and [35].) In this sense, Horn's study is incomplete.

We return to the case $\tilde{E}(3, 6)$. For this purpose, it is better to explain the results by separating the types of normal crossing points of $\mathscr{C}$ into the four cases:

(E1) The cases (NC.i) ($i=2, 4, 5, 6, 7$). As we have already shown, each of the fundamental solutions around normal crossing points whose type is one of (NC.i) ($i=2, 4, 5, 6, 7$) can be expressed in terms of $F_{(3,6),Z}$ ($Z=A, B, C$).

(E2) The case (NC.9). To construct fundamental solutions around normal crossing points of type (NC.9), we have to introduce four kinds of functions defined by power series in four variables which are of hypergeometric-Horn type, namely, their coefficients satisfy product formulas. (For details, refer to [32].)

(E3) The case (NC.1). Fundamental solutions at the point (NC.1) are given in [27]. We need a function defined by power series whose coefficients are expressed in terms of the special values of ${}_3F_2$ at $x=1$ as in the case of Appell's F_2.

(E4) The cases (NC.3), (NC.8). Normal crossing points of types (NC.3) and (NC.8) are contained in Y_r. The hypergeometric differential equation $\tilde{E}(3, 6)$ does not have singularities along the hypersurface Y_r which is the closure of the image of $p_{15}=0$ by the map $g: P(3, 6) \to \mathscr{C}$. For this reason, we do not enter into the determination of fundamental solutions around such points. We only note here that the solutions of $\tilde{E}(3, 6)$ on Y_r can be expressed in terms of determinants of the Lauricella functions F_D in three variables (cf. [38]).

6. Table of $b_\tau^{(ij)}$. We give the table of $b_\tau^{(ij)}$ for the triangulations T_a, T_b, T_c, T_d, T_e.

We explain notation in the table. To each simplex, there is associated a 3×3 matrix σ whose entries are asterisks as follows. If

$$\{i_1, j_1\},\ \{i_2, j_2\},\ \{i_3, j_3\},\ \{i_4, j_4\},\ \{i_5, j_5\}$$

is a simplex, the (i, j)-entry of σ is $*$ in the case of $(i, j)=(i_k, j_k)$ $(k=1, 2, 3, 4, 5)$ and is empty otherwise. Let T be a triangulation given in Theorem 1. Then the vector $(i_1, \ldots, i_6)$ following T means that the k-th simplex of T corresponds to the i_k-th series solution in Section 5. For example, the vector (1, 6, 4, 3, 5, 2) following T_a means that the first simplex corresponds to the first series solution in Section 5 and the second simplex corresponds to the sixth series solution and so on.

T_a, (1, 6, 4, 3, 5, 2)

$$\begin{pmatrix} * & * & * \\ & & * \\ & & * \end{pmatrix} : \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix},$$

$$\begin{pmatrix} * & * & \\ & * & * \\ * & & \end{pmatrix} : \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} * & & \\ * & * & * \\ * & & \end{pmatrix} : \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} & * & \\ & * & * \\ * & & * \end{pmatrix} : \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix},$$

$$\begin{pmatrix} & \ast & \\ & \ast & \\ \ast & \ast & \ast \end{pmatrix}: \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix},$$

$$\begin{pmatrix} \ast & \ast & \\ & & \ast \\ \ast & & \ast \end{pmatrix}: \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix}.$$

T_b, (1, 5, 6, 4, 3, 2)

$$\begin{pmatrix} \ast & \ast & \ast \\ & & \ast \\ & \ast & \end{pmatrix}: \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} \ast & \ast & \\ & \ast & \ast \\ & \ast & \end{pmatrix}: \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} \ast & & \\ \ast & \ast & \ast \\ & \ast & \end{pmatrix}: \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} \ast & & \\ \ast & & \ast \\ \ast & \ast & \end{pmatrix}: \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} \ast & & \\ & & \ast \\ \ast & \ast & \ast \end{pmatrix}: \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} \ast & & \ast \\ & & \ast \\ & \ast & \ast \end{pmatrix}: \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

T_c, (1, 4, 2, 5, 6, 3)

$$\begin{pmatrix} \ast & \ast & \ast \\ & & \ast \\ \ast & & \end{pmatrix}: \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} \ast & \ast & \\ & \ast & \ast \\ \ast & & \end{pmatrix}: \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} * & & \\ * & * & * \\ * & & \end{pmatrix} : \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} & * & \\ & * & * \\ * & * & \end{pmatrix} : \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} & & * \\ & & * \\ * & * & * \end{pmatrix} : \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} & * & * \\ & & * \\ * & * & \end{pmatrix} : \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

T_d, (1, 3, 2, 5, 6, 4)

$$\begin{pmatrix} * & * & * \\ & & * \\ * & & \end{pmatrix} : \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} * & * & \\ & * & * \\ * & & \end{pmatrix} : \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} * & & \\ * & * & * \\ * & & \end{pmatrix} : \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} & * & \\ & * & * \\ * & * & \end{pmatrix} : \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} & * & \\ & & * \\ * & * & * \end{pmatrix} : \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} & * & * \\ & & * \\ * & & * \end{pmatrix} : \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

T_e, (1, 4, 2, 6, 5, 3)

$$\begin{pmatrix} * & * & * \\ & & * \\ * & & \end{pmatrix} : \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} * & * & \\ & * & * \\ * & & \end{pmatrix} : \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} * & & \\ * & * & * \\ * & & \end{pmatrix} : \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} & * & \\ & * & * \\ * & & * \end{pmatrix} : \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix},$$

$$\begin{pmatrix} & * & \\ & * & \\ * & * & * \end{pmatrix} : \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix},$$

$$\begin{pmatrix} & * & * \\ & & * \\ * & & * \end{pmatrix} : \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

References

[1] K. Aomoto and N. Kita, Theory of hypergeometric functions, (in Japanese) Springer-Verlag, Tokyo, 1994.

[2] P. Appell and J. Kampé de Fériet, Fonctions Hypergéométriques et Hypershériques, Gauthier-Villars, Paris, 1926.

[3] N. Bourbaki, Groupes et Algèbres de Lie, Chaps. 4, 5, 6, Herman, Paris, 1968.

[4] L. J. Billera, F. Filliman and B. Sturmfels, Constructions and complexity of secondary polytopes, Adv. in Math. 83 (1990), 155–179.

[5] K. Cho, K. Matsumoto and M. Yoshida, Combinatorial structure of the configuration space of 6 points on the projective plane, Mem. Fac. Sci. Kyushu Univ., Ser A 47 (1993), 119–146.

[6] I. Dolgachev and D. Ortland, Point sets in projective spaces and theta functions, Astérisque 165 (1988).

[7] H. Edelsbrunner, Algorithms in Combinatorial Geometry, Springer-Verlag, Berlin-Heidelberg, 1987.

[8] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, Higher Transcendental Functions I, Robert E. Krieger Publishing Company, Malabas Florida, 1981.

[9] W. Fulton, Introduction to toric varieties, Princeton Univ. Press, 1993.

[10] I. M. Gel'fand and M. I. Graev, Hypergeometric functions associated with the Grassmannian $G_{3,6}$, Soviet Math. Dokl. 35 (1987), 298–303.

[11] I. M. Gel'fand, A. V. Zelevinskii and M. M. Kapranov, Hypergeometric functions and toral manifolds, Funct. Anal. and its Appl. 23 (1989), 94–106.

[12] I. M. Gel'fand, M. M. Kapranov and A. V. Zelevinskii, Discriminants, Resultants and Multidimensional Determinants, Birkhäuser, Boston, 1994.

[13] B. Hunt, A remarkable quintic fourfold in $\boldsymbol{P}^5$ and its dual variety, (Update: 7.1.1992), preprint.

[14] M. M. Kapranov, Chow quotients of Grassmannians I, Adv. Soviet Math. 16 (1993), 29–110.

[15] M. Kashiwara and T. Kawai, On holonomic systems of micro-differential equations, III, Publ. Res. Inst. Math. Sci. 17 (1981), 813–979.

[16] M. Kashiwara and T. Oshima, Systems with regular singularities and their boundary value problems, Ann. of Math. 106 (1977), 145–200.

[17] J. de Loera, Triangulations of polytopes and computational algebra, PhD. thesis, Cornell Univ. 1995.

[18] Yu. I. Manin, Cubic Forms, North-Holland, Amsterdam, 1974.

[19] T. Masada, Enumeration of regular triangulations, Master's thesis, Department of Information Science, Univ. of Tokyo, 1995.

[20] K. Matsumoto, T. Sasaki and M. Yoshida, The monodromy of the period map of a 4-parameter family of K3 surfaces and the hypergeometric function of type (3, 6), Internat. J. Math. 3 (1992), 1–164.

[21] I. Naruki, Cross ratio variety as a moduli space of cubic surfaces, Proc. London Math. Soc. 45 (1982), 1–30.

[22] I. Naruki and J. Sekiguchi, A modification of Cayley's family of cubic surfaces and birational action of $W(E_6)$ over it, Proc. Japan Acad., Ser. A, 56 (1980), 122–125.

[23] T. Oda, Convex bodies and algebraic geometry, Springer-Verlag, Berlin, 1988.

[24] Takayuki Oda, The canonical compactification of the configuration space of the pure braid group of $\boldsymbol{P}^1$, preprint.

[25] T. Oshima, Boundary value problems for systems of linear partial differential equations with regular singularities, Adv. Stud. Pure Math. 4 (1984), 391–432.

[26] Mutsumi Saito and N. Takayama, Restrictions of A-hypergeometric systems and connection formulas of the $\Delta_1 \times \Delta_{n-1}$ hypergeometric function, Internat. J. Math. 5 (1994), 537–560.

[27] T. Sasaki and T. Uehara, Power series solutions around a singular point of the system of hypergeometric differential equations of type (3, 6) by use of special values of ${}_3F_2$, Funkcial. Ekvac. 36 (1993), 405–431.

[28] J. Sekiguchi, The birational action of S_5 on $\boldsymbol{P}^2$ and the icosahedron, J. Math. Soc. Japan 44 (1992), 567–589.

[29] J. Sekiguchi, Appell hypergeometric function $F_2(\alpha, \beta, \beta', \gamma, \gamma', ; x, y)$ and the blowing up space of $\boldsymbol{P}^2$, Sūrikaisekikenkyūsho Kokyūroku 773 (1991), 66–77.

[30] J. Sekiguchi, The versal deformation of the E_6-singularity and a family of cubic surfaces, J. Math. Soc. Japan 46 (1994), 355–383.

[31] J. Sekiguchi, Cross ratio varieties for root systems, Kyushu J. Math. 48 (1994), 123–168.

[32] J. Sekiguchi, Hypergeometric function of type (3, 6) and Naruki's cross ratio variety, preprint.

[33] J. Sekiguchi and N. Takayama, A global representation of the solutions of the system of equations $E_{2,5}$ and the Appell function F_1, Notes available at http://www.math.s.kobe-u.ac.jp/HOME/taka/f1.ps (1991).

[34] J. Sekiguchi and M. Yoshida, $W(E_6)$-action on the configuration space of 6 points on the real projective plane, to appear in Kyushu J. Math.

[35] N. Takayama, Propagation of singularities of solutions of the Euler-Darboux equation and a global structure of the space of holonomic solutions II, Funkcial. Ekvac. 36 (1993), 187–234.

[36] N. Takayama, Hypergeometric functions and toric varieties (in Japanese), Sūrikaisekikenkyūsho Kokyūroku 934 (1996), 106–110.

[37] T. Terada, Fonction hypergéométriques F_1 et fonctions automorph I, J. Math. Soc. Japan 35 (1983), 451–475.

[38] T. Terasoma, Exponential Kummer coverings and determinants of hypergeometric functions, Tokyo J. Math. 16 (1993), 497–508.

Department of Mathematics
Himeji Institute of Technology
Himeji 671–22
Japan

Department of Mathematics
Faculty of Sciences
Kobe University
Kobe 657
Japan