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ON THE COMPLEXITY OF SMOOTH PROJECTIVE TORIC VARIETIES

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Abstract. In this paper we answer a question posed by Batyrev, which asks if there exists a complete regular fan with more than quadratically many primitive collections. We construct a smooth projective toric variety associated to a complete regular d-dimensional fan with n generators where the number of primitive collections is at least exponential in $n-d$. We also exhibit the connection between the number of primitive collections and the facet complexity of the Gröbner fan of the associated integer program.

1. Introduction. In this paper we give an affirmative answer to the following question posed by Batyrev [Bat]:

QUESTION. *Does there exist a complete regular d-dimensional fan Δ with n generators such that Δ has more than $(n-d-1)(n-d+2)/2$ primitive collections for $n-d>1$?*

In Section 2 we prove the following theorem which answers the above question.

THEOREM 1.1. *There exists a complete regular d-dimensional fan Δ with n generators where the number of primitive collections of Δ is more than $2^{(n-d)/2}$.*

A fan $\Delta\subset\boldsymbol{R}^d$ that covers $\boldsymbol{R}^d$ is a complete fan. If we require the full-dimensional cones in Δ to be simplicial with integral generators which form a $\boldsymbol{Z}$-basis for $\boldsymbol{Z}^d$, then Δ is said to be *regular* (see below for formal definitions). If a fan $\Delta\subset\boldsymbol{R}^d$ is generated by the one-dimensional cones defined by the vectors in $\mathscr{A}=\{a_1,a_2,\dots,a_n\}\subset\boldsymbol{Z}^d$, then the *primitive collections* of Δ are defined as follows:

DEFINITION 1.2. A nonempty subset $\mathscr{P}=\{a_{i_1},a_{i_2},\dots,a_{i_k}\}$ of $\mathscr{A}$ is called a *primitive collection* if for each generator $a_{i_s}\in\mathscr{P}$ the elements $\mathscr{P}\setminus a_{i_s}$ generate a $(k-1)$-dimensional cone in Δ, while $\mathscr{P}$ does not generate a k-dimensional cone in Δ.

The primitive collections of a complete regular fan $\Delta\subset\boldsymbol{R}^d$ with n generators are studied in [Bat] to classify d-dimensional smooth complete toric varieties with $n-d=3$. The question we study asks whether there exists a complete regular fan Δ where the number of primitive collections of Δ is at least *quadratic* in $n-d$. Theorem 1.1 constructs a complete regular fan with *exponentially* many primitive collections. The same theorem can be restated in the language of Gröbner bases of toric varieties: Theorem 3.1 shows that there exists a toric variety X with a square-free initial ideal whose number of

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minimal generators is exponential in the codimension of X. In the last section we make a connection between two conjectures: one of them appears in the context of the complexity of complete regular fans (Conjecture 7.1 in [Bat]) and the other one is about the complexity of *Gröbner fans* in the context of *integer programming* (Conjecture 6.1 in [ST]).

In this article we will use results from the theory of *coherent triangulations* of a vector configuration. In order to prove Theorem 1.1 we need the following definitions which connect coherent triangulations and complete regular projective fans: A complete fan $\Delta \in \boldsymbol{R}^d$ with n generators $\mathscr{A} = \{a_1, a_2, \ldots, a_n\} \subset \boldsymbol{Z}^d$ is said to be *regular* if every d-dimensional cone $\sigma \in \Delta$ is simplicial and the generators $\{a_{i_1}, a_{i_2}, \ldots, a_{i_d}\}$ of σ form a $\boldsymbol{Z}$-basis of $\boldsymbol{Z}^d$.

DEFINITION 1.3. A complete regular fan Δ is said to be *projective* if there exists a *support function* $\phi: \boldsymbol{R}^d \to \boldsymbol{R}$ such that

1. ϕ is convex and $\phi(\boldsymbol{Z}^d) \subset \boldsymbol{Z}$,
2. ϕ is linear on each cone of Δ with $\phi|_\sigma \neq \phi|_\tau$ for distinct d-dimensional cones σ and τ.

It is a well-known fact that if $V(\Delta)$ is the smooth complete d-dimensional toric variety that is associated with Δ then $V(\Delta)$ is projective if and only if Δ is projective (cf. [Oda]). In this paper we will use an equivalent definition of a projective toric variety via *coherent* triangulations.

DEFINITION 1.4. A triangulation T of a vector configuration $\mathscr{A} = \{a_1, a_2, \ldots, a_n\} \in \boldsymbol{R}^d$ is a polyhedral complex consisting of simplical cones which cover $\operatorname{pos}(\mathscr{A}) = \{x \in \boldsymbol{R}^d : x = \sum_{i=1}^n \lambda_i a_i, \lambda_i \geq 0\}$. A triangulation T of $\mathscr{A}$ is said to be *coherent* if there exists a support function ϕ on T as in Definition 1.3 (see [BFS], [GKZ]).

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2. Exponential lower bound. To give an exponential lower bound for the number of primitive collections of a complete regular fan we will use the example given in Proposition 6.7 of [ST].

DEFINITION 2.1. Given a matrix $B \in \boldsymbol{Z}^{d \times n}$, the *chamber complex* $\Gamma(B)$ of B is the coarsest polyhedral complex that refines all triangulations of B and covers $\operatorname{pos}(B)$.

PROPOSITION 2.2 ([BGS]). *Let A be a Gale transform of $B \in \boldsymbol{Z}^{d \times n}$, i.e. let $A \in \boldsymbol{Z}^{(n-d) \times n}$ such that*

$$0 \longrightarrow \boldsymbol{R}^d \xrightarrow{B^T} \boldsymbol{R}^n \xrightarrow{A} \boldsymbol{R}^{(n-d)} \longrightarrow 0$$

is exact. Then there is a bijection between the coherent triangulations of A and the d-dimensional chambers of $\Gamma(B)$ given by

$$T=\bigcup_{i=1}^{t}\sigma_i \Longleftrightarrow C=\bigcap_{i=1}^{t}\sigma_i^*$$

where $\sigma_i=\mathrm{pos}(a_{i_1}, a_{i_2}, \ldots, a_{i_{n-d}})$ are the cones of the coherent triangulation T and $\sigma_i^*=\mathrm{pos}(\{b_j\colon 1\le j\le n, j\ne i_1, i_2, \ldots, i_{n-d}\})$ are the cones of B containing the chamber C.

Now we construct a complete regular projective fan which has exponentially many primitive collections. Let B be the node-edge incidence matrix of the complete bipartite graph $K_{n,m}$ where $n=2k-1$ and $m=2k+1$. $B=\{e_i\times e_j'\colon 1\le i\le n, 1\le j\le m\}$ where $e_i\in \boldsymbol{R}^n$ and $e_j'\in \boldsymbol{R}^m$ are standard basis vectors. B has rank $n+m-1$ and is unimodular, i.e. any subdeterminant of B is 0 or ± 1 (cf. [Schr, p. 273]). The cone $\mathrm{pos}(B)$ consists of all non-negative vectors $(u_1, \ldots, u_n)\times(v_1, \ldots, v_m)$ such that $u_1+\cdots+u_n=v_1+\cdots+v_m$. Let $A\in \boldsymbol{Z}^{(n-1)(m-1)\times nm}$ be a Gale transform of B. By Proposition 2.2, for every chamber in the chamber complex $\Gamma(B)$ there exists a corresponding coherent triangulation of A. We consider a special chamber in $\Gamma(B)$. The one-dimensional cone generated by $(1/n, \ldots, 1/n)\times(1/m, \ldots, 1/m)$ is in $\mathrm{pos}(B)$ and we claim that it is in the interior of a full-dimensional chamber. The facets of full-dimensional chambers of $\Gamma(B)$ correspond to the cocircuits of the oriented matroid of B (cf. [DHSS, Lemma 2.7]). In our situation, a cocircuit of B corresponds to a cut $(C_+, C_-; D_+, D_-)$ in $K_{n,m}$ where (C_+, C_-) is a partition of $\{1, \ldots, n\}$ and (D_+, D_-) is a partition of $\{1, \ldots, m\}$. The corresponding hyperplane is defined by

$$\sum_{i\in C_+} u_i-\sum_{i\in C_-} u_i-\sum_{j\in D_+} v_j+\sum_{j\in D_-} v_j=0\,. \tag{1}$$

Since n and m are relatively prime, $(1/n, \ldots, 1/n)\times(1/m, \ldots, 1/m)$ cannot lie on any of these hyperplanes. So it must be in the interior of a full-dimensional chamber. This chamber is called the *central chamber* of $\mathrm{pos}(B)$. Let Δ be the corresponding coherent triangulation of A. Since B is unimodular, so is A, and therefore Δ consists of simplical cones whose generators form a $\boldsymbol{Z}$-basis for $\boldsymbol{Z}^{(n-1)(m-1)}$. Because there exists a strictly positive vector in $\mathrm{im}(B^T)=\ker(A)$, any vector in $\boldsymbol{R}^{(n-1)(m-1)}$ is in $\mathrm{pos}(A)$. This shows that Δ is a complete regular fan.

Now we show that every column of A is a generator of Δ. By Proposition 2.2 it is enough to show that for every column $b_{i,j}=e_i\times e_j'\in B$ there exists a cone τ which contains the central chamber but which does not have $b_{i,j}$ as a generator. Suppose τ is a cone that contains the central chamber and has $b_{i,j}$ as a generator. Since the natural action of the product of symmetric groups $S_n\times S_m$ on B fixes the central chamber, for any $\pi\times\sigma\in S_n\times S_m$, $(\pi\times\sigma)(\tau)$ covers the central chamber as well. We can pick π such that $\pi(i)=i$. If $b_{i,k}$, $k\ne j$, does not appear as a generator of τ we can choose σ so that $\sigma(k)=j$, and we would be done. So suppose $b_{i,k}$ is a generator of τ for $k=1, \ldots, m$.

Since these vectors are linearly independent and $\operatorname{rank}(B)=n+m-1$, there are exactly $n-1$ generators of τ which are not of the above form. But τ contains $(1/n, \ldots, 1/n) \times (1/m, \ldots, 1/m)$ in its interior, so these remaining $n-1$ generators are of the form b_{k,s_k}, $k=1, \ldots, n$, $k \neq i$. Each of these generators must have the coefficient $1/n$ in the unique expression that expresses $(1/n, \ldots, 1/n) \times (1/m, \ldots, 1/m)$ in terms of generators of τ. But $1/n > 1/m$, and this shows that Δ is a complete regular projective fan generated by the columns of A.

In order to give an exponential lower bound on the primitive collections of Δ constructed above we establish a link between its *circuits* and its primitive collections.

DEFINITION 2.3. Let $\mathscr{A}$ be a vector configuration in $\boldsymbol{Z}^d$. A collection of linearly dependent vectors $Z \subseteq \mathscr{A}$ is called a *circuit* if any proper subset of Z is linearly independent.

We will call the circuits of the generators of a complete fan Δ the circuits of Δ. If Z is a circuit of Δ, the unique (up to sign) dependence relation $\sum_i \lambda_i z_i = 0$ partitions Z into two subsets, namely $Z_+ = \{z_i \in Z : \lambda_i > 0\}$ and $Z_i = \{z_i \in Z : \lambda_i < 0\}$. In this case, there exist precisely two triangulations of $Z : t_+(Z) = \{Z \setminus z_i : z_i \in Z_+\}$ and $t_-(Z) = \{Z \setminus z_i : z_i \in Z_-\}$. Note that $\operatorname{relint}(\operatorname{pos}(Z_+)) \cap \operatorname{relint}(\operatorname{pos}(Z_-)) \neq \varnothing$ (see [BLSWZ]). Given a triangulation Δ and a circuit Z of Δ such that $t_+(Z)$ is a subcomplex of Δ, one can get via a *bistellar flip* another triangulation Δ' such that $t_-(Z)$ is a subcomplex of Δ'. For the details we refer to [GKZ, pp. 231–233]. The next lemma makes the connection between the circuits and primitive collections of Δ.

LEMMA 2.4. *Let $\Delta \subset \boldsymbol{R}^d$ be a complete regular fan (i.e. a triangulation) and let Z be a circuit such that $t_+(Z)$ is a subcomplex of Δ. Then Z_+ is a primitive collection. Moreover, if Z' is a different circuit where $t_+(Z')$ is a subcomplex of Δ, then $Z_+ \neq Z'_+$.*

PROOF. Clearly Z_+ does not generate a cone in Δ. By the definition of $t_+(Z)$, for all $z \in Z_+$, $\operatorname{pos}(Z_+ \setminus z)$ is a face of $t_+(Z)$, and hence is a cone in Δ. Each of these cones must be $(\operatorname{card}(Z_+)-1)$-dimensional, since otherwise Z cannot be a circuit. This shows that Z_+ is a primitive collection. For the second statement, assume $Z_+ = Z'_+$. Since $t_+(Z) \neq t_+(Z')$, the respective subcomplexes K and K' of Δ on which the bistellar flips are supported are different as well. But $\operatorname{pos}(Z_+) \cap \operatorname{relint}(K) \neq \varnothing$ and $\operatorname{pos}(Z'_+) \cap \operatorname{relint}(K') \neq \varnothing$, which implies $\operatorname{relint}(K) \cap \operatorname{relint}(K') \neq \varnothing$. This cannot happen since K and K' are distinct subcomplexes of Δ. This contradiction completes the proof.

For the main theorem we need the following result which provides the link between bistellar flips (and hence the primitive collections) of Δ and the corresponding chamber in the dual configuration.

THEOREM 2.5 ([GKZ, p. 233]). *Let $\mathscr{A} \subset \boldsymbol{Z}^d$ be a vector configuration and let $\mathscr{B}$ be a Gale transform of $\mathscr{A}$. If Δ and Δ' are two coherent triangulations of $\mathscr{A}$, then Δ and Δ' differ by a bistellar flip if and only if the corresponding chambers in $\Gamma(\mathscr{B})$ share a facet.*

PROOF OF THEOREM 1.1. Let B be the node-incidence matrix of $K_{n,m}$ where $n=2k-1$ and $m=2k+1$ and let A be a Gale transform of B. Let Δ be the coherent triangulation of A that corresponds to the central chamber in $\Gamma(B)$. As we established before Δ is a complete regular projective fan generated by the columns of A. Lemma 2.4 and Theorem 2.5 imply that the number of primitive collections of Δ should be at least the number of facets of the central chamber in $\Gamma(B)$. The following proposition shows that there are at least exponentially many such facets. The proof of the proposition can be found in [ST], but we include its proof for completeness.

PROPOSITION 2.6. *The central chamber in $\Gamma(B)$ which corresponds to Δ has at least 4^k facets.*

PROOF. If H is a hyperplane defined by the equation (1), we will call (card(C_+), card(D_+)) the *type* of H. Now starting at the point $(1/n,\dots,1/n)\times(1/m,\dots,1/m)$ and moving in the direction of $(-1,\dots,-1,n-1)\times(0,\dots,0)$ to a generic point $(a,\dots,a,a+1-na)\times(1/m,\dots,1/m)$ we cross a facet of type (r,s) with $n\in C_-$ whenever

$$r\cdot a-(n-r-1)\cdot a-(a+1-na)-s/m+(m-s)/m=2\cdot r\cdot a-2\cdot s/m=0\,.$$

From here we get $a=s/mr$ and since $a<1/n$ we like to find r and s which minimize the positive integer $m\cdot r-n\cdot s$. The unique solution is $r=k$ and $s=k+1$ and since $S_n\times S_m$ acts transitively on the set of hyperplanes of type (r,s), we conclude that every hyperplane of this type is a facet of the central chamber. When $k\geq 3$ there are $\binom{2k-1}{k}\binom{2k+1}{k+1}>2^k\cdot 2^k=4^k$ such facets. □

3. Connections to integer programming. In this section we will first state Theorem 1.1 in terms of the *Gröbner basis* (cf. [AL], [CLO]) of an *integer program.* Subsequently we will relate two conjectures, one that appears in the context of smooth projective toric varieties and the other one in the context of integer programming. An integer program can be stated as follows:

$$\text{minimize}\, c\cdot x \ \text{ subject to } \ A\cdot x=b,\ x\in \boldsymbol{N}^n$$

where $A\in\boldsymbol{Z}^{d\times n}$ with rank$(A)=d$, $b\in\boldsymbol{Z}^d$ and $c\in\boldsymbol{R}^n$. The reduced Gröbner basis of the *toric ideal* $I_A=\langle x^\alpha-x^\beta:\alpha,\beta\in\boldsymbol{N}^n, A\cdot\alpha=A\cdot\beta\rangle$ with respect to the term order induced by the cost vector c provides a *test set* for solving this integer program (see [AL], [CT], [ST], [Th] for details).

THEOREM 3.1. *Let A be a Gale transform of the node-edge incidence matrix B of $K_{n,m}$ with $n=2k-1$ and $m=2k+1$. Then I_A is generated by $x_{i1}x_{i2}\cdots x_{im}-1$, $i=1,\dots,n$ and $x_{1j}x_{2j}\cdots x_{nj}-1$, $j=1,\dots,m$, and the reduced Gröbner basis of I_A with respect to the degree lexicographic term order contains at least 4^k elements.*

PROOF. The rows of B constitute a $\boldsymbol{Z}$-basis for $\ker(A)\cap\boldsymbol{Z}^{nm}$. The above binomials correspond to the rows of B and the ideal they generate is contained in I_A. But since the sum of all the rows of B is a strictly positive vector, these binomials generate I_A (see Lemma 2.1 in [SWZ]). The degree lexicographic term order $\succ_{\text{deglex}}$ can be represented by the cost vector $c=(1, 1, \ldots, 1)$ refined by the lexicographic order. Since A is unimodular, the reduced Gröbner basis of I_A with respect to $\succ_{\text{deglex}}$ consists of square-free binomials (cf. [St, Corollary 8.9]) and the initial term of each binomial corresponds to a minimal non-face (i.e. a primitive collection) of the coherent triangulation Δ induced by c (cf. [St, Theorem 8.3]). As the vector $B \cdot c$ is in the central chamber of $\Gamma(B)$, Proposition 2.2 implies that the triangulation Δ induced by c is the same as the complete regular fan we considered in the proof of Theorem 1.1. □

EXAMPLE 3.2 (3×5 Complete Bipartite Graph). Let B be the node-incidence matrix of $K_{3,5}$. If we associate with every column $b_{ij}=e_i \times e'_j$, $i=1, 2, 3$, $j=1, 2, 3, 4, 5$, the variable x_{ij}, then

$$I_A=\langle x_{11}x_{21}x_{31}-1,\ x_{12}x_{22}x_{32}-1,\ x_{13}x_{23}x_{33}-1,\ x_{14}x_{24}x_{34}-1,\ x_{15}x_{25}x_{35}-1,$$
$$x_{11}x_{12}x_{13}x_{14}x_{15}-1,\ x_{21}x_{22}x_{23}x_{24}x_{25}-1,\ x_{31}x_{32}x_{33}x_{34}x_{35}-1\rangle\ .$$

There are 30 facets of the central chamber of B and indeed the reduced Gröbner basis of I_A with respect to $\succ_{\text{deglex}}$ consists of 50 binomials.

In relation to smooth complete projective varieties, the following conjecture is posed in [Bat].

CONJECTURE 3.3 ([Bat, Conjecture 7.1]). For any d-dimensional smooth complete toric variety defined by a complete regular fan Δ with n generators, there exists a constant $N(n-d)$ depending only on $n-d$ such that the number of primitive collections in Δ does not exceed $N(n-d)$.

Another conjecture with a similar flavor is stated about the complexity of *Gröbner cones* in the setting of integer programming in [ST] (Conjecture 6.1), and here we will give a connection between the two conjectures along the lines of the previous section. Given an integer program defined by a matrix A, two generic cost vectors c and c' are considered to be equivalent if the respective reduced Gröbner bases of I_A are the same. The set of all such equivalent cost vectors associated to a fixed reduced Gröbner basis of the toric ideal of A is an open polyhedral cone and the collection of the closures of all such cones and their faces constitute a fan called the *Gröbner fan* of A (cf. [MR], [BM], [St], [ST]).

CONJECTURE 3.4 ([ST, Conjecture 6.1]). There exists a function φ such that, for every matrix $A \in \boldsymbol{Z}^{d \times n}$ of rank d, every cone of the Gröbner fan of A has at most $\varphi(n-d)$ facets.

This conjecture is true for $n-d \leq 2$. For the case $n-d=3$, $\varphi(3)=4$ under certain genericity assumptions on the matrix A and this was proved by Bárány and Scarf in

[BS]. The following proposition points to a connection between N and φ:

PROPOSITION 3.5 ([ST, Corollary 3.18]). *Let $A \in \mathbf{Z}^{d \times n}$ with* $\operatorname{rank}(A) = d$ *be a unimodular matrix and let B be a Gale transform of A. Then the Gröbner fan of A and $\Gamma(B)$ coincide.*

In the light of this proposition one can formulate a specialized version of Conjecture 3.4.

CONJECTURE 3.6. There exists a function φ' such that, for every unimodular matrix $A \in \mathbf{Z}^{d \times n}$ of rank d, every cone of $\Gamma(B)$ has at most $\varphi'(n-d)$ facets where B is a Gale transform of A.

THEOREM 3.7. *If there exist N and φ' as above, then $N(n-d) \geq \varphi'(n-d)$ for all n and d.*

PROOF. Fix n and d, and suppose that $A \in \mathbf{Z}^{d \times n}$ is a unimodular matrix with a coherent triangulation Δ such that the corresponding chamber in $\Gamma(B)$, where B is a Gale transform of A, has $\varphi'(n-d)$ facets. If Δ uses all columns of A as generators, then by the results of the previous section we would be done. Otherwise we can refine Δ into another complete regular fan by adding the missing generators. This will not destroy the primitive collections in Δ associated with the bistellar flips as in Lemma 2.4. Hence $N(n-d) \geq \varphi'(n-d)$. □

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