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BOUNDS FOR THE ORDER OF AUTOMORPHISM GROUPS OF HYPERELLIPTIC FIBRATIONS

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Abstract. For a nonsingular complex algebraic surface with a pencil of hyperelliptic curves of genus g over a nonsingular algebraic curve, we take two approaches to get upper bounds for the order of its automorphism group as a generalization of Chen's results on genus two fibrations. If the genus of the base curve is neither one nor zero, we estimate the order of each automorphism group of the base curve and the general fiber by a theorem of Hurwitz and that of Tuji. In the cases of rational or elliptic base curves, we use the inequality of Horikawa-Persson to see the contribution of singular fibers.

1. Introduction. Let S be a nonsingular complex projective surface and C a nonsingular projective curve of genus π. Let $f:S\to C$ denote a relatively minimal fibration of curves of genus g.

An *automorphism* of f is, by definition, a pair of $\tilde{\sigma}\in\mathrm{Aut}(S)$ and $\sigma\in\mathrm{Aut}(C)$ which satisfies

$$f\tilde{\sigma}=\sigma f.$$

The group of automorphisms of f will be denoted by Aut(f). (cf. [3, Definition 0.1]).

Suppose S is a surface of general type and G a subgroup of Aut(f). Then Xiao [9] showed the following upper bounds for the order of G:

PROPOSITION 1.1 (cf. [9, Proposition 1]).

$$|G|\leq\begin{cases}882K_S^2 & \textit{if } \pi\geq 2\\ 168(2g+1)(K_S^2+8g-8) & \textit{otherwise}.\end{cases}$$

Furthermore Chen [3] obtained a more detailed estimate in the case of genus two fibrations:

PROPOSITION 1.2 (cf. [3, Theorem 0.1]). *Let $f: S\to C$ be a relatively minimal fibration of genus two. Then*

$$|G|\leq 504K_S^2$$

for $\pi\geq 2$. If f is not locally trivial, then

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$$|G| \leq \begin{cases} 126K_S^2 & \text{if } \pi \geq 2 \\ 144K_S^2 & \text{if } \pi = 1 \\ 120K_S^2 + 960 & \text{if } \pi = 0 \,. \end{cases}$$

In the present paper, we will attempt to generalize a part of Chen's results to hyperelliptic fibrations of higher genus.

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2. Preliminaries. Let us recall basic facts on hyperelliptic fibrations (cf. [5], [6], [7]):

Let $f: S \to C$ denote a relatively minimal hyperelliptic fibration of genus g over a nonsingular projective curve C of genus π. Then there exist a projective line bundle $\psi: W \to C$ and a divisor B on W with a double covering $\varpi: S' \to W$ branched along B such that we have a birational map $\mu: S \to S'$ which satisfies $f = \psi\varpi\mu$.

By canonical resolution, we get a smooth model S^b of S' with a composite $S^b \to S$ of blowing downs.

Let $F_1, F_2, \ldots$ denote the singular fibers of f. Then there exists a nonnegative rational number $\mathrm{Ind}(F_j)$ for each F_j such that the following holds:

PROPOSITION 2.1 (cf. [7, Theorem 2.1]).

$$K_S^2 = \frac{4(g-1)}{g} \{\chi + (g+1)(\pi - 1)\} + \sum_j \mathrm{Ind}(F_j)$$

where K_S is the canonical bundle and χ is the holomorphic Euler-Poincaré characteristic of the surface S.

In particular we have

$$K_S^2 \geq \frac{4(g-1)}{g} \{\chi + (g+1)(\pi - 1)\} \,. \tag{1}$$

A singular fiber of type 0 is, by definition, a singular fiber with $\mathrm{Ind} = 0$.

The following is obvious by the proof of [7, Theorem 2.1]:

LEMMA 2.1. *If a singular fiber F is not of type* 0, *then we have*

$$\mathrm{Ind}(F) \geq \frac{2g-4}{g} \,.$$

As for the topological Euler number e, we have:

PROPOSITION 2.2 (cf. [1, Proposition III.11.4]). *For each singular fiber F_j of f, the following hold*:

(i) $e(F_j) > 2-2g$.

(ii) $e(S) = (2-2g)(2-2\pi) + \sum_j \{e(F_j) + 2g - 2\}$.

Therefore we have

$$e(S) \geq (2-2g)(2-2\pi)\,. \tag{2}$$

By Propositions 2.1 and 2.2 and Noether's formula, we get the following:

PROPOSITION 2.3. (i)

$$\chi = (g-1)(\pi-1) + \frac{g}{8g+4} \sum_j \{\mathrm{Ind}(F_j) + e(F_j) + 2g - 2\}\,.$$

$$K_S^2 = 8(g-1)(\pi-1) + \frac{3g}{2g-1} \sum_j \mathrm{Ind}(F_j) + \frac{g-1}{2g+1} \sum_j \{e(F_j) + 2g - 2\}\,.$$

(ii) *In particular,*

$$\chi \geq (g-1)(\pi-1)\,, \quad K_S^2 \geq 8(g-1)(\pi-1)\,.$$

3. Upper bounds for $|G|$. Let G denote the automorphism group of a relatively minimal hyperelliptic fibration $f: S \to C$. By the same arguments as in [3], we conclude that there exist two exact sequences of groups:

$$1 \to K \to G \to H \to 1$$

$$1 \to Z_2 \to K \to \bar{K} \to 1\,,$$

where

$$K = \{(\tilde{\sigma}, \sigma) \in G;\ \sigma = \mathrm{id}_C\}\,,$$

$H \subset \mathrm{Aut}(C)$, $\bar{K} \subset \mathrm{Aut}(\boldsymbol{P}^1)$ and Z_2 is the cyclic group of order two coming from the hyperelliptic involution.

Suppose $\pi = \pi(C) \geq 2$. Then, by a theorem of Hurwitz, we have

$$|H| \leq 84(\pi - 1)\,.$$

On the other hand, since a general fiber of f is of genus g, we have the following upper bound for the order of $\bar{K}$:

LEMMA 3.1 (cf. [4], [8]). (i) *If $g \neq 2, 3, 5, 9$, then $|\bar{K}| \leq 4g + 4$.*

(ii) *If $g = 2, 3$, then $|\bar{K}| \leq 24$.*

(iii) *If $g = 5, 9$, then $|\bar{K}| \leq 60$.*

Therefore we get the following estimate for $|G| = 2|\bar{K}||H|$ by Proposition 2.3 (ii):

THEOREM 1. *If $\pi \geq 2$, then we have*

$$|G|\leq\begin{cases}84\dfrac{g+1}{g-1}K_S^2 & if\ g\neq 2,3,5,9\\ 504K_S^2 & if\ g=2\\ 252K_S^2 & if\ g=3\\ 315K_S^2 & if\ g=5\\ 157.5K_S^2 & if\ g=9\,.\end{cases}$$

To investigate the cases of $\pi\leq 1$, we assume that f has at least one singular fiber when $\pi=1$ and at least three singular fibers when $\pi=0$. Moreover we assume that G is a finite subgroup of Aut(f) which contains the hyperelliptic involution in the following arguments in this section.

Now suppose that the fiber F of f over $p\in C$ is singular and that $|H|=n$, $|Hp|=n/r$. Then, by Proposition 2.3 (i), we have

$$K_S^2\geq 8(g-1)(\pi-1)+\frac{n}{r}\left(\frac{g-1}{2g+1}\{e(F)+2g-2\}+\frac{3g}{2g+1}\operatorname{Ind}(F)\right),$$

which implies

$$|G|\leq\frac{2(2g+1)|\bar{K}|}{3g\operatorname{Ind}(F)+(g-1)\{e(F)+2g-2\}}\,rK_{S/C}^2\,, \tag{3}$$

where

$$K_{S/C}^2=K_S^2-8(g-1)(\pi-1)\,.$$

LEMMA 3.2. *Suppose that the horizontal part B_0 of the branch locus B is étale over C and $B\neq B_0$. Then we have*

$$|G|\leq\begin{cases}4\dfrac{g+1}{g-1}rK_{S/C}^2 & if\ g\neq 2,3,5,9\\ 24rK_{S/C}^2 & if\ g=2\\ 12rK_{S/C}^2 & if\ g=3\\ 15rK_{S/C}^2 & if\ g=5\\ 7.5rK_{S/C}^2 & if\ g=9\,.\end{cases}$$

PROOF. Since B has only a finite number of double points, we have $\operatorname{Ind}(F)=0$. On the other hand, the singular fiber F is of the form

$$F=2E_0+E_1+E_2+\cdots+E_{2g+2}\,,$$

where each E_j is a nonsingular rational curve with

$$E_0^2=-g-1\,,$$

$$E_iE_j=0\quad(1\leq i<j\leq 2g+2)\,,$$

$$E_0E_j=1\quad(1\leq j\leq 2g+2)\,.$$

Hence we have $e(F)=2g+4$ and the lemma follows. □

LEMMA 3.3. *Suppose that f has only singular fibers of type* 0 *and that we cannot choose the branch locus B on W in such a way that its horizontal part B_0 is étale over C. Then we have*

$$|G|\leq\frac{4(2g+1)}{g-1}rK_{S/C}^2 .$$

PROOF. Since $\mathrm{Ind}(F)=0$, we have to show

$$|\bar{K}|\leq 2(e(F)+2g-2) .$$

Let B_t denote the restriction of B to a general fiber Γ_t of ψ. Then B_t consists of $2(g+1)$ distinct points of $\Gamma_t\cong \boldsymbol{P}^1$ and $\bar{K}$ is nothing but a finite subgroup of $\mathrm{Aut}(\Gamma_t)$ with

$$\bar{K}B_t=B_t .$$

Hence $\bar{K}$ is isomorphic to one of the following:

- T_{12} (Tetrahedral group)
- O_{24} (Octahedral group)
- I_{60} (Icosahedral group)
- D_{2l} ($l=2g+2, 2g, g+1, \dots$) (Dihedral group)
- Z_l ($l=2g+2, 2g+1, \dots$) (Cyclic group),

where the suffix is the order of the group, and we may assume $|\bar{K}|$ is not 1.

A point of W is said to be *bad* if it is a singular point of B or B is tangent to the fiber at that point. Now let us look at bad points of B on $\Gamma=\psi^{-1}(p)\subset W$.

If there exists a bad point $z\in\Gamma$ such that $|\bar{K}z|=|\bar{K}|$, then we have $e(F)+2g-2\geq|\bar{K}|$ since each point of $\bar{K}z$ is also a bad point. So we assume that, for each bad point z on Γ, $|\bar{K}z|<|\bar{K}|$.

(a) $\bar{K}\cong T_{12}$. We have $|\bar{K}z|=4$ or 6. Let I_z denote the intersection number of B_0 and Γ at z. Then $I_z\geq 3$ if $|\bar{K}z|=4$ and $I_z\geq 2$ if $|\bar{K}z|=6$. Therefore we have $e(F)+2g-2\geq$ 8 or 6 and the claim follows.

(b) $\bar{K}\cong O_{24}$. We have $|\bar{K}z|=6$, 12 or 8 and if $|\bar{K}z|=6$ (resp. 8), $I_z\geq 4$ (resp. 3). Hence we have $e(F)+2g-2\geq 18$ (resp. 16) if $|\bar{K}z|=6$ (resp. 8).

(c) $\bar{K}\cong I_{60}$. We have $|\bar{K}z|=12$, 30 or 20. Moreover we have $I_z\geq 5$ (resp. 3) if $|\bar{K}z|=12$ (resp. 20). Hence we have $e(F)+2g-2\geq 48$ (resp. 40) if $|\bar{K}z|=12$ (resp. 20).

(d) $\bar{K}\cong D_{2l}$ or Z_l. Suppose $\bar{K}\cong D_{2l}$ and B_0 has bad points only at the north and south poles. Then we have $I_z\geq l$, and therefore

$$e(F)+2g-2\geq l-1+l-1=2l-2 .$$

If $\bar{K}\cong Z_l$ and B_0 has bad points only at the north or south poles, then we have $e(F)+2g-2\geq l-1$ by the same arguments as above. Since $l\geq 2$ the claim follows. □

Suppose that a singular fiber F of f over $p\in C$ is not of type 0. Then the horizontal part B_0 of the branch locus B cannot be étale over C and hence $\bar{K} \neq D_{4g+4}$, D_{4g} (cf. [3, Lemma 2.1]). Therefore we have $|\bar{K}| \leq 2g+2$. By Lemma 2.1, Proposition 2.2 (i) and the inequality (3), we have

$$|G| \leq \frac{4(g+1)(2g+1)}{7g-13} rK_{S/C}^2 . \tag{4}$$

Though this estimate may be far from being the best possible, it is not so bad when g is small.

PROPOSITION 3.1. *Let $f: S \to C$ denote a fibration of hyperelliptic curves of genus g. Suppose that there exists a singular fiber $F = f^{-1}(p)$ with $|\mathrm{Stab}_H(p)| = r$. Then we have*

$$|G| \leq \begin{cases} 24rK_{S/C}^2 & \text{if } g=2 \\ 14rK_{S/C}^2 & \text{if } g=3 \\ 12rK_{S/C}^2 & \text{if } g=4 \\ 15rK_{S/C}^2 & \text{if } g=5 \\ \dfrac{4(g+1)(2g+1)}{7g-13} rK_{S/C}^2 & \text{if } g \geq 6 . \end{cases}$$

PROOF. This is a direct consequence of Lemmas 3.2 and 3.3, and the inequarity (4). □

Now we estimate the value of r. It is well known that if $\pi = 1$, then r is at most 6. So we assume that $\pi = 0$. Then $H \subset \mathrm{Aut}(C)$ is isomorphic to one of T_{12}, O_{24}, I_{60}, D_{2l} and Z_l. If $H \cong T_{12}$, O_{24} or I_{60}, we have $|\mathrm{Stab}_H(p)| \leq 5$ for each p on C. Suppose $H \cong D_{2l}$ or Z_l. Then by the assumption that f has at least three singular fibers, we conclude that there exists at least one singular fiber not over the north nor south pole of C, which implies that $r = 1$ or 2 for that singular fiber. Hence we get the following lemma:

LEMMA 3.4. *There exists a singular fiber $f^{-1}(p)$ of f such that*

$$r = |\mathrm{Stab}_H(p)| \leq 5 .$$

Consequently we get the following:

THEOREM 2. (i) *If $\pi = 1$ and f has at least one singular fiber, we have*

$$|G| \leq \begin{cases} 144K_S^2 & \text{if } g=2 \\ 84K_S^2 & \text{if } g=3 \\ 72K_S^2 & \text{if } g=4 \\ 90K_S^2 & \text{if } g=5 \\ \dfrac{24(g+1)(2g+1)}{7g-13} K_S^2 & \text{if } g \geq 6 . \end{cases}$$

(ii) *If $\pi=0$ and f has at least three singular fibers, we have*

$$|G|\leq\begin{cases}120(K_S^2+8) & \text{if } g=2\\ 70(K_S^2+16) & \text{if } g=3\\ 60(K_S^2+24) & \text{if } g=4\\ 75(K_S^2+32) & \text{if } g=5\\ \dfrac{20(g+1)(2g+1)}{7g-13}\{K_S^2+8(g-1)\} & \text{if } g\geq 6\,.\end{cases}$$

REMARK. Beauville [2] showed that a family of curves *over* $\boldsymbol{P}^1$ with at most two singular fibers is isotrivial (cf. [2, Proposition 1.1]). In this situation, there exist hyperelliptic fibrations with arbitrarily large automorphism groups.

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