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IRREDUCIBLE CONSTANT MEAN CURVATURE 1 SURFACES IN HYPERBOLIC SPACE WITH POSITIVE GENUS

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Abstract. In this work we give a method for constructing a one-parameter family of complete CMC-1 (i.e. constant mean curvature 1) surfaces in hyperbolic 3-space that correspond to a given complete minimal surface with finite total curvature in Euclidean 3-space. We show that this one-parameter family of surfaces with the same symmetry properties exists for all given minimal surfaces satisfying certain conditions. The surfaces we construct in this paper are irreducible, and in the process of showing this, we also prove some results about the reducibility of surfaces.

Furthermore, in the case that the surfaces are of genus 0, we are able to make some estimates on the range of the parameter for the one-parameter family.

1. Introduction. Recently, new examples of immersed CMC-1 surfaces of finite total curvature in the hyperbolic 3-space $H^3(-1)$ of curvature -1 have been found (cf. [UY1], [UY2], [S], and [UY4]). One method used to make these new examples is the following: The set of all conformal branched CMC-c (i.e. constant mean curvature c) immersions in $H^3(-c^2)$ of finite total curvature with hyperbolic Gauss map G defined on a compact Riemann surface $\bar{M}$ corresponds bijectively to the set of conformal pseudometrics of constant curvature 1 with conical singularities on $\bar{M}$. (cf. [UY4].) By the work of Small [S], this correspondence can be explicitly written when the immersion can be lifted to a null curve in $PSL(2, \boldsymbol{C}) = SL(2, \boldsymbol{C})/\{\pm 1\}$. This gives a method for constructing new examples. However, to construct non-branched CMC-c surfaces is still difficult, because the method above does not give any control over branch points.

In this paper we use a new method to construct new examples without branch points, which have higher genus, many symmetries, and embedded ends. More precisely, we prove that for each complete symmetric finite-total-curvature minimal surface in $\boldsymbol{R}^3$ with a non-degenerate period problem, there exists a corresponding one-parameter family of CMC-1 surfaces in $H^3(-1)$. We define the terms "symmetric" and "non-degenerate" later. To prove the existence of these corresponding one-parameter families, we begin by using a small deformation from the original minimal surface in $\boldsymbol{R}^3$, preserving its (hyperbolic) Gauss map G and Hopf differential Q. This gives us CMC-c surfaces in $H^3(-c^2)$, for $c \approx 0$. Finally, we rescale the surfaces into CMC-1

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surfaces in $H^3(-1)$. The method is somewhat similar to that of [UY2], the main difference being that we use the duality on CMC-1 surfaces to keep the symmetry properties of the initial minimal surfaces.

Here, we briefly outline the construction: Let $f_0 : M \to \boldsymbol{R}^3$ be a conformal minimal immersion defined on a Riemann surface M. We set

$$G = \frac{\partial x_3}{\partial x_1 - i\partial x_2}\,, \quad \omega = \partial x_1 - i\partial x_2\,, \quad Q = \omega \cdot dG\,,$$

where the dot means the symmetric product. Then G is the Gauss map of f_0, and Q is the Hopf differential, namely the $(2, 0)$-part of the complexification of the second fundamental form. The pair (G, ω) is called the Weierstrass data of f_0. For CMC-c surfaces in $H^3(-c^2)$, the Weierstrass data can also be defined. The CMC-c surface f_c with the same Weierstrass data $(G, -\omega)$ as $-f_0$ is only defined on the universal cover $\tilde{M}$. We set

$$\mathfrak{D}_M^{(c)}(G, Q) = \left\{ f : \tilde{M} \to H^3(-c^2)\,;\ \begin{array}{l} f \text{ is a conformal CMC-}c \text{ immersion} \\ \text{with hyperbolic Gauss map } G \\ \text{and Hopf differential } Q\,. \end{array} \right\}$$

In Section 3, we show that the set $\mathfrak{D}_M^{(c)}(G, Q)$ can be identified with the hyperbolic 3-space $\mathscr{H}^3$. (We will use two different notations for hyperbolic space. The notation H^3 will be used to represent the ambient space for CMC surfaces, and the notation $\mathscr{H}^3$ will be used to represent the parameter space of $\mathfrak{D}(G, Q)$.) We show that any two CMC-c immersions in $\mathfrak{D}_M^{(c)}(G, Q)$ are non-congruent, and they are dual surfaces of the CMC-c surface f_c. (As mentioned above, $(G, -\omega)$ is the Weierstrass data of f_c. The duality exchanges the roles of the hyperbolic Gauss map and the secondary Gauss map, and at the same time the sign of the Hopf differential is reversed.)

Then, defining a subset $I_M^{(c)}(G, Q)$ of immersions which are single-valued on M:

$$I_M^{(c)}(G, Q) = \{ f \in \mathfrak{D}_M^{(c)}(G, Q);\ f \text{ is single-valued on } M \text{ itself.} \}\,, \tag{1.1}$$

we show that $I_M^{(c)}(G, Q)$ is either empty, or is a connected totally geodesic subspace of dimension 0, 1 or 3 in $\mathscr{H}^3$. (The two-dimensional case does not occur.) When $\dim I_M^{(c)}(G, Q) = 0$, the unique CMC-c immersion in $I_M^{(c)}(G, Q)$ is irreducible in the sense of [UY1]. For initial minimal surfaces f_0, which are symmetric, and have non-degenerate period problems, we can construct a one-parameter family of CMC-1 surfaces $f_c : M \to H^3(-c^2)$ with the same symmetry properties as f_0. Moreover, if the initial minimal surface is generated from a fundamental domain by reflections with respect to three non-parallel planes, we can show that $I_M^{(c)}(G, Q)$ consists of only one point, whenever c is sufficiently small. This unique surface coincides with the above f_c. If we consider $H^3(-c^2)$ as the Poincaré model of radius $1/|c|$, then it converges to the initial surface f_0 as $c \to 0$. (See Remark 5.8.) The method we have just described can be applied to virtually all of the well-known symmetric minimal surfaces in $\boldsymbol{R}^3$. (See Section 5.) When the hyperbolic Gauss map G and the Hopf differential Q have certain properties, our method is valid even without initial minimal surfaces. In Section 7, we demonstrate this by constructing hyperbolic correspondences of the catenoid fence and Jorge-Meeks fences (cf. [Kar], [R]). Taking limits of these surfaces as c approaches 0, we obtain alternative proofs of the existence of the corresponding minimal surfaces in $\boldsymbol{R}^3$.

Other useful applications of duality for CMC-1 surfaces can be found in [UY5].

2. Preliminaries.

In this paper, we use the following identification:

$$H^3(-c^2)=\left\{\frac{1}{c}aa^*;\, a\in SL(2,\boldsymbol{C})\right\}, \tag{2.1}$$

where $a^*={}^t\bar{a}$. The complex Lie group $PSL(2,\boldsymbol{C})=SL(2,\boldsymbol{C})/\{\pm 1\}$ acts isometrically on $H^3(-c^2)$ by $a\circ p=apa^*$, where $a\in PSL(2,\boldsymbol{C})$ and $p\in H^3(-c^2)$. (See [Bry], [UY1], or [UY2].)

Let M be a Riemann surface and $f: M\to H^3(-c^2)$ a complete conformal CMC-c immersion of finite total curvature. Then there is a null holomorphic immersion F: $\tilde{M}\to SL(2,\boldsymbol{C})$ defined on the universal cover $\tilde{M}$ of M such that $f=(1/c)FF^*$. Such F is uniquely determined up to the ambiguity Fb, for $b\in SU(2)$. Define a meromorphic function G by

$$G=\frac{dF_{11}}{dF_{21}}=\frac{dF_{12}}{dF_{22}}, \tag{2.2}$$

where $F=(F_{ij})_{i,j=1,2}$. Then G is a meromorphic single-valued function on M. The function G is called the *hyperbolic Gauss map* of f (cf. [Bry]). We can set

$$F^{-1}dF=c\begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix}\omega, \tag{2.3}$$

where g is a meromorphic function on $\tilde{M}$ and ω is a holomorphic 1-form defined on $\tilde{M}$. We call the pair (g,ω) the *Weierstrass data* of the CMC-c immersion f, and g the *secondary Gauss map* of f. The pair (g,ω) also has an $SU(2)$-ambiguity with respect to that of F (cf. Remark 2.1). By using the Weierstrass data (g,ω), the first fundamental form ds^2 and the second fundamental form h are written as

$$ds^2=(1+|g|^2)^2\omega\cdot\bar{\omega}, \quad h=-Q-\bar{Q}+c\,ds^2, \tag{2.4}$$

where

$$Q=\omega\cdot dg. \tag{2.5}$$

Conversely, if the Weierstrass data (g,ω) on M satisfying the compatibility condition

$$ds^2=(1+|g|^2)^2\omega\cdot\bar{\omega}>0 \tag{2.6}$$

is given abstractly, then there is a CMC-c immersion $f: \tilde{M}\to H^3(-c^2)$ with Weierstrass data (g, ω).

REMARK 2.1. Let $\tilde{f}: \tilde{M} \to H^3(-c^2)$ be another conformal CMC-c immersion with Weierstrass data $(\tilde{g}, \tilde{\omega})$. Then $\tilde{f}$ and f are congruent if and only if their fundamental forms mutually coincide, which, by (2.4), is equivalent to the following condition:

$$\tilde{g} = \frac{pg - \bar{q}}{qg + \bar{p}}, \quad \tilde{\omega} = (qg + \bar{p})^2 \omega, \tag{2.7}$$

for some matrix

$$b = \begin{pmatrix} p & -\bar{q} \\ q & \bar{p} \end{pmatrix} \in SU(2),$$

where $|p|^2 + |q|^2 = 1$ (cf. [UY1, (1.6)]). We call this *$SU(2)$-equivalence*. Transforming (g, ω) into $(\tilde{g}, \tilde{\omega})$ in (2.3) transforms F into Fb^{-1}. Thus

$$\tilde{f} = \frac{1}{c}(Fb^{-1})(Fb^{-1})^* = \frac{1}{c} FF^* = f$$

holds.

The holomorphic quadratic differential Q defined by (2.5) is called the *Hopf differential* of f. By (2.4), Q is single-valued on M. The hyperbolic Gauss map G, the secondary Gauss map g and the Hopf differential Q satisfy the identity (cf. [UY1], [UY4])

$$S(g) - S(G) = 2cQ, \tag{2.8}$$

where $S(g) = S_z(g)dz^2$ is the Schwarzian derivative of g, namely $S_z(g)$ is

$$S_z(g) = \left(\frac{g''}{g'}\right)' - \frac{1}{2}\left(\frac{g''}{g'}\right)^2 \qquad ({}' = d/dz). \tag{2.9}$$

DEFINITION 2.2. Let (g, ω) be a Weierstrass data on M with the compatibility condition (2.6). Then there exists a conformal CMC-c immersion $f: \tilde{M} \to H^3(-c^2)$ with the Weierstrass data (g, ω). Let G be the hyperbolic Gauss map of f. Then there exists a unique null holomorphic immersion $F: \tilde{M} \to SL(2, \mathbf{C})$ satisfying (2.2) and (2.3) (cf. [UY4, Theorem 1.6]). We call F the *lift* of f with respect to the Weierstrass data (g, ω).

For a meromorphic function G and a holomorphic 2-differential Q defined on M, the set $\mathfrak{D}_M^{(c)}(G, Q)$ is defined as in the introduction.

LEMMA 2.3. *The following assertions are equivalent*:

(1) *$\mathfrak{D}_M^{(c)}(G, Q)$ is nonempty.*

(2) *The symmetric tensor*

$$ds_G^2=(1+|G|^2)^2\,\frac{Q}{dG}\cdot\overline{\left(\frac{Q}{dG}\right)}$$

is positive definite on M.

Moreover, any immersion f in $\mathfrak{D}_M^{(c)}(G, Q)$ has a complete induced metric whenever ds_G^2 is a complete metric of finite total curvature and Q has poles of order at most 2.

PROOF. Assume that (2) holds. Then there is a conformal CMC-c immersion $f:\tilde{M}\to H^3(-c^2)$ with the Weierstrass data $(G, -Q/dG)$. Let g be the hyperbolic Gauss map of f. Take the lift $F:\tilde{M}\to SL(2, \boldsymbol{C})$ of f. We set

$$f^{\#}=\frac{1}{c}F^{-1}(F^{-1})^{*}:\tilde{M}\to H^3(-c^2)\,,$$

which is called the dual surface in [UY5]. By taking the dual, the hyperbolic Gauss map G and the secondary Gauss map g are exchanged, and the sign of Q is reversed. So the conformal CMC-c immersion $f^{\#}$ has the hyperbolic Gauss map G and the Hopf differential Q. (See Proposition 4 of [UY5], with special attention to the sign of Q.) Hence $f^{\#}\in\mathfrak{D}_M^{(c)}(G, Q)$, and (2) implies (1). Moreover the completeness of the induced metric of $f^{\#}$ follows from [UY5, Lemma 5].

Conversely, suppose that $\hat{f}\in\mathfrak{D}_M^{(c)}(G, Q)$ is given. Let $(\hat{g}, \hat{\omega})$ be the Weierstrass data of $\hat{f}$. Taking the lift $\hat{F}$ of $\hat{f}$ with respect to the Weierstrass data $(\hat{g}, \hat{\omega})$, the conformal CMC-1 immersion defined by $\hat{f}^{\#}=(\hat{F}^{-1})(\hat{F}^{-1})^{*}$ has the Weierstrass data $(G, -Q/dG)$. Since $\hat{F}^{-1}$ is an immersion, the first fundamental form ds_G^2 of $\hat{f}^{\#}$ is positive definite. □

COROLLARY 2.4. *Suppose that $f\in\mathfrak{D}_M^{(c)}(G, Q)$ satisfies $f=(1/c)FF^{*}$, where $F: M\to SL(2, \boldsymbol{C})$ is a null holomorphic immersion. Then F satisfies the differential equation*

$$dF\cdot F^{-1}=c\begin{pmatrix} G & -G^2\\ 1 & -G\end{pmatrix}\frac{Q}{dG}\,. \tag{2.10}$$

PROOF. As seen above, the dual immersion $f^{\#}=(1/c)(F^{-1})(F^{-1})^{*}$ has the Weierstrass data $(G, -Q/dG)$. Since $(F^{-1})^{-1}d(F^{-1})=-dFF^{-1}$, the assertion immediately follows from (2.3). □

LEMMA 2.5. *If $\mathfrak{D}_M^{(c)}(G, Q)$ is not empty, then it is identified with the hyperbolic 3-space $\mathscr{H}^3$.*

In the above statement, we used the notation $\mathscr{H}^3$ for the hyperbolic 3-space to distinguish it from the hyperbolic 3-space as the ambient space for CMC surfaces.

PROOF. Choose a CMC-c immersion $f_0\in\mathfrak{D}_M^{(c)}(G, Q)$, and fix a null holomorphic immersion $F_0:\tilde{M}\to SL(2, \boldsymbol{C})$ such that $f_0=(1/c)F_0F_0^{*}$. Consider any immersion $f_1\in\mathfrak{D}_M^{(c)}(G, Q)$. Then there exists a null holomorphic immersion $F_1:\tilde{M}\to SL(2, \boldsymbol{C})$

satisfying $f_1=(1/c)F_1F_1^*$. Let g_0 and g_1 be the secondary Gauss maps of f_0 and f_1, respectively. By (2.8), $S_z(g_0)=S_z(g_1)$. Thus a well-known property of the Schwarzian derivative yields

$$g_1=\frac{a_{11}g_0+a_{12}}{a_{21}g_0+a_{22}} \qquad \text{for some} \quad a=(a_{kj})_{k,j=1,2}\in SL(2, \boldsymbol{C}) .$$

Since the hyperbolic Gauss maps of f_0 and f_1 coincide, we have $F_1=F_0a^{-1}$ (cf. [UY4, (1.6)]). Thus $f_1=(1/c)F_0(a^{-1})(a^{-1})^*F_0^*$, and

$$\begin{aligned}\mathfrak{D}_M^{(c)}(G, Q)&=\left\{\frac{1}{c}F_0(a^{-1})(a^{-1})^*F_0^*;\, a\in SL(2, \boldsymbol{C})\right\}\\ &\simeq\{(a^{-1})(a^{-1})^*;\, a\in SL(2, \boldsymbol{C})\}=\mathscr{H}^3 .\end{aligned}$$

□

By Remark 2.1, the following assertion is immediately obtained.

COROLLARY 2.6. *For any two distinct* CMC-c *immersions* $f_1, f_2 : \tilde{M}\to H^3(-c^2)$ *in* $\mathfrak{D}_M^{(c)}(G, Q)$, *there is no isometry* $T\in SL(2, \boldsymbol{C})$ *such that* $T(f_1)=f_2$.

3. Reducibility. Let M be a Riemann surface and $p : \tilde{M}\to M$ the universal cover of M. We fix a reference point $\tilde{z}_0\in\tilde{M}$ and identify canonically the fundamental group $\pi_1(M)=\pi_1(M, p(\tilde{z}_0))$ with the deck transformation group on $\tilde{M}$. Take a meromorphic function G and a holomorphic 2-differential Q defined on M. We identify the lifts $G\circ p$ and $Q\circ p$ on $\tilde{M}$ with G and Q themselves.

We consider the subset $I_M^{(c)}(G, Q)$ of $\mathfrak{D}_M^{(c)}(G, Q)$ as defined in (1.1). When $\mathfrak{D}_M^{(c)}(G, Q)$ is non-empty, as seen in Lemma 2.5, we can consider $I_M^{(c)}(G, Q)$ as a subset of the hyperbolic 3-space $\mathscr{H}^3$. The set $I_M^{(c)}(G, Q)$ is closely related to the reducibility of CMC-c surfaces. We recall the definition (cf. [UY1, Definition 3.1]): For $f\in\mathfrak{D}_M^{(c)}(G, Q)$, we define a pseudometric $d\sigma_f^2$ by $d\sigma_f^2=(-K)ds^2$, where K is the sectional curvature of the first fundamental form ds^2. Then (cf. [UY4, (2.8)])

$$d\sigma_f^2=\frac{4dg\cdot d\bar{g}}{(1+|g|^2)^2}, \tag{3.1}$$

where g is the secondary Gauss map. By (2.4), (2.5) and (3.1), one can easily get the relation

$$ds^2\cdot d\sigma_f^2=4Q\cdot\bar{Q}. \tag{3.2}$$

By (3.1), $d\sigma_f^2$ is the pull back of the canonical metric on the unit sphere $S^2\cong\boldsymbol{C}\cup\{\infty\}$ by the secondary Gauss map g. For each deck transformation $\tau\in\pi_1(M)$, there exists a matrix $\tilde{\rho}(\tau)=(a_{jk})_{j,k=1,2}$ in $PSL(2, \boldsymbol{C})$ such that

$$g\circ\tau^{-1}=\tilde{\rho}(\tau)*g=\frac{a_{11}g+a_{12}}{a_{21}g+a_{22}}, \tag{3.3}$$

where the asterisk means the action of $\tilde{\rho}(\tau)$ as a Möbius transformation. Then a representation $\tilde{\rho}: \pi_1(M) \to PSL(2, \boldsymbol{C})$ is induced. Let $F: \tilde{M} \to SL(2, \boldsymbol{C})$ be the lift of f with respect to the Weierstrass data $(g, Q/dg)$ (cf. Definition 2.2). Since $F \circ \tau$ and $F\tilde{\rho}(\tau)$ have the same hyperbolic Gauss map and secondary Gauss map $g \circ \tau$, Theorem 1.6 in [UY4] yields

$$F \circ \tau = s_\tau F \tilde{\rho}(\tau) \qquad (\tau \in \pi_1(M)), \tag{3.4}$$

where $s_\tau = 1$ or -1. We set

$$\rho(\tau) = s_\tau \tilde{\rho}(\tau) \qquad (\tau \in \pi_1(M)).$$

Then $\tilde{\rho}: \pi_1(M) \to PSL(2, \boldsymbol{C})$ can be lifted to a representation

$$\rho: \pi_1(M) \to SL(2, \boldsymbol{C}).$$

Since the representation ρ depends on the choice of the Weierstrass data, we will also use the notation ρ_F as well as ρ when such an explicit notation is required.

LEMMA 3.1. *Suppose $f \in \mathfrak{D}_M^{(c)}(G, Q)$. Then $f \in I_M^{(c)}(G, Q)$ if and only if $\rho(\tau) \in SU(2)$ for all $\tau \in \pi_1(M)$.*

PROOF. Since G and Q are single-valued on M, by Corollary 2.4 in [UY4], $f \in I_M^{(c)}(G, Q)$ if and only if ds^2 is single-valued on M. The condition is equivalent to $d\sigma_f^2$ being single-valued on M by (3.2). Thus $f \in I_M^{(c)}(G, Q)$ if and only if each $\rho(\tau)$ preserves $d\sigma_f^2$, that is, $\rho(\tau) \in SU(2)$. □

A CMC-c immersion $f \in I_M^{(c)}(G, Q)$ is said to be *reducible* if $\rho(\tau_1)\rho(\tau_2) = \rho(\tau_2)\rho(\tau_1)$ holds for all $\tau_1, \tau_2 \in \pi_1(M)$. In Theorem 3.3 of [UY1], the second and third authors showed that reducible CMC-c surfaces have a nontrivial deformation associated with the deformation of the Weierstrass data $(\lambda g, (1/\lambda)\omega)$ $(\lambda \in \boldsymbol{R} \setminus \{0\})$. This deformation preserves the Hopf differential $Q = \omega \cdot dg$. Moreover, by (2.8), $S(G)$ is also preserved. Since the Schwarzian derivative $S(G)$ is invariant under the Möbius transformations of G, we can place each of these one-parameter family of surfaces by a suitable rigid motion in $H^3(-c^2)$ so that the hyperbolic Gauss map is not changed. Thus we get a non-trivial deformation in $I_M^{(c)}(G, Q)$ for reducible CMC-c surfaces. As a refinement of this observation, we prove the following theorem.

THEOREM 3.2. *Assume that the subset $I_M^{(c)}(G, Q)$ of $\mathfrak{D}_M^{(c)}(G, Q)$ is not empty. Then the set $I_M^{(c)}(G, Q)$ is a point, a geodesic line $\mathscr{H}^1$ or all of $\mathscr{H}^3 = \mathfrak{D}_M^{(c)}(G, Q)$. Moreover, each $f \in I_M^{(c)}(G, Q)$ is irreducible if and only if $I_M^{(c)}(G, Q)$ is a point.*

REMARK 3.3. In other words, $I_M^{(c)}(G, Q)$ is a connected totally geodesic n-dimensional subset of $\mathscr{H}^3$ with $n = 0$, 1, or 3. In the case $I_M^{(c)}(G, Q) = \mathscr{H}^3$, any null holomorphic immersion $F: \tilde{M} \to PSL(2, \boldsymbol{C})$ with the hyperbolic Gauss map G and the Hopf differential Q is single-valued on M (cf. [UY3, Theorem 1.6]).

We set $\Gamma := \rho(\pi_1(M)) \subset SU(2)$. Then Theorem 3.2 is an immediate consequence of the lemma in the appendix.

4. A representation of the fundamental group. In this section, we consider how to find a CMC-c immersion $f \in \mathfrak{D}_M^{(c)}(G, Q)$ for any given hyperbolic Gauss map G and Hopf differential $Q(\not\equiv 0)$ on a Riemann surface generated by reflections. As in the previous section, we fix a reference point $\tilde{z}_0 \in \tilde{M}$ and identify canonically the fundamental group $\pi_1(M) = \pi_1(M, p(\tilde{z}_0))$ with the deck transformation group on $\tilde{M}$. Let $\tilde{\mu}_1, \dots, \tilde{\mu}_m$ be reflections of $\tilde{M}$, that is, conformal orientation-reversing involutions on $\tilde{M}$. Suppose that $\tilde{\mu}_1, \dots, \tilde{\mu}_m$ generate the deck transformation group on $\tilde{M}$ and induce reflections $\mu_1, \dots, \mu_m$ of M such that

$$p \circ \tilde{\mu}_j = \mu_j \circ p \qquad (j = 1, \dots, m), \tag{4.1}$$

where $p\colon \tilde{M} \to M$ is the covering projection.

Take a pair (G, Q) of a meromorphic function and a holomorphic 2-differential on M such that $ds_G^2 = (1 + |G|^2)^2 (Q/dG) \cdot \overline{(Q/dG)}$ is positive definite. In addition, we assume that

$$\overline{Q \circ \mu_j} = Q \quad \text{and} \quad ds_G^2 \circ \mu_j = ds_G^2 \tag{4.2}$$

hold. By Lemma 2.3, the set $\mathfrak{D}_M^{(c)}(G, Q)$ is non-empty.

LEMMA 4.1. *There exist matrices $\sigma(\mu_j) \in SU(2)$ $(j = 1, \dots, m)$ such that*

$$\overline{G \circ \mu_j} = \sigma(\mu_j)^{-1} * G = \frac{a_{11}^{(j)} G + a_{12}^{(j)}}{a_{21}^{(j)} G + a_{22}^{(j)}},$$

where $\sigma(\mu_j)^{-1} = (a_{k,l}^{(j)})$.

PROOF. Consider a pseudometric $d\rho_G^2 = 4dG \cdot d\bar{G}/(1 + |G|^2)^2$ on $\tilde{M}$. By (4.2) and (3.2), $d\rho_G^2 \circ \mu_j = d\rho_G^2$ holds. This occurs if and only if there exists a matrix $\sigma(\mu_j) \in SU(2)$ such that $\overline{G \circ \mu_j} = \sigma(\mu_j)^{-1} * G$, since μ_j is orientation reversing. $\square$

Each matrix $\sigma(\mu_j)$ $(j = 1, \dots, m)$ is determined up to sign. From now on, we fix a sign of $\sigma(\mu_j)$.

LEMMA 4.2. *Let $f \in \mathfrak{D}_M^{(c)}(G, Q)$ and $F\colon \tilde{M} \to SL(2, \boldsymbol{C})$ be a null holomorphic immersion such that $f = (1/c) F F^*$. Then there exists a unique matrix $\hat{\rho}_F(\tilde{\mu}_j) \in SL(2, \boldsymbol{C})$ $(j = 1, \dots, m)$ such that*

$$\overline{F \circ \tilde{\mu}_j} = \sigma(\mu_j)^{-1} F \hat{\rho}_F(\tilde{\mu}_j) \tag{4.3}$$

holds for $\sigma(\mu_j)$ in Lemma 4.1.

PROOF. Consider two CMC-1 immersions defined by

$$f_1 := (F^{-1})(F^{-1})^*, \quad f_2 := (\overline{F\circ\tilde{\mu}_j})^{-1}((\overline{F\circ\tilde{\mu}_j})^{-1})^*.$$

Then by assumption (4.2), the two immersions have the same first fundamental form ds_G^2 and the same second fundamental form $-Q-\bar{Q}+cds_G^2$. Hence these immersions are congruent by the fundamental theorem for surfaces, and so there is a unique isometry $a\in SL(2, \boldsymbol{C})$ such that $f_1 = af_2a^*$. In particular, there exists $b\in SU(2)$ such that (cf. Remark 2.1) $F^{-1} = a(\overline{F\circ\tilde{\mu}_j})^{-1}b$, that is, $\overline{F\circ\tilde{\mu}_j} = bFa$. Since $\overline{F\circ\tilde{\mu}_j}$ has the hyperbolic Gauss map $\overline{G\circ\mu_j}$, we have $b=\varepsilon\sigma(\mu_j)^{-1}$ ($\varepsilon = \pm 1$) by Lemma 4.1 and by [UY4, Theorem 1.6]. Finally, if we set $\hat{\rho}_F(\tilde{\mu}_j) = \varepsilon a$, we have the desired expression. The uniqueness of $\hat{\rho}_F(\tilde{\mu}_j)$ follows from that of the matrix a and the sign ε. □

REMARK 4.3. For any two reflections $\tilde{\mu}_j$, $\tilde{\mu}_k$, the following relation holds:

$$F\circ\tilde{\mu}_j\circ\tilde{\mu}_k = \overline{\sigma(\mu_k)^{-1}(F\circ\tilde{\mu}_j)\hat{\rho}_F(\tilde{\mu}_k)} = (\sigma(\mu_j)\overline{\sigma(\mu_k)})^{-1}F\hat{\rho}_F(\tilde{\mu}_j)\overline{\hat{\rho}_F(\tilde{\mu}_k)}.$$

COROLLARY 4.4. *Let $f\in\mathfrak{D}_M^{(c)}(G, Q)$ and $F\colon \tilde{M}\to SL(2, \boldsymbol{C})$ be a null holomorphic immersion such that $f=(1/c)FF^*$. Then $\hat{\rho}_{F_a}(\mu_j) = a^{-1}\hat{\rho}_F(\mu_j)\bar{a}$ holds for any $a\in SL(2, \boldsymbol{C})$.*

LEMMA 4.5. $\sigma(\mu_j)\overline{\sigma(\mu_j)} = \mathrm{id}$.

PROOF. Since $\tilde{\mu}_j\circ\tilde{\mu}_j = \mathrm{id}$, $G = G\circ\mu_j\circ\mu_j = \{\sigma(\mu_j)\overline{\sigma(\mu_j)}\}^{-1} * G$, and we see that $\sigma(\mu_j)\overline{\sigma(\mu_j)} = \pm\mathrm{id}$. Suppose that $\sigma(\mu_j)\overline{\sigma(\mu_j)} = -\mathrm{id}$. Since $\sigma(\mu_j)\in SU(2)$, we have

$$\sigma(\mu_j) = \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and so $G\circ\mu_j = \sigma(\mu_j)^{-1} * G = -1/\bar{G}$. Let z be a fixed point of μ_j. Then $|G(z)|^2 = -1$ holds, a contradiction. □

REMARK 4.6. By Remark 4.3 and the previous lemma, we have $\hat{\rho}_F(\mu_j)\overline{\hat{\rho}_F(\mu_j)} = \mathrm{id}$.

Recall that for each $\tau\in\pi_1(M)$, there is a representation $\rho_F\colon \pi_1(M)\to SL(2, \boldsymbol{C})$ such that $F\circ\tau = F\rho_F(\tau)$. (See Section 3.) Each τ is represented as

$$\tau = \tilde{\mu}_{j_1}\circ\tilde{\mu}_{j_2}\circ\cdots\circ\tilde{\mu}_{j_{2k-1}}\circ\tilde{\mu}_{j_{2k}}. \tag{4.4}$$

Then by Remark 4.3, we have

$$F\circ\tau = \{\sigma(\mu_{j_1})\overline{\sigma(\mu_{j_2})}\cdots\sigma(\mu_{j_{2k-1}})\overline{\sigma(\mu_{j_{2k}})}\}^{-1}\cdot F\cdot\hat{\rho}_F(\tilde{\mu}_{j_1})\overline{\hat{\rho}_F(\tilde{\mu}_{j_2})}\cdots\hat{\rho}_F(\tilde{\mu}_{j_{2k-1}})\overline{\hat{\rho}_F(\tilde{\mu}_{j_{2k}})}. \tag{4.5}$$

On the other hand, by Lemma 4.1,

$$G\circ\tau = \{\sigma(\mu_{j_1})\overline{\sigma(\mu_{j_2})}\cdots\sigma(\mu_{j_{2k-1}})\overline{\sigma(\mu_{j_{2k}})}\}^{-1} * G$$

holds. Since G is single-valued on M, we have

$$\sigma(\mu_{j_1})\overline{\sigma(\mu_{j_2})}\cdots\sigma(\mu_{j_{2k-1}})\overline{\sigma(\mu_{j_{2k}})} = \varepsilon_\tau\mathrm{id}, \tag{4.6}$$

where $\varepsilon_\tau = 1$ or -1. So we have an expression

(4.7) $$\rho_F(\tau)=\varepsilon_\tau\hat{\rho}_F(\tilde{\mu}_{j_1})\overline{\hat{\rho}_F(\tilde{\mu}_{j_2})}\cdots\hat{\rho}_F(\tilde{\mu}_{j_{2k-1}})\overline{\hat{\rho}_F(\tilde{\mu}_{j_{2k}})}\,.$$

The following criterion is useful to construct examples.

PROPOSITION 4.7. *Let $f\in\mathfrak{D}_M^{(c)}(G,Q)$ and $F\colon\tilde{M}\to SL(2,\boldsymbol{C})$ be a null holomorphic immersion such that $f=(1/c)FF^*$. Suppose $\hat{\rho}_F(\tilde{\mu}_j)\in SU(2)$ $(j=1,\dots,m)$ holds. Then the immersion f is an element of $I_M^{(c)}(G,Q)$. Moreover, each reflection μ_j extends to an isometry of $H^3(-c^2)$ preserving the image of f.*

PROOF. The first assertion immediately follows from Lemma 3.1 and (4.7). Since $\overline{F\circ\mu_j}=\sigma(\mu_j)^{-1}F\hat{\rho}_F(\tilde{\mu}_j)$ and $\hat{\rho}_F(\tilde{\mu}_j)\in SU(2)$, we have $f\circ\mu_j=\sigma(\mu_j)^{-1}f(\sigma(\mu_j)^{-1})^*$. This implies the last assertion. □

The following lemma plays an important role in later sections:

LEMMA 4.8. *Let $F_c\colon M\to SL(2,\boldsymbol{C})$ be a family of null holomorphic immersions such that $\lim_{c\to 0}F_c=\mathrm{id}$ and $(1/c)F_cF_c^*\in\mathfrak{D}_M^{(c)}(G,Q)$. Let l be a loop on M and $\tau\in\pi_1(M)$ the deck transformation induced from l. Then*

$$\left.\frac{\partial}{\partial c}\right|_{c=0}\rho_{F_c}(\tau)=\oint_l\begin{pmatrix}G & -G^2\\ 1 & -G\end{pmatrix}\frac{Q}{dG}\,.$$

PROOF. Let z_0 be a point of l in M and $\tilde{z}_0\in\tilde{M}$ a lift of z_0. Put $\tilde{z}_1=\tau(\tilde{z}_0)$. Then there exists a lift $\tilde{l}$ of l joining $\tilde{z}_0$ and $\tilde{z}_1$. We set $F_c'=(\partial F_c/\partial c)|_{c=0}$. By Corollary 2.4, F_c is a solution of the equation

(4.8) $$dF_c=c\alpha F_c\,,\quad \alpha=\begin{pmatrix}G & -G^2\\ 1 & -G\end{pmatrix}\frac{Q}{dG}\,.$$

Since $F_0=\lim_{c\to 0}F_c=\mathrm{id}$, we have $\rho_{F_0}=\mathrm{id}$, and since $dF_c'=\alpha F_c+c\alpha F_c'$, we have $dF_0'=\alpha$, where $'=\partial/\partial c|_{c=0}$. Integrating this, we have

$$F_0'(\tilde{z}_1)=\int_{\tilde{z}_0}^{\tilde{z}_1}\alpha+F_0'(\tilde{z}_0)\,.$$

Hence

(4.9) $$\left.\frac{\partial}{\partial c}\right|_{c=0}(F_c\circ\tau)(\tilde{z}_0)=\oint_l\alpha+F_0'(\tilde{z}_0)\,.$$

On the other hand,

(4.10) $$\left.\frac{\partial}{\partial c}\right|_{c=0}(F_c\circ\tau)(\tilde{z}_0)=\left.\frac{\partial}{\partial c}\right|_{c=0}(F_c\rho_{F_c}(\tau))(\tilde{z}_0)=\left(\left.\frac{\partial}{\partial c}\right|_{c=0}\rho_{F_c}(\tau)\right)+F_0'(\tilde{z}_0)$$

holds. By (4.9) and (4.10), we are done. □

The above lemma is a generalization of [UY2, Lemma 3.3] in which the same statement for a loop l surrounding an end is shown.

5. CMC-1 **surfaces of finite total curvature.** In [UY4], the hyperbolic correspondence of the Jorge-Meeks n-oid and CMC-1 surfaces of genus 0 with Platonic symmetries have been constructed. (See also Section 6.) In this section, we construct CMC-1 surfaces of both genus zero and positive genus in $H^3(-1)$, which correspond to minimal surfaces in Euclidean 3-space $\boldsymbol{R}^3$ described in [BR], [R], [W].

In fact, we show that for any complete finite-total-curvature minimal surface in $\boldsymbol{R}^3$ satisfying certain conditions, there exists a one-parameter family of corresponding CMC-1 surfaces in $H^3(-1)$. The conditions are quite general in the sense that they are satisfied by almost all known examples of complete finite-total-curvature minimal surfaces.

Osserman [O] showed that any complete finite-total-curvature minimal surface in $\boldsymbol{R}^3$ can be represented as a conformal immersion $f: M \to \boldsymbol{R}^3$, where M is a compact Riemannian manifold with a finite number of points removed, i.e. $M = \bar{M} \setminus \{e_1, \dots, e_r\}$. Let $S = \{x \in M \mid \exists y \neq x, f(y) = f(x)\}$ be the intersection set of f. The following definition describes minimal surfaces that are embedded everywhere except in neighborhoods of the ends.

Definition 5.1. If there exist r disjoint open disks $U_j \subset \bar{M}$ $(j = 1, \dots, r)$ such that $e_j \in U_j$ and $S \subset \bigcup_{j=1}^{r} U_j$, then the immersion f is *almost embedded.*

Next we define what we mean by a symmetric minimal immersion.

Definition 5.2. A minimal immersion f is *symmetric* if there is a subregion $D \subset f(M)$ that is a disk bounded by non-straight planar geodesics, and does not contain any non-straight planar geodesics in its interior.

If f is symmetric with subdisk D, then D generates the entire surface by reflections across planes containing boundary planar geodesics. (Since each curve in the boundary of D is a planar geodesic, the surface can be smoothly extended across these boundary curves, by the Schwartz reflection principle.) Note that since f has finite total curvature, f is not periodic. Therefore, if any two boundary curves of D lie in parallel planes, they must actually lie in the same plane.

Lemma 5.3. *If f is almost embedded, and if f is symmetric with subdisk D, then there are at most three planes that contain all of the boundary planar geodesics of D. These (at most three) planes lie in general position, and meet pairwise at angles of the form π/m, where $m \geq 2$ is an integer.*

Proof. Let $P_1, \dots, P_n$ be the smallest set of distinct planes that contain ∂D, and let μ_j $(j = 1, \dots, n)$ be the reflection with respect to the plane P_j. Since f is of finite total curvature, the group Γ generated by the symmetries $\mu_1, \dots, \mu_n$ is finite. It is well-known that Γ has a fixed point and the number of planes of the fundamental chamber is at most three (see [Bou, Chapters 4–6]). □

Let us name these distinct planes $P_1, \dots, P_s$, where s ($=2$ or 3) is the number of the planes. Let the boundary planar geodesics of D contained in P_j be called $S_{j,1}$, $S_{j,2}, \dots, S_{j,d_j}$ ($j=1, \dots, s$). We now define what we mean by a non-degenerate set of period problems. Let d equal the number of smooth boundary planar geodesics minus the number of planes. Thus $d=d_1+d_2+d_3-3$ if $s=3$, and $d=d_1+d_2-2$ if $s=2$.

DEFINITION 5.4. Let $f: M \to \boldsymbol{R}^3$ be an almost embedded minimal immersion that is symmetric with subdisk D. Then the set of period problems for f is *non-degenerate* if there exists a continuous d-parameter family of disks D_λ, ($\lambda=(\lambda_1, \dots, \lambda_d)$, $|\lambda|<\varepsilon$) such that

(1) $D_{(0,0,\cdots,0)}=D$.

(2) $\partial D_\lambda=\bigcup_{j=1}^{s}(\bigcup_{k=1}^{d_j} S_{j,k}(\lambda))$ such that each $S_{j,k}(\lambda)$ is a planar geodesic lying in a plane $P_{j,k}(\lambda)$ parallel to P_j.

(3) Letting $\mathrm{Per}_{j,k}(\lambda)$ ($j=1, \dots, s$, $k=2, \dots, d_j$) be the oriented distance between the plane $P_{j,k}(\lambda)$ and $P_{j,1}(\lambda)$, the map $\Lambda: \lambda=(\lambda_1, \dots, \lambda_d) \mapsto (\mathrm{Per}_{j,k}(\lambda)) \in \boldsymbol{R}^d$ is an open map onto a neighborhood of 0.

We reflect D_λ infinitely often to get a simply connected complete surface $\tilde{M}_\lambda$. Let M be the initial minimal surface. Then the universal cover $\tilde{M}$ of M coincides with $\tilde{M}_0$. The initial fundamental disk D_λ is contained in $\tilde{M}_\lambda$. Also we have associated reflections $\tilde{\mu}_{\lambda,j,k}$ on $\tilde{M}_\lambda$ with respect to D_λ, and we have the properties

$$\tilde{\mu}_{\lambda,j,k} \circ \tilde{\mu}_{\lambda,j,k}=\mathrm{id}\,, \qquad (\tilde{\mu}_{\lambda,j,k} \circ \tilde{\mu}_{\lambda,j',k'})^{m_{j,k,j',k'}}=\mathrm{id}\,, \tag{5.1}$$

whenever the planes containing $S_{j,k}$ and $S_{j',k'}$ meet at an angle of $\pi/m_{j,k,j',k'}$. Let $\tilde{\Gamma}(\lambda)$ be the group generated by these reflections, with the above properties (5.1). Since $m_{j,k,j',k'}$ does not depend on λ, $\tilde{\Gamma}(\lambda)$ is isomorphic to $\tilde{\Gamma}(0)$, so let $\iota: \tilde{\Gamma}(\lambda) \to \tilde{\Gamma}(0)$ be the canonical isomorphism. Let Γ be the deck transformation of the universal cover $\tilde{M}$ of the initial minimal surface M. Γ is considered as a subgroup of $\tilde{\Gamma}(0)$. Let $\Gamma_\lambda=\iota^{-1}(\Gamma)$, and let $M_\lambda=\tilde{M}_\lambda/\Gamma_\lambda$. Then M_λ is diffeomorphic to the initial minimal surface $M=\tilde{M}_0/\Gamma$. On the initial minimal surface, each reflection $\tilde{\mu}_{0,j,k}$ on $\tilde{M}$ induces the reflection $\mu_{j,k}: M \to M$. Thus for any $\tau \in \Gamma$, there exists a $\tau' \in \Gamma$ such that $\tilde{\mu}_{0,j,k} \circ \tau=\tau' \circ \tilde{\mu}_{0,j,k}$. By the definition of Γ_λ, any $\tau \in \Gamma_\lambda$ has also an element $\tau' \in \Gamma_\lambda$, such that $\tilde{\mu}_{\lambda,j,k} \circ \tau=\tau' \circ \tilde{\mu}_{\lambda,j,k}$. This implies that $\tilde{\mu}_{\lambda,j,k}$ also induces a reflection $\mu_{\lambda,j,k}: M_\lambda \to M_\lambda$.

Since D_λ is a minimal disk bounded by planar geodesics, the Gauss map G_λ and the Hopf differential Q_λ are well-defined on D_λ. Thus G_λ and Q_λ can be extended to $\tilde{M}_\lambda$, so that they satisfy the properties $\overline{G_\lambda \circ \tilde{\mu}_{j,k}}=\sigma(\mu_{j,k})^{-1} * G_\lambda$ and $\overline{Q_\lambda \circ \tilde{\mu}_{j,k}}=Q_\lambda$. Here, $\sigma(\mu_{\lambda,j,k}) \in SU(2)$ is explicitly given by

$$\sigma(\mu_{\lambda,j,k})^{-1}=\begin{pmatrix} v_2+iv_1 & -iv_3 \\ -iv_3 & v_2-iv_1 \end{pmatrix}, \tag{5.2}$$

where $v=(v_1, v_2, v_3)$ is the unit normal vector perpendicular to the plane $P_{j,k}$. (Since $P_{j,k}$ is parallel to $P_{j,1}$, the unit vector v does not depend on the choice of k.) One can easily check the above formula as follows: Since the left hand matrix is in $SU(2)$, the transformation

$$T\colon \boldsymbol{C}\cup\{\infty\}\ni z\mapsto \overline{\frac{(v_2+iv_1)z-iv_3}{-iv_3z+(v_2-iv_1)}}\in \boldsymbol{C}\cup\{\infty\}$$

preserves the metric $d\sigma_0^2=4dzd\bar{z}/(1+|z|^2)^2$. Moreover, the inverse matrix is the conjugate matrix. Thus T is an isometric involution. Let $\tilde{T}\colon S^2(1)\to S^2(1)$ be the induced isometric involution. By straightforward calculation, $\tilde{T}$ corresponds to a reflection with respect to the plane passing through the origin perpendicular to v.

LEMMA 5.5. *G_λ and Q_λ are single-valued on M_λ for any λ.*

PROOF. It suffices to show that $G_\lambda\circ\tau=G_\lambda$ and $Q_\lambda\circ\tau=Q_\lambda$ for each $\tau\in\Gamma_\lambda$. Since τ is an orientation preserving diffeomorphism on $\tilde{M}_\lambda$, it can be written as $\tau=\tilde{\mu}_{\lambda,j_1,k_1}\circ\cdots\circ\tilde{\mu}_{\lambda,j_{2n},k_{2n}}$. Since $\overline{Q_\lambda\circ\tilde{\mu}_{\lambda,j,k}}=Q_\lambda$, obviously $Q\circ\tau=Q$ holds. On the other hand, we have

$$G\circ\tau=\{\sigma(\mu_{\lambda,j_1,k_1})\overline{\sigma(\mu_{\lambda,j_2,k_2})}\cdots\sigma(\mu_{\lambda,j_{2n-1},k_{2n-1}})\overline{\sigma(\mu_{\lambda,j_{2n},k_{2n}})}\}^{-1}*G\,.$$

Note that $G\circ\tau=G$ holds for $\lambda=0$, because G coincides with the Gauss map of the initial minimal surface when $\lambda=0$. This implies that

$$\sigma(\mu_{\lambda,j_1,k_1})\overline{\sigma(\mu_{\lambda,j_2,k_2})}\cdots\sigma(\mu_{\lambda,j_{2n-1},k_{2n-1}})\overline{\sigma(\mu_{\lambda,j_{2n},k_{2n}})}=\pm\mathrm{id}$$

where $\lambda=0$. Thus it holds for any λ because $\sigma(\mu_{j,k})$ does not depend on λ, by (5.2). $\square$

Thus we have an *abstract* Riemann surface M_λ. Moreover, the induced conformal metric of $\tilde{M}_\lambda$ defined by $ds_\lambda^2:=(1+|G_\lambda|^2)^2Q_\lambda/dG_\lambda\cdot\overline{(Q_\lambda/dG_\lambda)}$ is single-valued on M_λ. Since $\tilde{M}_\lambda$ is complete and M_λ consists of a finite number of congruent copies of the finite-total-curvature disk D_λ, M_λ is complete and has finite total curvature. Hence M_λ can be written as

$$M_\lambda=\bar{M}_\lambda\setminus\{e_1,\dots,e_r\}\,,$$

where $\bar{M}_\lambda$ is a compact Riemann surface, and $e_1,\dots,e_r$ are points corresponding to the ends of M_λ. Let $f_\lambda\colon\tilde{M}_\lambda\to\boldsymbol{R}^3$ be the immersion of the (periodic) minimal surface containing D_λ. Then G_λ and $\omega_\lambda=Q_\lambda/dG_\lambda$ are the Weierstrass data of f_λ. Since D_λ is continuous in λ, we see that G_λ, and Q_λ are also continuous in λ. *For the sake of simplicity, we express each reflection by $\mu_{j,k}$ instead of $\mu_{\lambda,j,k}$.* We can place D_λ in $\boldsymbol{R}^3$ so that the boundary curves $S_{1,k}$ lie in the plane $\{x_2=0\}$, the boundary curves $S_{2,k}$ lie in a vertical plane containing the x_3-axis, and the boundary curves $S_{3,k}$ lie in a nonvertical plane. Since the angle between P_1 and P_2 is of the form π/m (see Lemma 5.3), we have

$$(5.3)\qquad \sigma(\mu_{1,k})=\mathrm{id}\,,\quad \sigma(\mu_{2,k})=\begin{pmatrix} e^{\pi i/m} & 0 \\ 0 & e^{-\pi i/m}\end{pmatrix},\quad \sigma(\mu_{3,k})=\begin{pmatrix} p_0 & q_0 \\ -\bar{q}_0 & \bar{p}_0\end{pmatrix},$$

where m is an integer, $|p_0|^2+|q_0|^2=1$, and $q_0\neq 0$, $\mathrm{Re}(q_0)=0$.

PROPOSITION 5.6. *Suppose $m\geq 2$. Then, for a sufficiently small $\varepsilon>0$, there exists a $(d+1)$-parameter family $\{F_{c,\lambda}\}_{|c|<\varepsilon,\lambda\in B_\varepsilon(0)}$, $(B_\varepsilon(0)\subset \boldsymbol{R}^d)$ of null holomorphic immersions of $\tilde{M}_\lambda$ into $SL(2,\boldsymbol{C})$ with the following properties.*

(1) $f_{c,\lambda}=(1/c)F_{c,\lambda}F^*_{c,\lambda}\in\mathfrak{D}^{(c)}_{M_\lambda}(G_\lambda,Q_\lambda)$ *for each* (c,λ) $(c\neq 0)$.

(2) $F_{c,\lambda}$ *is smooth in* c *and continuous in* λ.

(3) $\lim_{c\to 0}F_{c,\lambda}=\mathrm{id}$.

(4) $\lim_{c\to 0}\hat{\rho}_{F_{c,\lambda}}(\tilde{\mu}_{j,k})=\sigma(\mu_{j,k})$, *where* $\rho_{F_{c,\lambda}}(\tilde{\mu}_{j,k})$ *is the matrix given in Lemma* 4.2.

(5) $\hat{\rho}_{F_{c,\lambda}}(\tilde{\mu}_{1,1})=\mathrm{id}$.

(6) *There exists a smooth function* $\xi=\xi(c,\lambda)$ *such that* $|\xi|=1$ *and*

$$\hat{\rho}_{F_{c,\lambda}}(\tilde{\mu}_{2,1})=\begin{pmatrix}\xi & 0\\ 0 & \xi^{-1}\end{pmatrix}.$$

(7) *When* $s=3$,

$$\hat{\rho}_{F_{c,\lambda}}(\tilde{\mu}_{3,1})=\begin{pmatrix}p_{3,1} & i\beta\\ i\beta & \overline{p_{3,1}}\end{pmatrix},$$

where $p_{3,1}=p_{3,1}(c,\lambda)$ *is a complex-valued function and* $\beta=\beta(c,\lambda)$ *is a real-valued function.*

(8) *For each* $j=1,\dots,s$, *and each* $k\geq 2$,

$$\hat{\rho}_{F_{c,\lambda}}(\tilde{\mu}_{j,k})=\begin{pmatrix}p_{j,k} & i\gamma_{1,j,k}\\ i\gamma_{2,j,k} & \overline{p_{j,k}}\end{pmatrix},$$

where $p_{j,k}=p_{j,k}(c,\lambda)$ *is a complex-valued function, and* $\gamma_{l,j,k}=\gamma_{l,j,k}(c,\lambda)$ $(l=1,2)$ *are real-valued functions.*

The following lemma is easily proved in view of the fact that $a\bar{a}=\mathrm{id}$ implies $\bar{a}=a^{-1}$.

LEMMA 5.7. *Let a be a matrix in $SL(2,\boldsymbol{C})$. Then $a\bar{a}=\mathrm{id}$ holds if and only if a is of the form*

$$a=\begin{pmatrix}p & i\gamma_1\\ i\gamma_2 & \bar{p}\end{pmatrix}\qquad(\gamma_1,\gamma_2\in\boldsymbol{R}).$$

PROOF OF PROPOSITION 5.6. We prove the proposition in the case of $s=3$. When $s=2$, the assertions are proved if we simply ignore the discussions after (5.7) in Step III.

Step I. Let $z_{0,\lambda}$ be a fixed point in the curve $S_{1,1}$, which depends continuously on λ. Consider the initial value problem according to Corollary 2.4:

$$(5.4)\qquad d\check{F}_{c,\lambda}\check{F}_{c,\lambda}^{-1}=c\alpha_\lambda\,,\quad \check{F}_{c,\lambda}(\tilde{z}_{0,\lambda})=\mathrm{id}\,,\qquad \text{where}\quad \alpha_\lambda=\begin{pmatrix} G_\lambda & -G_\lambda^2 \\ 1 & -G_\lambda \end{pmatrix}\frac{Q_\lambda}{dG_\lambda}\,.$$

G_λ is real on the curve $S_{1,1}$, since $S_{1,1}$ is a planar geodesic in a plane perpendicular to the x_2-axis. Q is real on the curve $S_{1,1}$, since $\overline{Q_\lambda\circ\mu_{1,1}}=Q_\lambda$. Thus, the solutions $\check{F}_{c,\lambda}$ are also real on $S_{1,1}$. For the solutions $\check{F}_{c,\lambda}$ of (5.4), $\check{f}_{c,\lambda}=(1/c)\check{F}_{c,\lambda}\check{F}_{c,\lambda}^{*}$ satisfies (1), (2).

For $c=0$, (5.4) reduces to $d\check{F}_{0,\lambda}=0$. So $\check{F}_{0,\lambda}$ satisfies (3). Moreover, since the $\check{F}_{c,\lambda}$ are real on $S_{1,1}$, $\overline{\check{F}_{c,\lambda}\circ\mu_{1,1}}=\check{F}_{c,\lambda}$ on $\tilde{S}_{1,1}$, where $\tilde{S}_{1,1}$ is the lift of $S_{1,1}$ that contains the point $\tilde{z}_0$. So by the holomorphicity of $\check{F}_{c,\lambda}$,

$$\overline{\check{F}_{c,\lambda}\circ\mu_{1,1}}=\check{F}_{c,\lambda}\qquad\text{on}\quad \tilde{M}_\lambda\,.$$

This shows that $\hat{\rho}_{\check{F}}(\tilde{\mu}_{1,1})$ is the identity and (5) is proved. By (5.4), we have $\check{F}_{0,\lambda}=\mathrm{id}$. This proves (3). By $\overline{G_\lambda\circ\tilde{\mu}_{j,k}}=\sigma(\mu_{j,k})^{-1}*G_\lambda$, we have

$$\overline{\check{F}_{0,\lambda}\circ\tilde{\mu}_{j,k}}=\sigma(\mu_{j,k})^{-1}\check{F}_{0,\lambda}\hat{\rho}_{\check{F}}(\tilde{\mu}_{j,k})\,,$$

where $\sigma(\mu_{j,k})$ is the matrix given by (5.2). Since $\check{F}_{0,\lambda}=\mathrm{id}$, we have $\lim_{c\to0}\hat{\rho}_{\check{F}_{c,\lambda}}(\tilde{\mu}_{j,k})=\sigma(\mu_{j,k})$. This proves (4).

Step II. By (4.5), we have

$$(5.5)\qquad \hat{\rho}_{\check{F}_{c,\lambda}}(\tilde{\mu}_{j,k})\overline{\hat{\rho}_{\check{F}_{c,\lambda}}(\tilde{\mu}_{j,k})}=\mathrm{id}\,.$$

By Lemma 5.7, we have the following expression:

$$\hat{\rho}_{\check{F}_{c,\lambda}}(\tilde{\mu}_{2,1})=\begin{pmatrix} p(c,\lambda) & i\gamma_1 \\ i\gamma_2 & \overline{p(c,\lambda)} \end{pmatrix}(\in SL(2,\boldsymbol{C}))\,.$$

Since we suppose that $m\ge2$ and $\lim_{c\to0}p(c,\lambda)=e^{\pi i/m}$, there exists a sufficiently small neighborhood $U\subset\boldsymbol{R}^k$ and a positive $\varepsilon>0$ such that the imaginary part $\mathrm{Im}(p(c,\lambda))>0$ and $|\mathrm{Re}(p)|<1$ for any $|c|<\varepsilon$ and $\lambda\in U$. In such a small neighborhood, $\hat{\rho}_{F_{c,\lambda}}(\tilde{\mu}_{2,1})$ has two distinct eigenvalues $\xi(c,\lambda)$ and $1/\xi(c,\lambda)$. Since $|\mathrm{Re}(p)|<1$, $\xi(c,\lambda)$ is not a real number, and thus $|\xi(c,\lambda)|=1$ by straightforward calculation. Moreover $\xi=\xi(c,\lambda)$ is a smooth function. Now we define a real matrix $u(c,\lambda)\in SL(2,\boldsymbol{R})$ by

$$u(c,\lambda)=\left\{1+\frac{\gamma_1\gamma_2}{(\mathrm{Im}(p)+\sqrt{1-\mathrm{Re}(p)^2})^2}\right\}^{-1}\begin{pmatrix} 1 & -i\gamma_1/(p-\bar{\xi}) \\ -i\gamma_2/(\bar{p}-\xi) & 1 \end{pmatrix}.$$

Since $\mathrm{Im}(p(c,\lambda))>0$, we can easily see that $\bar{p}-\xi$ is a non-vanishing imaginary number for any $\lambda\in U$ and $|c|<\varepsilon$. Thus $u=u(c,\lambda)$ is defined as a smooth $SL(2,\boldsymbol{R})$-valued function. By straightforward calculation, we have

$$(5.6)\qquad \hat{\rho}_{\check{F}_{c,\lambda}}(\tilde{\mu}_{2,1})=u(c,\lambda)\begin{pmatrix} \xi(c,\lambda) & 0 \\ 0 & \overline{\xi(c,\lambda)} \end{pmatrix}u(c,\lambda)^{-1}\,.$$

We set $\hat{F}_{c,\lambda}=\check{F}_{c,\lambda}u(c,\lambda)$ and $\hat{f}_{c,\lambda}=(1/c)\hat{F}_{c,\lambda}\hat{F}_{c,\lambda}^{*}$. Since

$$\overline{\check{F}_{0,\lambda}\circ\tilde{\mu}_{j,k}}=\sigma(\mu_{j,k})^{-1}\check{F}_{0,\lambda}\hat{\rho}_{\check{F}}(\tilde{\mu}_{j,k})$$

and $u(c,\lambda)\in SL(2,\boldsymbol{R})$, we have by Corollary 4.4

$$\hat{\rho}_{\hat{F}_{c,\lambda}}(\mu_{j,k})=u(c,\lambda)^{-1}\hat{\rho}_{\check{F}_{c,\lambda}}(\mu_{j,k})\overline{u(c,\lambda)} \qquad (j=1,\dots,s,\ k=1,\dots,d_j)\,.$$

By (5.6), we have (6) for $\hat{f}_{c,\lambda}$. Since $u(0,\lambda)=\mathrm{id}$, $\hat{f}_{c,\lambda}$ satisfies the same properties (1)–(5) as $\check{f}_{c,\lambda}$.

Step III. By (5.5) and Lemma 5.7, the matrices $\hat{\rho}_{\hat{F}_{c,\lambda}}(\tilde{\mu}_{3,1})$ and $\hat{\rho}_{\hat{F}_{c,\lambda}}(\tilde{\mu}_{j,k})$ $(j\geq 2)$ are written in the form

$$\hat{\rho}_{\hat{F}_{c,\lambda}}(\tilde{\mu}_{3,1})=\begin{pmatrix} p_{3,1} & i\beta_1 \\ i\beta_2 & \overline{p_{3,1}}\end{pmatrix},\quad \hat{\rho}_{\hat{F}_{c,\lambda}}(\tilde{\mu}_{j,k})=\begin{pmatrix} p_{j,k} & i\gamma_{1,j,k} \\ i\gamma_{2,j,k} & \overline{p_{j,k}}\end{pmatrix}, \tag{5.7}$$

where β_1, β_2, $\gamma_{1,j,k}$ and $\gamma_{2,j,k}$ are real-valued functions of c and λ. Since $\lim_{c\to 0}\hat{\rho}_{\hat{F}_{c,\lambda}}(\tilde{\mu}_{3,1})=\sigma(\mu_{3,1})$, we have

$$p_{3,1}(0,\lambda)=p_0\,,\quad \beta_1(0,\lambda)=\beta_2(0,\lambda)=-iq_0\neq 0\,. \tag{5.8}$$

By (5.8), there exists a positive number ε such that $\beta_1/\beta_2>0$ holds for $|c|<\varepsilon$. For c in such a range, we set $f_{c,\lambda}=(1/c)F_{c,\lambda}F_{c,\lambda}^{*}$, where

$$F_{c,\lambda}=\hat{F}_{c,\lambda}\begin{pmatrix} t_{c,\lambda} & 0 \\ 0 & t_{c,\lambda}^{-1}\end{pmatrix},\qquad t_{c,\lambda}=\sqrt[4]{\frac{\beta_1}{\beta_2}}\,.$$

Since $t_{0,\lambda}=1$ and $t=t_{c,\lambda}$ is a real-valued function, $F_{c,\lambda}$ and $f_{c,\lambda}$ satisfy (1)–(6). By Corollary 4.4,

$$\hat{\rho}_{F_{c,\lambda}}(\tilde{\mu}_{3,1})=\begin{pmatrix} t_{c,\lambda}^{-1} & 0 \\ 0 & t_{c,\lambda}\end{pmatrix}\hat{\rho}_{\hat{F}_{c,\lambda}}(\tilde{\mu}_{3,1})\begin{pmatrix} t_{c,\lambda} & 0 \\ 0 & t_{c,\lambda}^{-1}\end{pmatrix}=\begin{pmatrix} p_{3,1} & i\beta \\ i\beta & \overline{p_{3,1}}\end{pmatrix},$$

where $\beta=\sqrt{\beta_1\beta_2}$. Thus $f_{c,\lambda}$ satisfies (7). By (5.7), we have

$$\hat{\rho}_F(\tilde{\mu}_{j,k})=\begin{pmatrix} t_{c,\lambda}^{-1} & 0 \\ 0 & t_{c,\lambda}\end{pmatrix}\hat{\rho}_{\hat{F}_{c,\lambda}}(\tilde{\mu}_{j,k})\begin{pmatrix} t_{c,\lambda} & 0 \\ 0 & t_{c,\lambda}^{-1}\end{pmatrix}=\begin{pmatrix} p_{j,k} & it_{c,\lambda}^{-2}\gamma_{1,j,k} \\ it_{c,\lambda}^{2}\gamma_{2,j,k} & \overline{p_{j,k}}\end{pmatrix}.$$

Replacing $\gamma_{1,j,k}$ and $\gamma_{2,j,k}$ by $t_{c,\lambda}^{-2}\gamma_{1,j,k}$ and $t_{c,\lambda}^{2}\gamma_{2,j,k}$, we get (8). □

REMARK 5.8. In the Poincaré model of radius $1/c$, the immersion $f_{c,\lambda}$ converges to the initial minimal immersion f_λ. Indeed, by (5.4) we have $dF_{c,\lambda}F_{c,\lambda}^{-1}=d\check{F}_{c,\lambda}\check{F}_{c,\lambda}^{-1}=c\alpha_\lambda$. By the same argument as in Lemma 4.8, we have $dF'_{0,\lambda}=\alpha_\lambda$, where $F'_{0,\lambda}=(\partial F_c/\partial c)|_{c=0}$. In the complexified Poincaré plane of radius $1/c$, $F_{c,\lambda}$ converges to $F'_{0,\lambda}$ by Lemma 2.1 of [UY2]. On the other hand, the initial minimal immersion f_λ can be expressed as $f_\lambda=F'_{0,\lambda}+\overline{F'_{0,\lambda}}$, where we identify a point (x_1,x_2,x_3) in $\boldsymbol{C}^3$ with the matrix

$$\begin{pmatrix} x_3 & x_1+ix_2 \\ x_1-ix_2 & -x_3\end{pmatrix}$$

in the Lie algebra of $SL(2, \boldsymbol{C})$. Thus by Corollary 2.2 of [UY2], we can conclude $f_{c,\lambda} \to f_\lambda$.

It should be remarked that $\hat{\rho}_{F_{c,\lambda}}(\tilde{\mu}_{1,1})$, $\hat{\rho}_{F_{c,\lambda}}(\tilde{\mu}_{2,1})$, $\hat{\rho}_{F_{c,\lambda}}(\tilde{\mu}_{3,1}) \in SU(2)$. Moreover, $\hat{\rho}_{F_{c,\lambda}}(\tilde{\mu}_{j,k}) \in SU(2)$ if and only if $\gamma_{1,j,k} = \gamma_{2,j,k}$. By Proposition 4.7, the period problem in the hyperbolic case reduces to showing that $\gamma_{1,j,k} = \gamma_{2,j,k}$. To show that this can be done, we have the following lemma. Recall that $o(c)$ denotes any function of c that tends to zero faster than c itself as $c \to 0$.

LEMMA 5.9. *Let $F_{c,\lambda}$ be as in Proposition 5.6. Then*

$$(\gamma_{1,j,k} - \gamma_{2,j,k})(c, \lambda) = 2c \cdot \mathrm{Per}_{j,k}(\lambda) + o(c) \qquad (j = 1, \dots, s, k = 2, \dots, d_j).$$

PROOF. Let $l_{j,k}^{(1)}$ be a curve in D_λ starting from a point on $S_{j,1}$ and ending at a point on $S_{j,k}$, and let $l_{j,k}^{(2)}$ be its reflected curve across the plane containing $S_{j,1}$. Then the curve $l_{j,k}$ obtained as the composite of the reversed oriented curve $-l_{j,k}^{(2)}$ with $l_{j,k}^{(1)}$ can be considered as a closed loop in M_λ. Let $\tau_{j,k}$ be the corresponding element of $\pi_1(M_\lambda)$. It can be identified with an element of the deck transformations of $\tilde{M}_\lambda$, if we choose the base point in $\tilde{M}_\lambda$ as the initial point of $l_{j,k}$. Then we have $\tau_{j,k} = \tilde{\mu}_{j,k} \circ \tilde{\mu}_{j,1}$. We have from Remark 4.3 and (4.6) that

$$F_{c,\lambda} \circ \tau_{j,k} = F_{c,\lambda} \circ \tilde{\mu}_{j,k} \circ \tilde{\mu}_{j,1} = F_{c,\lambda} \hat{\rho}_{F_{c,\lambda}}(\tilde{\mu}_{j,k}) \overline{\hat{\rho}_{F_{c,\lambda}}(\tilde{\mu}_{j,1})}.$$

By the Weierstrass representation formula for the initial minimal surface with the Weierstrass data $(G_\lambda, \omega_\lambda)$, (where $\omega_\lambda = Q_\lambda / dG_\lambda$), we have

$$\mathrm{Re} \oint_{l_{j,k}} (1 - G_\lambda^2, i(1 + G_\lambda^2), 2G_\lambda)\omega_\lambda = 2 \cdot \mathrm{Per}_{j,k}(\lambda) \cdot (v_1, v_2, v_3), \tag{5.9}$$

where (v_1, v_2, v_3) is a unit vector perpendicular to the planes containing $S_{j,1}$ and $S_{j,k}$. If we set

$$\mathrm{Im} \oint_{l_{j,k}} (1 - G_\lambda^2, i(1 + G_\lambda^2), 2G_\lambda)\omega_\lambda = (\eta_1, \eta_2, \eta_3),$$

then we have from Lemma 4.8

$$\left. \frac{\partial}{\partial c} \right|_{c=0} \rho_{F_{c,\lambda}}(\tau_{j,k}) = \oint_{l_{j,k}} \begin{pmatrix} G_\lambda & -G_\lambda^2 \\ 1 & -G_\lambda \end{pmatrix} \omega_\lambda = \mathrm{Per}_{j,k}(\lambda) A + iB, \tag{5.10}$$

where A and B are the matrices given by

$$A = \begin{pmatrix} v_3 & v_1 + iv_2 \\ v_1 - iv_2 & -v_3 \end{pmatrix}, \quad B = \begin{pmatrix} \eta_3 & \eta_1 + i\eta_2 \\ \eta_1 - i\eta_2 & -\eta_3 \end{pmatrix}. \tag{5.11}$$

Since $\rho_{F_{c,\lambda}}(\tau) = \hat{\rho}_{F_{c,\lambda}}(\tilde{\mu}_{j,k}) \overline{\hat{\rho}_{F_{c,\lambda}}(\tilde{\mu}_{j,1})}$, by (4) of Proposition 5.6, we have

$$\rho_{F_{c,\lambda}}(\tau_{j,k})' = (\rho_k)' \bar{\sigma} + \sigma \overline{(\rho_1)}',$$

where we simplify the notation as $\rho_k=\hat{\rho}_{F_{c,\lambda}}(\tilde{\mu}_{j,k})$, $\sigma=\sigma(\mu_{j,1})=\sigma(\mu_{j,k})$ and the prime denotes differentiation with respect to c evaluated at $c=0$. Since $\rho_j\overline{\rho_j}=\mathrm{id}$, we have $(\rho_j)'\bar{\sigma}+\sigma\overline{(\rho_j)}'=0$. Thus by (5.10) we have $\rho_{F_{c,\lambda}}(\tau_{j,k})'=\{(\rho_k)'-(\rho_1)'\}\bar{\sigma}$. By Lemma 4.5, $\sigma=\bar{\sigma}^{-1}$ holds. Thus we have

$$(\rho_k)'-(\rho_1)'=\mathrm{Per}_{j,k}(\lambda)A\sigma+iB\sigma\,.$$

For a matrix $a=(a_{ij})_{i,j=1,2}$, we define a function $\Delta[a]$ by $\Delta[a]=a_{12}-a_{21}$. Since $\Delta[\rho_1]=0$, we have $\Delta[(\rho_1)']=0$. Hence

$$i\{(\gamma_{1,j,k})'-(\gamma_{2,j,k})'\}=\Delta[(\rho_k)']=\Delta[(\rho_k)']-\Delta[(\rho_1)']=\mathrm{Per}_{j,k}(\lambda)\Delta[A\sigma]+i\Delta[B\sigma]\,.$$

By (5.2) and (5.11), we can directly compute that $\Delta[A\sigma]=2i$ and $\Delta[B\sigma]\in i\boldsymbol{R}$. Since $(\gamma_{1,j,k})'-(\gamma_{2,j,k})'\in\boldsymbol{R}$, we have

$$(\gamma_{1,j,k})'-(\gamma_{2,j,k})'=2\cdot\mathrm{Per}_{j,k}(\lambda)\,. \tag{5.12}$$

On the other hand, by (4) of Proposition 5.6, it holds that

$$\lim_{c\to 0}(\gamma_{1,j,k}-\gamma_{2,j,k})=\lim_{c\to 0}\Delta(\rho_j)=\Delta(\sigma)=0\,. \tag{5.13}$$

By (5.12) and (5.13), we have the conclusion. □

Thus we have equated the period problems for the minimal surface in $\boldsymbol{R}^3$ with the period problems for the CMC-c surface in $H^3(-c^2)$. If the period problem on the minimal surface is non-degenerate, the period map $\lambda\mapsto\mathrm{Per}_{j,i}(\lambda)$ is an open map onto a neighborhood of the origin. Since F is smooth in c and continuous in λ, for any sufficiently small c there exists a λ_c such that $\gamma_{2,j,k}(c,\lambda_c)=\gamma_{1,j,k}(c,\lambda_c)$ for all k and j. For such a pair (c,λ_c), we see that f_{c,λ_c} satisfies Proposition 4.7. Thus the CMC-c surface f_{c,λ_c} is an element of $I_M^{(c)}(G_{\lambda_c},Q_{\lambda_c})$. By Lemma 2.3, the surface has a complete induced metric, whenever the Hopf differential Q has a pole of order at most 2. Moreover, in this case the surface has finite total curvature, because its secondary Gauss map has at most a pole at each end. Multiplying the Poincaré model by c, $H^3(-c^2)$ becomes $H^3(-1)$ and f_{c,λ_c} becomes a CMC-1 surface. As we can do this for any c sufficiently close to 0, we have found a one-parameter family of CMC-1 surfaces in $H^3(-1)$. Thus we get the following:

THEOREM 5.10. *Let f_0 be a complete almost-embedded minimal surface in $\boldsymbol{R}^3$ with finite total curvature and non-degenerate period problem, and suppose that f_0 is symmetric with respect to a subdisk D. Then there exists a one-parameter family of* CMC-1 *surfaces f_c in $H^3(-1)$ with the same reflectional symmetry properties. Moreover it has the following properties*:

(1) *If the Hopf differential Q of the initial minimal surface has poles of order at most* 2, *then the surfaces also have complete induced metrics with finite total curvature.*

(2) *The* CMC-1 *surfaces are irreducible if $s=3$, where s is the number of planes*

containing the boundary of the minimal subdisk D.

The final statement follows as follows: Suppose that one of the CMC-1 surfaces are reducible. Then by definition, $\hat{\rho}_{F_{c,\lambda}}(\tilde{\mu}_{2,1})$ and $\hat{\rho}_{F_{c,\lambda}}(\tilde{\mu}_{3,1})$ must be commutative, a contradiction to the fact that $\hat{\rho}_{F_{c,\lambda}}(\tilde{\mu}_{3,1})$ is not diagonal.

Now we apply this theorem to construct several examples.

5.1. Jorge-Meeks n-oid with one handle. Let $n \geq 3$ be an integer. Then there exist complete minimal surfaces of genus one with n catenoid ends in $\boldsymbol{R}^3$ (cf. [BR]). These surfaces look like Jorge-Meeks surfaces [JM] with one attached handle, and so they admit $D_n \times \boldsymbol{Z}_2$ symmetry, where D_n is the dihedral group of order n. The fundamental piece of the surface is given in Figure 1. As shown in Berglund-Rossman [BR], the period problem of this example is non-degenerate. (Here $s=3$, $d=1$ and so the period problem is one-dimensional). Thus, by Theorem 5.10, we have the existence of a one-parameter family of CMC-1 irreducible n-oid cousins in $H^3(-1)$ with finite total

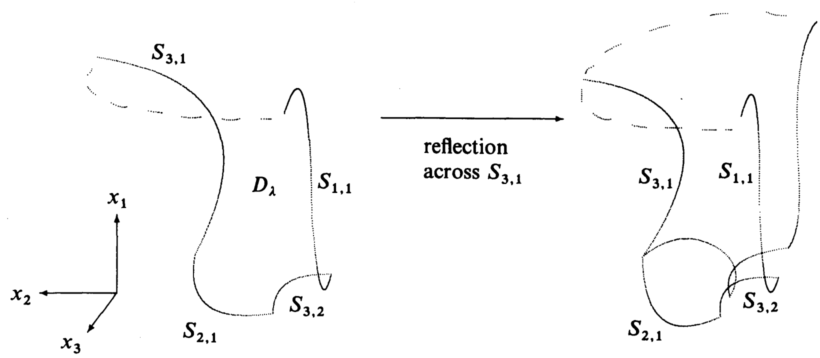

FIGURE 1. The fundamental piece D_λ.

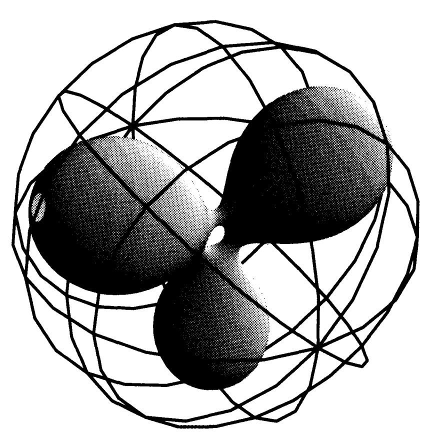

FIGURE 2. A genus-1 CMC-1 trinoid in $H^3(-1)$. (This picture is due to Katsunori Sato of the Tokyo Institute of Technology.)

curvature and symmetry $D_n \times \boldsymbol{Z}_2$.

5.2. Genus-0 Jorge-Meeks n-oid. The genus zero verison of the previous example can be shown to exist as a CMC-1 surface in $H^3(-1)$ by the same method. It has been constructed in [UY4]. (There is no period problem ($s=3$, $d=0$), so the non-degenerate condition is trivially satisfied.) We examine this example in more detail in the next section.

5.3. Genus-0 and higher genus Platonoids. With only slight modifications of the two previous examples, we have CMC-c surfaces with Platonic symmetries, which correspond to the minimal surfaces in Section 4 of [BR]. Higher genus Platonoids were also constructed by Berglund-Rossman [BR]. Since they have non-degenerate period problem, we get the following irreducible examples by Theorem 5.10:

- a CMC-c surface of genus 3 with 4-ends and tetrahedral symmetry,
- a CMC-c surface of genus 5 with 8-ends and octahedral symmetry,
- a CMC-c surface of genus 7 with 6-ends and octahedral symmetry,
- a CMC-c surface of genus 11 with 20-ends and icosahedral symmetry, and
- a CMC-c surface of genus 19 with 20-ends and icosahedral symmetry.

5.4. Enneper's surface. Enneper's cousin f was found in [Bry], which is a complete simply connected CMC-1 immersion on $\boldsymbol{C}$ with the same Weierstrass data (z, dz) as the original Enneper surface. Since it has no period problem, we can trivially apply Theorem 5.10 and get the daul $f^{\#}$ of Enneper's cousin. Since the hyperbolic Gauss map of f is given by $G=\tanh z$ (see [Bry]), the dual surface $f^{\#}$ is complete. (Since G has an essential singularity at infinity, the total curvature of $f^{\#}$ is infinite.) Compared to Enneper's cousin, its dual surface $f^{\#}$ is aesthetically more appealing (see Figure 3). Note that for every planar geodesic on the minimal Enneper's surface, there is a corresponding planar geodesic on Enneper's cousin. However, for the straight lines in minimal Enneper's surface, there are not corresponding geodesic lines in

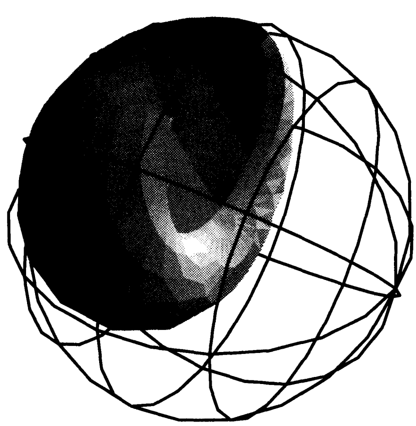

FIGURE 3. Half of the dual of Enneper's cousin.

Enneper's cousin. This is because our construction utilizes the planar geodesics of the minimal surface, but does not concern itself with straight lines in the minimal surface. The result is that the symmetry group of the dual of Enneper's cousin is a subgroup of that of minimal Enneper's surface.

Recently, Sato [Sa] has shown the existence of higher genus Enneper-type minimal surfaces for any positive genus, where he showed a certain non-degeneracy of the period problems. However, his work does not imply that the period problems are non-degenerate in our sense, since his fundamental domain is smaller than ours.

5.5. Higher genus prismoid. Let $n \geq 3$ be an integer. There are so called "higher genus prismoid" minimal surfaces as shown in Figure 4 (cf. [R]). These surfaces have $D_n \times \boldsymbol{Z}_2$ symmetry. In these examples, we have two period-killing problems ($s=3$, $d=2$), which correspond to the two parameters λ_1 and λ_2 in Figure 5. The fundamental domain D_{λ_1,λ_2} as in Figure 5 is a minimal surface in $\boldsymbol{R}^3$ which is obtained as the conjugate surface of the minimal disk bounded by $\hat{C}_{\lambda_1,\lambda_2}$ in Figure 5. Each segment of the boundary D_{λ_1,λ_2} is a planar geodesic contained in the plane $P_{j,k}$, where $P_{3,1}$, $P_{3,2}$ and $P_{3,3}$ are parallel to the x_1x_3-plane, $P_{2,1}$ is perpendicular to $(\sin(\pi/n), \cos(\pi/n), 0)$, and $P_{1,1}$ is parallel to the x_1x_2-plane. We define c_k ($k=1, 2, 3$) by $P_{3,k}=\{x_2=c_k\}$. Then the c_k's are continuous functions in λ_1 and λ_2. Consider two functions $f_1=c_1-c_3$ and $f_2=c_2-c_3$. The following statement implies that the period problem is non-degenerate: *There exists an open disk in the (λ_1, λ_2)-plane such that its image under the map $(\lambda_1, \lambda_2) \mapsto (f_1(\lambda_1, \lambda_2), f_2(\lambda_1, \lambda_2))$ contains the origin in the (f_1, f_2)-plane as an interior point.* This can be proved by arguments found in [BR]. By Theorem 5.10, there exist CMC-1 higher genus prismoid cousins.

5.6. Genus-0 prismoids. This case is a simpler version of the above example. In this case, there is still a period problem, but now it is only one-dimensional ($s=3$, $d=1$), and it is known to be non-degenerate (cf. [R]).

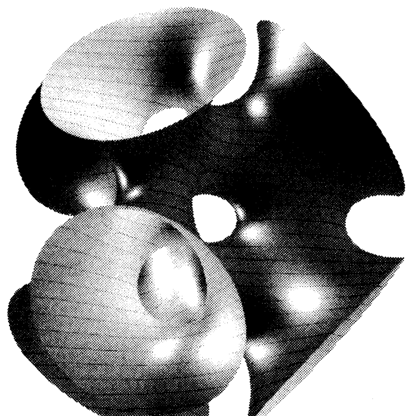

FIGURE 4. A 6-ended genus-2 prismoid in Euclidean space.

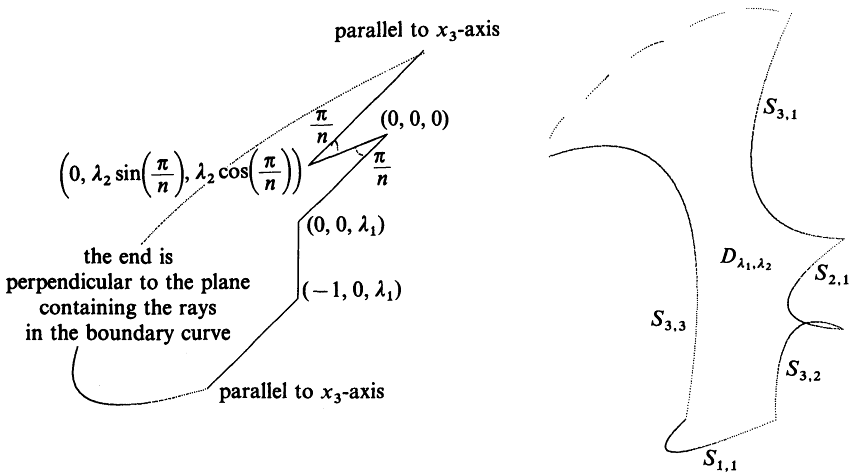

FIGURE 5. The contour $\hat{C}_{\lambda_1,\lambda_2}$ and the fundamental piece D_{λ_1,λ_2} for the prismoids.

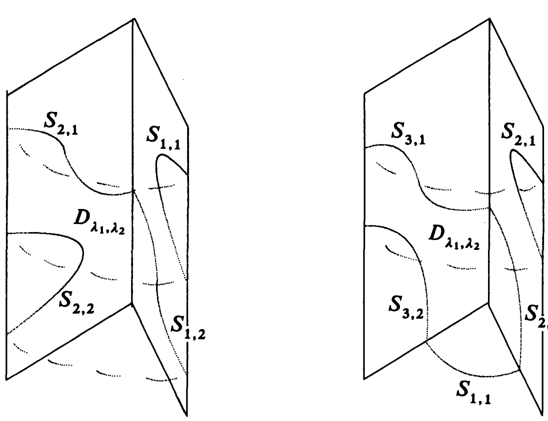

FIGURE 6. The disks for the 3-ended and 4-ended Costa surfaces.

5.7. 4-ended Costa surfaces. We refer to a complete minimal surface of finite total curvature with four parallel ends as a "4-ended Costa surface". Wohlgemuth [W] has constructed several types of 4-ended Costa surfaces, which can have arbitrarily high genus. He has shown that the period problems are non-degenerate for all of his examples. Thus our method applies to all of these examples of Wohlgemuth, producing corresponding CMC-1 surfaces in $H^3(-1)$.

We describe one of the types here (named CSSCFF and CSSCCC by Wohlgemuth). It is a one-parameter family of embedded minimal surfaces of genus $2k-2$ (for any $k \geq 2$), and with four parallel ends. The two outermost ends are always catenoid ends, and the two innermost ends are either catenoid ends (CSSCCC) or planar ends (CSSCFF).

The fundamental piece D_{λ_1,λ_2} for this 4-ended Costa example is shown in Figure 6. The boundary curves $S_{1,1}$, $S_{2,1}$, $S_{2,2}$, $S_{3,1}$, $S_{3,2}$ are planar geodesics. The planes $P_{2,1}$ and $P_{2,2}$ are parallel to the x_1x_3-plane, and $P_{3,1}$ and $P_{3,2}$ are parallel to the x_2x_3-plane. We set $P_{2,1}=\{x_2=c_1\}$, $P_{2,2}=\{x_2=c_2\}$, $P_{3,1}=\{x_1=c_3\}$, and $P_{3,2}=\{x_1=c_4\}$. Then the functions c_1, c_2, c_3, and c_4 are continuous in λ_1 and λ_2, and $f_1=c_2-c_1$ and $f_2=c_4-c_3$ are non-degenerate [W]. Thus the period problems are non-degenerate and by Theorem 5.10, we get the 4-ended Costa cousin.

It seems to be an interesting problem to construct 3-ended Costa surfaces. Unlike the 4-ended Costa cousin, the fundamental piece of the 3-ended Costa cousin contains a straight line in its boundary. Thus, like the case of Enneper's surface, we must use the union of two fundamental disks in our construction. Even though the period problem is non-degenerate on the fundamental disk of a minimal 3-ended Costa surface, this does not imply the period problem is non-degenerate on the union of two fundamental disks. However, existence of a 3-ended Costa cousin has been verified numerically. (Clearly, this CMC-1 surface has less symmetries than the minimal 3-ended Costa surface.)

6. Long time deformations of genus-0 CMC-c surfaces. The examples in the previous section and in Section 7 are obtained through a small perturbation of c, and in general we cannot explicitly determine the range of c such that the corresponding CMC-c surfaces exist. In this section, we find that we can determine the range under a certain situation less general than before. We now assume that we have at most three smooth planar geodesics in the boundaries of the fundamental pieces. So we will use the simpler notation μ_j instead of the notation $\mu_{j,i}$.

6.1. Restricted situation. Consider a complete minimal immersion $f_0: M \to \boldsymbol{R}^3$ of a Riemann surface M. Throughout this section, we assume that f_0 is symmetric with

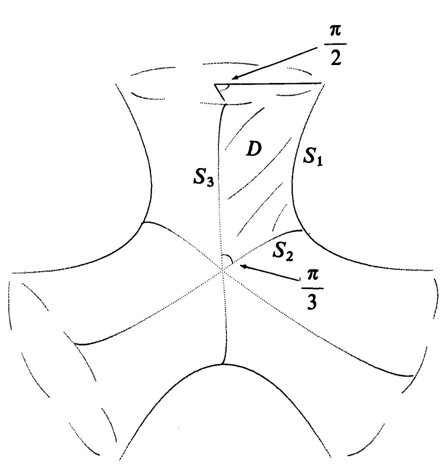

FIGURE 7. Assumption 6.1 ($m=2, n=3$).

respect to a disk $D \subset f(M)$ in the sense of Definition 5.2 and almost embedded in the sense of Definition 5.1. Moreover, we assume the following (cf. Figure 7)

ASSUMPTION 6.1. The boundary ∂D consists of three non-straight planar geodesics S_1, S_2 and S_3 contained in the planes P_1, P_2 and P_3, respectively, and

- S_1 and S_3 are infinite rays, and S_2 is a curve of finite length;
- S_1 and S_2 meet at p_1 with angle π/n, where $n \geq 3$ is an integer;
- S_2 and S_3 meet at p_2 with angle $\pi/2$;
- S_1 and S_2 bound a $1/(2m)$-piece of a catenoid end which is asymptotic to the standard catenoid, where $m \geq 2$ is an integer;
- $1/2 < 1/m + 1/n$.

Here, the *standard catenoid* is the catenoid in $\boldsymbol{R}^3$ obtained from the Gauss map $G_{\mathrm{std}} = z$ and the Hopf differential $Q_{\mathrm{std}} = z^{-2}(dz)^2$.

REMARK 6.2. Since f_0 is almost embedded, the planes P_1, P_2 and P_3 are in general position (cf. Lemma 5.3). Hence we have:

- Each pair of planes among P_1, P_2 and P_3 are not parallel.
- Each pair of lines among $P_1 \cap P_2$, $P_2 \cap P_3$ and $P_3 \cap P_1$ are not parallel.

We have the following theorem.

THEOREM 6.3. *Let $f_0 : M \to \boldsymbol{R}^3$ be a conformal minimal immersion satisfying Assumption* 6.1, *which is almost embedded, and has Gauss map G and Hopf differential Q.*

Then for each $c \neq 0$ satisfying

$$-\frac{1}{4}\left(\left(2-\frac{m}{2}+\frac{m}{n}\right)^2-1\right)<c<0 \quad or \quad 0<c<\frac{1}{4}\left(1-\left(\frac{m}{2}-\frac{m}{n}\right)^2\right), \tag{6.1}$$

there exists an irreducible complete conformal CMC-c *immersion $f_c : M \to H^3(-c^2)$ whose hyperbolic Gauss map and Hopf differential are G and Q, respectively.*

We give a proof in Section 6.3. (About the interval (6.1), see Remark 6.11.)

REMARK 6.4. Let TA denote the total absolute curvature $\int_M (-K)dA$, where dA is the area element with respect to the induced metric ds^2. When c varies over the range (6.1), TA is given by

$$TA = 2\pi[N(\sqrt{1-4c}-1)+2N-2]\,,$$

where N denotes the number of the ends. This is verified as follows: The total absolute curvature is the area with respect to the pseudo-metric $d\sigma_f^2$ in (3.1). Thus, by the Gauss-Bonnet formula, we have $TA/2\pi = \chi(\bar{M}) + \sum_{p \in \bar{M}} \mathrm{ord}_p d\sigma_f^2$. Moreover, by (3.2), $\mathrm{ord}_p d\sigma_f^2 = \mathrm{ord}_p Q$ on M because ds^2 is non-degenerate. Thus,

$$\frac{TA}{2\pi} = \chi(\bar{M}) + \sum_{p \in \bar{M} \setminus M} \mathrm{ord}_p d\sigma_f^2 + \sum_{p \in M} \mathrm{ord}_p Q\,.$$

Since $\bar{M}$ is of genus 0 and Q has poles of order 2 at the ends, the sum of $\mathrm{ord}_p Q$ over $p \in M$ is $2N-4$. On the other hand, by the upcoming equation (6.14), $\mathrm{ord}_p d\sigma_f^2$ is $\lambda-1$, where $\lambda = \sqrt{1-4c}$. Hence, we have the formula. We remark that the total absolute curvature of the corresponding minimal surface in $\boldsymbol{R}^3$ is $4\pi(N-1)$. Thus $TA > 4\pi(N-1)$ (resp. $TA < 4\pi(N-1)$) if $c<0$ (resp. $c>0$).

6.2. Examples. The following two examples have already been constructed in [UY4]. However the method in this section is more explicit and suggests an algorithm to draw the surfaces by numerical calculations.

EXAMPLE 6.5 (Genus-0 Jorge-Meeks surface). Let $f_0 : M \to \boldsymbol{R}^3$ be the Jorge-Meeks n-oid, with each end asymptotic to a standard catenoid. Then the fundamental region D of f_0 satisfies Assumption 6.1, where $m=2$ and n is the number of ends. Then by Theorem 6.3, for each c satisfying $-(n+1)/n^2 < c < 0$ or $0 < c < (n-1)/n^2$, there exists a conformal CMC-c immersion $f_c : M \to H^3(-c^2)$ whose hyperbolic Gauss map and Hopf differential coincide with those of f_0. The total absolute curvature TA of f_c varies over the range $(4\pi(n-2), 4\pi(n-1)) \cup (4\pi(n-1), 4\pi n))$ (cf. Remark 6.4).

EXAMPLE 6.6 (Genus-0 surfaces with Platonic symmetry). There are minimal surfaces with symmetries of the Platonoids. For such minimal surfaces, we can apply Theorem 6.3. In these cases, n is the number of edges of the Platonic solid bounding each face, and m is the number of edges with a common vertex. Table 1 shows the range of c for which we know the minimal surfaces can be deformed to CMC-c surfaces.

TABLE 1.

Symmetry	Number of ends	m	n	Range of c	Range of TA
Tetrahedra	4	3	3	$\left(-\frac{5}{16}, 0\right) \cup \left(0, \frac{3}{16}\right)$	$(8\pi, 12\pi) \cup (12\pi, 16\pi)$
Hexahedra	8	3	4	$\left(-\frac{9}{64}, 0\right) \cup \left(0, \frac{7}{64}\right)$	$(24\pi, 28\pi) \cup (28\pi, 32\pi)$
Octahedra	6	4	3	$\left(-\frac{7}{36}, 0\right) \cup \left(0, \frac{5}{36}\right)$	$(16\pi, 20\pi) \cup (20\pi, 24\pi)$
Dodecahedra	20	3	5	$\left(-\frac{21}{400}, 0\right) \cup \left(0, \frac{19}{400}\right)$	$(72\pi, 76\pi) \cup (76\pi, 80\pi)$
Icosahedra	12	5	3	$\left(-\frac{13}{144}, 0\right) \cup \left(0, \frac{11}{144}\right)$	$(40\pi, 44\pi) \cup (44\pi, 48\pi)$

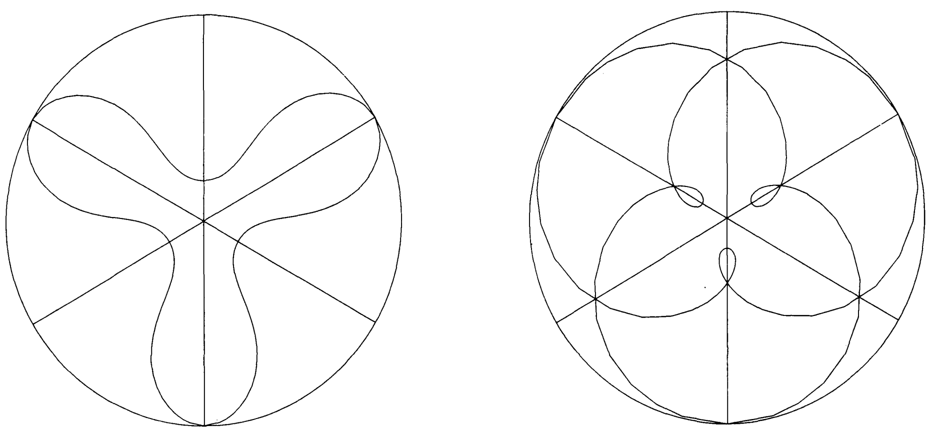

FIGURE 8. Profile curves for the genus-0 Jorge-Meeks cousin ($c>0$ on the left, $c<0$ on the right).

REMARK 6.7. The corresponding minimal surfaces of CMC-c surfaces in Examples 6.5 and 6.6 are non-embedded. Although f_c has self-intersections for $c<0$, it seems to be embedded for some positive c (the total absolute curvature is less than that of the corresponding minimal surface), by our numerical calculations (see Figure 8).

6.3. PROOF OF THEOREM 6.3. Let μ_j be the reflection across the plane P_j containing the geodesic S_j. In the same way as in the previous section, we can induce the reflections $\tilde{\mu}_j$ on the universal cover $\tilde{M}$. Let l be a loop on M surrounding the end that intersects with D, and let $\tau\in\pi_1(M)$ be the deck transformation on $\tilde{M}$ induced by l. Then the reflections satisfy the following relations:

$$\tilde{\mu}_j\circ\tilde{\mu}_j=\mathrm{id}\ (j=1,2,3)\,,\quad (\tilde{\mu}_1\circ\tilde{\mu}_2)^n=\mathrm{id}\,,\quad (\tilde{\mu}_2\circ\tilde{\mu}_3)^2=\mathrm{id}\,,\quad (\tilde{\mu}_1\circ\tilde{\mu}_3)^m=\tau\,. \tag{6.2}$$

The Gauss map G and the Hopf differential Q of f can be lifted to $\tilde{M}$, where we will continue to denote them by G and Q, respectively. Then, by symmetry, there exist matrices $\sigma(\mu_j)\in SU(2)$ $(j=1,2,3)$ such that

$$\overline{G\circ\tilde{\mu}_j}=\sigma(\mu_j)^{-1}*G\,,\quad \overline{Q\circ\tilde{\mu}_j}=Q\qquad (j=1,2,3)\,. \tag{6.3}$$

LEMMA 6.8. *By a suitable choice of the coordinate system of $\boldsymbol{R}^3$, we can choose*

$$\sigma(\mu_1)=\mathrm{id}\,,\quad \sigma(\mu_2)=\begin{pmatrix} e^{\pi i/n} & 0 \\ 0 & e^{-\pi i/n}\end{pmatrix},\quad \sigma(\mu_3)=i\begin{pmatrix} \alpha_0 e^{\pi i/n} & \beta_0 \\ \beta_0 & -\alpha_0 e^{-\pi i/n}\end{pmatrix}, \tag{6.4}$$

where $\alpha_0=\cos(\pi/m)/\sin(\pi/n)\in\boldsymbol{R}$ and $\alpha_0^2+\beta_0^2=1$, $\beta_0\in\boldsymbol{R}$.

PROOF. Take a coordinate system of $\boldsymbol{R}^3$ such that the plane P_1 is the x_1x_3-plane and $P_1\cap P_2$ is the x_3-axis. Since P_1 and P_2 form an angle π/n, $\sigma(\mu_1)$ and $\sigma(\mu_2)$ satisfies (6.4). By $\tilde{\mu}_3\circ\tilde{\mu}_3=\mathrm{id}$ and Lemma 4.5, we have $\sigma(\mu_3)\overline{\sigma(\mu_3)}=\mathrm{id}$. This shows that

$$\sigma(\mu_3)=\begin{pmatrix} a & i\beta_0 \\ i\beta_0 & \bar{a} \end{pmatrix} \qquad (a\bar{a}+\beta_0^2=1,\ \beta_0\in\boldsymbol{R})\,. \tag{6.5}$$

By the relation $(\tilde{\mu}_2\circ\tilde{\mu}_3)^2=\mathrm{id}$, we have $(\sigma(\mu_2)\overline{\sigma(\mu_3)})^2=\pm\mathrm{id}$. If $(\sigma(\mu_2)\overline{\sigma(\mu_3)})^2=\mathrm{id}$, by (6.5) and the form of $\sigma(\mu_2)$, we have $a=\pm e^{-\pi i/n}$ and $\beta_0=0$. This shows that $\sigma(\mu_3)=\sigma(\mu_2)$, and that P_2 and P_3 are parallel. This is impossible by Remark 6.2. Hence

$$(\sigma(\mu_2)\overline{\sigma(\mu_3)})^2=-\mathrm{id}\,, \tag{6.6}$$

and then

$$\sigma(\mu_3)=i\begin{pmatrix} \alpha_0 e^{\pi i/n} & \beta_0 \\ \beta_0 & -\alpha_0 e^{-\pi i/n} \end{pmatrix} \qquad (\alpha_0^2+\beta_0^2=1)\,, \tag{6.7}$$

where α_0 and β_0 are real numbers. Since the rays S_1 and S_3 span a $1/(2m)$-piece of a catenoid end, the angle between the planes P_1 and P_3 is π/m. Hence the eigenvalues of $\sigma(\mu_3)$ are $e^{\pm\pi i/m}$. Thus, we have trace $\sigma(\mu_3)=2\cos(\pi/m)=2\alpha_0\sin(\pi/n)$. □

The metric ds_G^2 defined in Lemma 2.3 is positive definite on M. Hence the set $\mathfrak{D}_M^{(c)}(G, Q)$ is not empty.

PROPOSITION 6.9. *For each c satisfying* (6.1), *there exists a one-parameter family of null holomorphic immersions of* $\tilde{M}$ *into* $SL(2, \boldsymbol{C})$ *with the following properties.*

(1) $f_c=(1/c)F_cF_c^*\in\mathfrak{D}_M^{(c)}(G, Q)$ *for each c.*
(2) F_c *is smooth in c.*
(3) $\lim_{c\to 0}F_c=\mathrm{id}$.
(4) $\lim_{c\to 0}\hat{\rho}_{F_c}(\tilde{\mu}_j)=\sigma(\mu_j)$ $(j=1, 2, 3)$.
(5) $\hat{\rho}_{F_c}(\tilde{\mu}_1)=\mathrm{id}$.
(6) *There exists real-valued continuous functions* α *and* β *in c such that*

$$\hat{\rho}_{F_c}(\tilde{\mu}_2)=\begin{pmatrix} e^{\pi i/n} & 0 \\ 0 & e^{-\pi i/n} \end{pmatrix}, \quad \hat{\rho}_{F_c}(\tilde{\mu}_3)=i\begin{pmatrix} \alpha e^{\pi i/n} & \beta \\ \beta & -\alpha e^{-\pi i/n} \end{pmatrix}\in SU(2)\,,$$

$\alpha^2+\beta^2=1$, *and* $\lim_{c\to 0}\alpha(c)=\alpha_0$.

Once the above proposition is proven, $f_c\in I_M^{(c)}(G, Q)$ by Proposition 4.7. Thus f_c is (after an appropriate 'dilation') the desired CMC-1 immersion as in Theorem 6.3.

PROOF OF PROPOSITION 6.9. In the same way as in Step I of the proof of Proposition 5.6, we can choose $\{\check{F}_c\}$ which satisfies (3) and (5).

The eigenvalues of $\hat{\rho}_{\check{F}_c}(\tilde{\mu}_2)$ are $\{e^{\pi i/n}, e^{-\pi i/n}\}$. Indeed, since $(\overline{\sigma(\mu_1)}\sigma(\mu_2))^n=-\mathrm{id}$ and $\hat{\rho}_{\check{F}_c}(\tilde{\mu}_1)=\mathrm{id}$, we have $\hat{\rho}_{\check{F}_c}(\tilde{\mu}_2)^n=-\mathrm{id}$. Then the eigenvalues of $\hat{\rho}_{\check{F}_c}(\tilde{\mu}_2)$ are n-th roots of -1. Then, by the continuity of $\hat{\rho}_{\check{F}_c}(\tilde{\mu}_2)$, they must be constant. Here, $\lim_{c\to 0}\hat{\rho}_{\check{F}_c}(\tilde{\mu}_2)=\sigma(\mu_2)$, so the eigenvalues of $\hat{\rho}_{\check{F}_c}(\tilde{\mu}_2)$ coincide with those of $\sigma(\mu_2)$. In particular, the eigenvalues of $\hat{\rho}_{\check{F}_c}(\tilde{\mu}_2)$ are not real. Then, by the same method as in Step II of the proof of Proposition

5.6, we can diagonalize it with a real matrix $u=u(c)$:

$$u(c)^{-1}\hat{\rho}_{\check{F}_c}(\tilde{\mu}_2)u(c)=\begin{pmatrix} e^{\pi i/n} & 0 \\ 0 & e^{-\pi i/n} \end{pmatrix}.$$

Setting $\hat{F}_c=\check{F}_c u(c)$, we obtain a family $\{\hat{F}_c\}$ satisfying (3), (5) and (6). Moreover, by using $(\tilde{\mu}_2\circ\tilde{\mu}_3)^2=\mathrm{id}$ and (6.6), $(\hat{\rho}_{\hat{F}_c}(\tilde{\mu}_2)\overline{\hat{\rho}_{\hat{F}_c}(\tilde{\mu}_3)})^2=-\mathrm{id}$ holds and then $\hat{\rho}_{\hat{F}_c}(\tilde{\mu}_3)$ can be written in the form

$$\hat{\rho}_{\hat{F}_c}(\tilde{\mu}_3)=i\begin{pmatrix} e^{\pi i/n}\alpha & \beta_1 \\ \beta_2 & -e^{-\pi i/n}\alpha \end{pmatrix} \qquad (\alpha, \beta_1, \beta_2\in\boldsymbol{R}, \alpha^2+\beta_1\beta_2=1)\,. \tag{6.8}$$

When c is sufficiently small, we can prove the proposition by applying the same argument as in Step III of the proof of Proposition 5.6, because of $\beta_1/\beta_2>0$. So it is sufficient to show that $\beta_1/\beta_2>0$ for any c satisfying (6.1). Thus the proof of the proposition reduces to the following lemma. □

LEMMA 6.10. *If c satisfies* (6.1), *then* $\beta_1/\beta_2>0$.

PROOF. Since $\sigma(\mu_3)$ has plus-minus ambiguity, we have two choices for $\sigma(\mu_3)$. We choose $\sigma(\mu_3)$ so that the eigenvalues of $\sigma(\mu_3)$ are $e^{\pm\pi i/m}$, and then

$$(\sigma(\mu_1)\overline{\sigma(\mu_3)})^m=\sigma(\mu_3)^m=-\mathrm{id}\,.$$

By the last relation in (6.2), we have

$$\rho_{F_c}(\tau)=-(\hat{\rho}_{F_c}(\tilde{\mu}_1)\overline{\hat{\rho}_{F_c}(\tilde{\mu}_3)})^m=-\overline{\hat{\rho}_{F_c}(\tilde{\mu}_3)}^m\,, \tag{6.9}$$

where τ is the deck transformation induced from the loop l surrounding the catenoid end.

The eigenvalues of $\hat{\rho}_{F_c}(\tilde{\mu}_3)$ are $\xi_\pm=s\pm i\sqrt{1-s^2}$, where $s=\alpha\sin(\pi/n)$. Thus, if

$$|\alpha|<\frac{1}{\sin(\pi/n)}\,, \tag{6.10}$$

$\xi_\pm$ are complex numbers of absolute value 1, and then, there exists $\theta\in(0,\pi)$ such that $\xi_\pm=e^{\pm i\theta}$ and $\theta=\cos^{-1}(\alpha\sin(\pi/n))$. Since $\alpha(0)=\alpha_0$ satisfies (6.10), there exists an interval $I_0\subset\boldsymbol{R}$ which contains the origin and (6.10) holds for each $c\in I_0$. We now restrict $c\in I_0$ in order to make $|\xi_k|=1$. Later, we restrict the range of c again to make $|\alpha(c)|<1$. Hence, for each $c\in I_0$, it holds that

$$\mathrm{trace}\,\rho_{F_c}(\tau)=-\overline{\mathrm{trace}(\hat{\rho}_{F_c}(\tilde{\mu}_3))^m}=-(e^{-im\theta}+e^{im\theta})=-2\cos m\theta\,. \tag{6.11}$$

Let $\bar{M}$ be the compactification of M and $p\in\bar{M}$ a point corresponding to an end. Since the end is asymptotic to a standard catenoid, the Weierstrass data are also asymptotic to those of a catenoid. Hence there exists a coordinate z of $\bar{M}$ such that $z(p)=0$ and (G, Q) are expanded as

$$G = z + \cdots, \quad Q = \left(\frac{1}{z^2} + \cdots\right)(dz)^2. \tag{6.12}$$

Let g_c be the secondary Gauss map of F_c. Then, by (3.3), we have

$$g \circ \tau = \rho_{F_c}(\tau) * g = \frac{a_{11}g + a_{12}}{a_{21}g + a_{22}} \quad \text{where} \quad \rho_{F_c}(\tau) = (a_{kj}). \tag{6.13}$$

So, by (2.8) and the fact that the Schwarzian derivative is invariant under Möbius transformations, there exists a matrix $b \in SL(2, \boldsymbol{C})$ such that

$$b * g = z^\lambda(g_0 + g_1 z + g_2 z^2 + \cdots), \qquad g_0 \neq 0, \tag{6.14}$$

where $\lambda = \sqrt{1-4c}$. Hence, by (6.13), we have

$$b^{-1}\rho_{F_c}(\tau) b = \pm\begin{pmatrix} e^{\pi\lambda i} & 0 \\ 0 & e^{-\pi\lambda i}\end{pmatrix};$$

that is, the eigenvalues of $\rho_{F_c}(\tau)$ are $\{\pm e^{\pi\lambda i}, \pm e^{-\pi\lambda i}\}$. So, for each $c < 1/4$,

$$\operatorname{trace} \rho_{F_c}(\tau) = \pm 2\cos \pi\lambda \tag{6.15}$$

holds. Combining (6.11) and (6.15), we have $\pm\cos\pi\lambda = -\cos m\theta$. Letting $c \to 0$, θ tends to $\theta(0) = \pi/m$ and $\lambda \to 1$. Thus the sign on the left-hand side is "minus":

$$\cos\pi\lambda = \cos m\theta. \tag{6.16}$$

Now we assume c satisfies (6.1), that is, $c \in I$, where $I \setminus \{0\}$ is the open interval

$$I := \left(-\frac{1}{4}\left\{\left(2 - \frac{m}{2} + \frac{m}{n}\right)^2 - 1\right\}, \frac{1}{4}\left\{1 - \left(\frac{m}{2} - \frac{m}{n}\right)^2\right\}\right). \tag{6.17}$$

So, if $c \in I$, we have

$$\pi m((1/2) - (1/n)) < m\theta < 2\pi - \pi m((1/2) - (1/n)),$$

because of (6.16).

Here, $\theta = \theta(c)$ is a continuous function, and $m\theta(0) = \pi$, and $m\theta(I)$ is an interval containing π. Thus, we have

$$\theta(I) \subset \left(\pi\left(\frac{1}{2} - \frac{1}{n}\right), \pi\left(\frac{2}{m} - \frac{1}{2} + \frac{1}{n}\right)\right) \subset (0, \pi).$$

On $(0, \pi)$, $\cos\theta$ is a decreasing function. Thus

$$\sin\frac{\pi}{n} = \cos\left(\pi\left(\frac{1}{2} - \frac{1}{n}\right)\right) > \cos\theta(c) > \cos\left(\pi\left(\frac{2}{m} - \frac{1}{2} + \frac{1}{n}\right)\right) \geq -\sin\frac{\pi}{n}.$$

This shows

$$-\sin(\pi/n) < \cos\theta(c) = \alpha\sin(\pi/n) < \sin(\pi/n).$$

Therefore, $|\alpha(c)| < 1$ if $c \in I$. For such a value of c, $\beta_1\beta_2 = 1 - \alpha^2 > 0$. □

REMARK 6.11. The interval (6.1) in Theorem 6.3 gives only a sufficient condition for the existence of deformation. However, in the case of the genus 0 Jorge-Meeks surfaces, one can determine a necessary and sufficient condition on c so that there exists a corresponding CMC-c surface. Indeed, for the Jorge-Meeks n-oid $f_0 : M \to \boldsymbol{R}^3$ as in Example 6.5, there exists a corresponding CMC-c immersion into $H^3(-c^2)$ *if and only if* $c \in I_{0,+}$, $I_{0,-}$ or I_k $(k = 1, 2, \dots)$, where

$$\text{(6.18)}\quad \begin{aligned} &I_{0,+} = \left(0, \frac{n-1}{n^2}\right), \quad I_{0,-} = \left(-\frac{n+1}{n^2}, 0\right) \\ &I_k = \left(-\left(k+\frac{1}{n}\right)\left(k+1+\frac{1}{n}\right), \ \left(k-\frac{1}{n}\right)\left(k+1-\frac{1}{n}\right)\right) \qquad (k = 1, 2, \dots). \end{aligned}$$

Since $m = 2$ in this case, the condition (6.16) can be rewritten as $\cos\pi\lambda = \cos 2\theta = 2\cos^2\theta - 1$, and recalling that $\cos\theta = \alpha\sin(\pi/n)$ (see the first part in the proof of Lemma 6.10), we have $\cos\pi\lambda = 2\alpha^2\sin^2(\pi/n) - 1$, and then, $\alpha^2 = (\cos\pi\lambda + 1)/(2\sin^2(\pi/n))$ holds. Here, a corresponding CMC-c immersion exists if and only if the conclusion of Lemma 6.10, i.e., $\beta_1\beta_2 = 1 - \alpha^2 > 0$ holds. (Otherwise, one cannot find f_c with $SU(2)$-condition.) Using the above relation, this necessary and sufficient condition is rewritten as $\cos\pi\lambda + 1 < 2\sin^2(\pi/n)$, where $\lambda = \sqrt{1-4c}$. This inequality holds if and only if (6.18) holds.

By the same argument as in Remark 6.4, the total absolute curvature for f_c $(c \in I_k)$ varies over the interval $(4\pi\{n(k+1) - 2\}, 4\pi n(k+1))$. Numerical investigations suggest that the surface is embedded for $c \in I_{0,+}$ sufficiently near $(n-1)/n^2$.

7. Periodic CMC-c **surfaces.** The construction in Section 5 depends on the properties of minimal surfaces in $\boldsymbol{R}^3$. However, our method can sometimes be applied even if the corresponding minimal surfaces do not exist. We believe that the following example is of interest, because our construction method here is independent of the corresponding minimal surfaces.

In this section, we shall construct singly-periodic CMC-c surfaces in $H^3(-c^2)$. The corresponding minimial surfaces are well-known as the "catenoid fence" and "Jorge-Meeks n-oid fence" (cf. [Kar], [R], Figure 9). Although, in the following argument, we borrow geometric intuition from the corresponding minimal surfaces, our construction will not depend on them. Thus, by letting $c \to 0$, we have another way to construct the minimal catenoid (resp. n-oid) fence.

We denote the shaded region in Figure 9 by M_0. It is the fundamental region of the catenoid fence, that is, the whole surface is obtained by reflections of M_0 about its boundary curves. In the CMC-c case, we define the underlying Riemann surface and

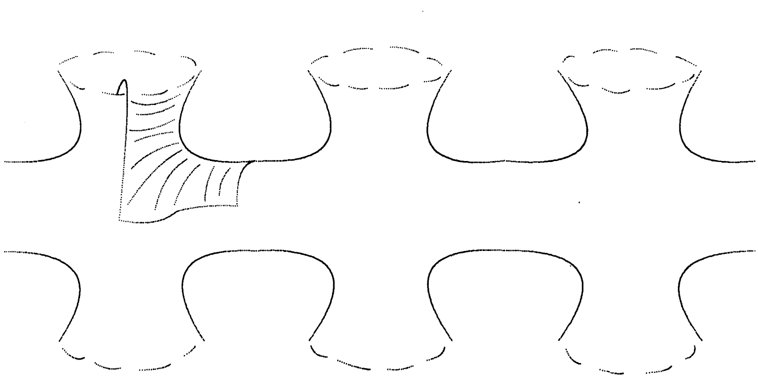

FIGURE 9. The catenoid fence and its fundamental disk.

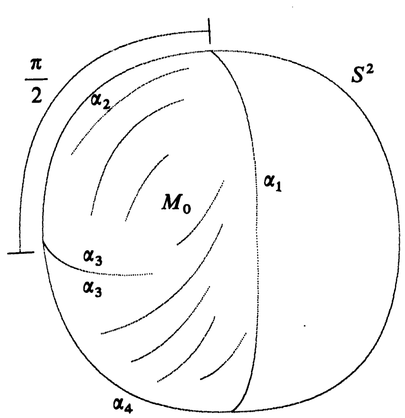

FIGURE 10. The image of M_0 by G_0.

the data g and Q by modifying M_0. Let G_0 and Q_0 be the Gauss map and the Hopf differential of the minimal surface M_0. We consider G_0 as a map to the unit sphere S^2. So the image of G_0 in S^2 looks like Figure 10, where each α_j is the image of a planar geodesic in the boundary of M_0. We identify M_0 with the image of M_0 by G_0.

REMARK 7.1. If the "catenoid cousin fence" exists, its hyperbolic Gauss map is not singly periodic. Indeed, if G is singly periodic, any fundamental pieces coincide mutually completely (not only are congruent). This contradiction implies that G cannot be singly periodic.

For each number $\delta \in (-\pi/2, \pi/2)$, we define a region $M_{\delta,0}$ on S^2 as in Figure 11, where α_1 and α_3 are segments of great circles through P, α_4 is a segment of a great circle centered at P, and α_2 is one-fourth of a *smaller* circle centered at P with radius $\pi/2+\delta$.

The boundary curves of $M_{\delta,0}$ are circles, so we reflect $M_{\delta,0}$ infinitely often by

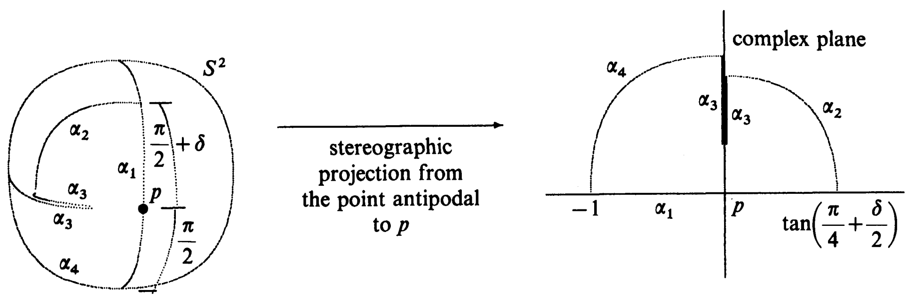

FIGURE 11. Modified Fundamental Region $M_{\delta,0}$.

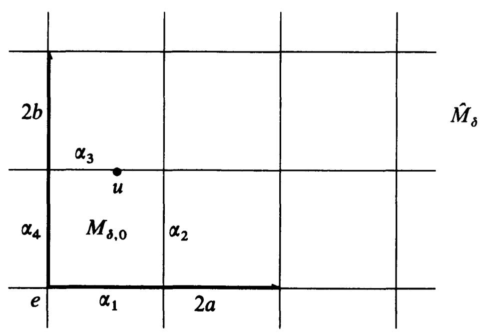

FIGURE 12. The Riemann surface M_δ.

inversions with respect to the boundaries of it and of its copies. Then we get an abstract Riemann surface $\hat{M}_\delta$ which is biholomorphic to $\boldsymbol{C}$. We identify $\hat{M}_\delta$ with $\boldsymbol{C}$ and take $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\boldsymbol{C}$ as in Figure 12. We denote

$$\begin{aligned} E &= e + 2\boldsymbol{Z}\boldsymbol{a} + 2\boldsymbol{Z}\boldsymbol{b}\,, \\ U &= (u + 2\boldsymbol{Z}\boldsymbol{a} + 2\boldsymbol{Z}\boldsymbol{b}) \cup (-u + 2\boldsymbol{Z}\boldsymbol{a} + 2\boldsymbol{Z}\boldsymbol{b})\,, \end{aligned} \tag{7.1}$$

and define $M_\delta = \boldsymbol{C} \setminus E$. We need to find a $2\boldsymbol{b}$-periodic CMC-c immersion of a cylinder $f\colon M_\delta/(2\boldsymbol{Z}\boldsymbol{a}) \to H^3(-c^2)$ which has ends at E and umbilic points at U.

Let $G_{\delta,0}\colon M_{\delta,0} \to M_{\delta,0} \subset S^2$ be the *identity map*. Since the boundary curves of $M_{\delta,0}$ are circles, $G_{\delta,0}$ can be extended to the holomorphic map $G_\delta\colon \boldsymbol{C} = \hat{M}_\delta \to S^2 = \boldsymbol{C} \cup \{\infty\}$.

Let μ_j be a reflection of the universal cover $\tilde{M}_\delta$ of M_δ, which is the lift of the reflection about α_j. By the definition of G_δ, it satisfies $\overline{G_\delta \circ \tilde{\mu}_j} = \sigma(\mu_j)^{-1} * G_\delta$, where

$$\begin{aligned} &\sigma(\mu_1) = \mathrm{id}\,, && \sigma(\mu_2) = \begin{pmatrix} 0 & i\tan((\pi/4)+(\delta/2)) \\ i\cot((\pi/4)+(\delta/2)) & 0 \end{pmatrix}, \\ &\sigma(\mu_3) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, && \sigma(\mu_4) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \end{aligned} \tag{7.2}$$

(see Figure 12). We give the following holomorphic 2-differential Q_δ as a Hopf differential on M_δ.

LEMMA 7.2. *There exists a meromorphic 2-differential Q_δ which has the following properties.*

(1) *Q_δ is doubly periodic with respect to the periods generated by $2\boldsymbol{a}$ and $2\boldsymbol{b}$.*
(2) *Q_δ has poles of order 2 at E, and is regular on $\boldsymbol{C}\setminus E$.*
(3) *Q_δ has zeroes of order 1 at U, and is non-zero on $\boldsymbol{C}\setminus U$.*
(4) $\overline{Q_\delta \circ \tilde{\mu}_j} = Q_\delta$.

Moreover, Q_δ is unique up to a real constant factor.

PROOF. If we set $Q_\delta = q_\delta(z)dz^2$, then the proof reduces to finding a suitable elliptic function $q_\delta(z)$, which can be done by using elementary elliptic function theory. □

Hence, we can choose G_δ and Q_δ continuously on δ so that $ds_\delta^2 = (1+|G_\delta|^2)^2 \omega_\delta \bar{\omega}_\delta$ is positive definite, where $\omega_\delta = Q_\delta / dG_\delta$.

THEOREM 7.3. *For a sufficiently small c, there exist a number δ and a $2\boldsymbol{b}$-periodic complete conformal* CMC-*c immersion $f: M_\delta/(2\boldsymbol{Z}\boldsymbol{a}) \to H^3(-c^2)$ which has hyperbolic Gauss map G_δ and Hopf differential Q_δ.*

By the same argument as in the proof of Proposition 5.6, we have the following:

LEMMA 7.4. *There exists a two-parameter family $\{F_{c,\delta}\}_{|c|<\varepsilon, |\delta|<\delta_0}$ of null holomorphic immersion into $SL(2, \boldsymbol{C})$ with the following properties*:

(1) $f_{c,\delta} = (1/c) F_{c,\delta} F_{c,\delta}^* \in \mathfrak{D}^{(c)}_{M_\delta / 2\boldsymbol{Z}\boldsymbol{a}}(G_\delta, Q_\delta)$ *for each* (c, δ), *where* $c \neq 0$.
(2) *$F_{c,\delta}$ is smooth in c and continuous in δ.*
(3) $\lim_{c\to 0} F_{c,\delta} = \mathrm{id}$.
(4) $\hat{\rho}_{F_{c,\delta}}(\tilde{\mu}_1) = \mathrm{id}$, *and*

$$\hat{\rho}_{F_{c,\delta}}(\tilde{\mu}_3) = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}, \quad \hat{\rho}_{F_{c,\delta}}(\tilde{\mu}_4) = \begin{pmatrix} p & i\beta \\ i\beta & \bar{p} \end{pmatrix}, \quad \hat{\rho}_{F_{c,\delta}}(\tilde{\mu}_2) = \begin{pmatrix} q & i\gamma_1 \\ i\gamma_2 & -\bar{q} \end{pmatrix},$$

where $\xi = \xi(c, \delta)$, $p = p(c, \delta)$ and $q = q(c, \delta)$ are complex-valued functions such that $|\xi| = 1$, and β, γ_1 and γ_2 are real-valued functions of c and δ. They satisfy

$$p(0,\delta)=0\,, \quad q(0,\delta)=0\,, \quad \beta(0,\delta)=1\,, \quad \xi(0,\delta)=i\,,$$

$$\gamma_1(0,\delta)=\tan\left(\frac{\pi}{4}+\frac{\delta}{2}\right), \quad \gamma_2(0,\delta)=\cot\left(\frac{\pi}{4}+\frac{\delta}{2}\right).$$

PROOF OF THEOREM 7.3. In Lemma 7.4, let $\phi(c, \delta) = \gamma_1(c, \delta)/\gamma_2(c, \delta)$. So $\phi(0, \delta) > 1$ if $\delta > 0$, and $\phi(0, \delta) < 1$ if $\delta < 0$. Then, by the continuity of γ_j, there exist δ_+ and δ_- such that $\phi(c, \delta_+) > 1$ and $\phi(c, \delta_-) < 1$ hold for a sufficiently small c. Hence there exists δ such that $\phi(c, \delta) = 1$. For such c and δ, all matrices in Lemma 7.4 are in $SU(2)$. □

REMARK 7.5. The same argument can be applied to the Jorge-Meeks n-oid fence (see Section 4 in [R]), and this produces a Jorge-Meeks n-oid fence cousin.

8. Appendix. Let Γ be a subgroup of $SU(2)$. In this appendix, we prove a property of a set of groups conjugate to Γ in $SL(2, C)$ defined by

$$C_\Gamma := \{\sigma \in SL(2, C);\ \sigma\Gamma\sigma^{-1} \subset SU(2)\}.$$

The authors wish to thank Hiroyuki Tasaki for valuable comments on the first draft of the appendix.

If $\sigma \in C_\Gamma$, it is obious that $a\sigma \in C_\Gamma$ for all $a \in SU(2)$. So if we consider the quotient space

$$I_\Gamma := C_\Gamma / SU(2),$$

the structure of the set C_Γ is completely determined. Define a map $\tilde{\phi}: C_\Gamma \to \mathscr{H}^3$ by

$$\tilde{\phi}(\sigma) := \sigma^*\sigma,$$

where $\mathscr{H}^3$ is the hyperbolic 3-space defined by $\mathscr{H}^3 := \{aa^*;\ a \in SL(2, C)\}$. Then it induces an injective map $\phi: I_\Gamma \to \mathscr{H}^3$ such that $\phi \circ \pi = \tilde{\phi}$, where $\pi: C_\Gamma \to I_\Gamma$ is the canonical projection. So we can identify I_Γ with a subset $\tilde{\phi}(I_\Gamma) = \tilde{\phi}(C_\Gamma)$ of the hyperbolic 3-space $\mathscr{H}^3$. The following assertion holds.

LEMMA. *The subset $\tilde{\phi}(I_\Gamma)$ is a point, a geodesic line, or all $\mathscr{H}^3$.*

REMARK. Theorem 3.2 in Section 3 is directly obtained if we set $\Gamma := \rho(\pi_1(M))$, where ρ is the representation defined in Section 3. Indeed, the set $I_M^{(c)}(G, Q)$ defined in Section 3 coincides with the set $\tilde{\phi}(I_{\rho(\pi_1(M))})$.

PROOF. For each $\gamma \in \Gamma$, we set

$$C_\gamma := \{c \in SL(2, C);\ \sigma\gamma\sigma^{-1} \in SU(2)\}.$$

Then we have

$$C_\Gamma := \bigcap_{\gamma \in \Gamma} C_\gamma. \tag{8.1}$$

The condition $\sigma\gamma\sigma^{-1} \in SU(2)$ is rewritten as $\sigma^*\sigma\gamma = \gamma\sigma^*\sigma$. So we have

$$\tilde{\phi}(C_\gamma) = \mathscr{H}^3 \cap Z_\gamma, \tag{8.2}$$

where Z_γ is the center of $\gamma \in \Gamma$.

Assume $\gamma \neq \pm\mathrm{id}$. If γ is a diagonal matrix, it can easily be checked that Z_γ consists of diagonal matrices in $SL(2, C)$. Since any $\gamma \in \Gamma$ can be diagonalized by a matrix in $SU(2)$, we have $Z_\gamma = \{\exp(zT);\ z \in C\}$, where $T \in \mathfrak{su}(2)$ is chosen so that $\gamma = \exp(T)$. ($\mathfrak{su}(2)$ is the Lie algebra of $SU(2)$.) Hence we have

$$\tilde{\phi}(C_\gamma) = \mathscr{H}^3 \cap Z_\gamma = \exp(i\boldsymbol{R}T), \tag{8.3}$$

because $\exp(i\mathfrak{su}(2)) = \mathscr{H}^3$.

Suppose now that Γ is not abelian. Then there exist $\gamma, \gamma' \in \Gamma$ such that $\gamma\gamma' \neq \gamma'\gamma$.

Set $\gamma=\exp(T)$ and $\gamma'=\exp(T')$, where $T, T' \in \mathfrak{su}(2)$. Then we have $i\boldsymbol{R}T \cap i\boldsymbol{R}T'=\{0\}$. It is well-known that the restriction of the exponential map $\exp|_{i\mathfrak{su}(2)}: i\mathfrak{su}(2) \to \mathscr{H}^3$ is bijective. Hence we have

$$\tilde{\phi}(C_\gamma) \cap \tilde{\phi}(C_{\gamma'})=\exp(i\boldsymbol{R}T) \cap \exp(i\boldsymbol{R}T')=\{\mathrm{id}\}\ .$$

By (8.1), (8.2) and (8.3), we have

$$\tilde{\phi}(I_\Gamma)=\{\mathrm{id}\} \qquad (\text{if } \Gamma \text{ is not abelian})\ .$$

Next we consider the case Γ is abelian. If $\Gamma \subset \{\pm\mathrm{id}\}$, then obviously

$$\tilde{\phi}(I_\Gamma)=\mathscr{H}^3 \qquad (\text{if } \Gamma \subset \{\pm\mathrm{id}\})\ .$$

Suppose $\Gamma \not\subset \{\pm\mathrm{id}\}$. Then there exists $\gamma \in \Gamma$ such that $\gamma \neq \pm\mathrm{id}$. We set $\gamma=\exp T\ (T \in \mathfrak{su}(2))$. Since $\exp(\boldsymbol{R}T)$ is a maximal abelian subgroup containing γ, we have $\Gamma \subset \exp(\boldsymbol{R}T)$. Then by (8.3), we have

$$\tilde{\phi}(I_\Gamma)=\exp(i\boldsymbol{R}T) \qquad (\text{if } \Gamma \not\subset \{\pm\mathrm{id}\} \text{ is abelian})\ .$$

$\square$

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