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THE MINIMAL MODEL THEOREM FOR DIVISORS OF TORIC VARIETIES

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Abstract. The minimal model conjecture says that if a proper variety has non-negative Kodaira dimension, then it has a minimal model with abundance and if the Kodaira dimension is $-\infty$, then it is birationally equivalent to a variety which has a fibration with the relatively ample anti-canonical divisor. In this paper, first we prove this conjecture for a Δ-regular divisor on a proper toric variety by means of successive contractions of extremal rays and flips of the ambient toric variety. Then we prove the main result: for such a divisor with the non-negative Kodaira dimension there is an algorithm to construct concretely a projective minimal model with abundance by means of "puffing up" the polytope.

Introduction. Let k be an algebraically closed field of arbitrary characteristic. Varieties in this paper are all defined over k. Let X be a proper algebraic variety. A proper algebraic variety Y is called a minimal model of X, if (1) Y is birationally equivalent to X, (2) Y has at worst terminal singularities and (3) the canonical divisor K_Y is nef. A minimal model Y is said to have abundance if the linear system $|mK_Y|$ is base point free for sufficiently large m. The minimal model conjecture states: an arbitrary proper variety with $\kappa \geq 0$ has a minimal model with abundance while an arbitrary proper variety with $\kappa = -\infty$ has a birationally equivalent model Y with at worst terminal singularities and a fibration $Y \to Z$ to a lower dimensional variety with $-K_Y$ relatively ample.

The conjecture is classically known to hold in the 2-dimensional case. In the 3-dimensional case the conjecture for $k = \boldsymbol{C}$ is proved by Mori [4] and Kawamata [3], while it is not yet proved in higher dimension. As a special case of higher dimension, Batyrev [1] proved, among other results, the existence of a minimal model for a Δ-regular anti-canonical divisor of a Gorenstein Fano toric variety $T_N(\Delta)$.

In this paper, first in Section 1 we prove the minimal model conjecture for every Δ-regular divisor X on a toric variety of arbitrary dimension by means of successive contractions of extremal rays and flips which are introduced by Reid [7]. By Bertini's theorem, for a field k of characteristic 0, the minimal model conjecture thus holds for a general member of a base point free linear system on a proper toric variety over k. An important point of this part is providing with a technical statement Corollary 1.17 which is used in the following sections. Then in Sections 2 and 3 we prove the main result: for a Δ-regular divisor with $\kappa \geq 0$, there exists an algorithm to construct concretely a projective minimal model with abundance by means of "puffing up" the polytope corresponding to the adjoint divisor. The advantage of

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working on toric varieties is that one can see every possible exceptional divisor as a vector in N. We can make use of it for concrete construction of minimal model. In Section 4 we show some examples of projective minimal models constructed by this algorithm.

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1. The minimal model theorem for divisors on toric varieties.

DEFINITION 1.1 ([1]). A divisor X of a toric variety $T_N(\Delta)$ defined by a fan Δ is called Δ-regular, if for every $\tau \in \Delta$ the intersection $X \cap \operatorname{orb}(\tau)$ is either a smooth divisor of $\operatorname{orb}(\tau)$ or empty. In particular, $X \cap T \neq \emptyset$ for the maximal orbit T in $T_N(\Delta)$.

DEFINITION 1.2. Let V and V' be toric varieties defined by fans Δ and Δ', respectively and $f : V' - \rightarrow V$ a toric birational map: i.e. Δ' is obtained by successive subdivisions and converse of subdivisions from Δ. If f is a morphism, the proper transform of a divisor X of V is defined as usual. For general f, we can define the proper transform for a certain divisor. Namely, let T be the maximal orbit in V. If an irreducible divisor X on V satisfies $X \cap T \neq \emptyset$, the divisor $X' = \overline{f^{-1}(X \cap T)}$ on V' is called the proper transform of X on V'.

DEFINITION 1.3. If a proper variety X has a resolution $f : X' \rightarrow X$ of singularities, define $\kappa(X)$ as the Kodaira dimension $\kappa(X')$ of X'.

1.4. For a Δ-regular divisor X on a proper toric variety $V = T_N(\Delta)$, $\kappa(X)$ is defined. Indeed, let Δ' be a non-singular subdivision and $f : V' = T_N(\Delta') \rightarrow V$ the corresponding morphism; then the proper transform X' of X on V' is Δ'-regular by 3.2.1 of [1], therefore it is non-singular.

DEFINITION 1.5. Let X be a divisor on a normal variety V such that $K_V + X$ is a $\boldsymbol{Q}$-Cartier divisor and $f : V' \rightarrow V$ a birational morphism. Let X' be the proper transform of X. If

$$K_{V'} + X' = f^*(K_V + X) + \sum_i a_i E_i ,$$

where E_i's are the exceptional divisors of f, then a_i is called the discrepancy of $K_V + X$ at E_i.

DEFINITION 1.6. Let V be a toric variety defined by a simplicial fan Δ and X an irreducible divisor on V. The divisor $K_V + X$ is called toric terminal, if the following hold:

(1) There exists a morphism $f : V' = T_N(\Delta') \rightarrow V$ corresponding to a non-singular subdivision Δ' of Δ such that the proper transform X' of X on V' is Δ'-regular, in particular $X \cap T \neq \emptyset$ for the maximal orbit T in V, and

(2) for every such morphism as in (1) the discrepancy of $K_V + X$ at every exceptional divisor on V' is positive.

LEMMA 1.7. *If $V = T_N(\Delta)$ is non-singular and an irreducible divisor X on V is Δ-regular, then $K_V + X$ is toric terminal.*

PROOF. For every non-singular subdivision Δ' of Δ, where $\Delta' \neq \Delta$, the proper transform X' of X by the corresponding morphism $f : V' = T_N(\Delta') \to V$ is Δ'-regular. Since $X' = f^*X$ and $K_{V'} = f^*K_V + \sum_i a_i E_i$, where $a_i > 0$ for every exceptional divisor E_i on V', it follows that the discrepancy of $K_V + X$ at each E_i is positive. □

PROPOSITION 1.8. *Let V be a toric variety defined by a simplicial fan Δ and X an irreducible divisor on V. Then the divisor $K_V + X$ is toric terminal if and only if the following hold*:

(i) *There exists a morphism $f : V' = T_N(\Delta') \to V$ corresponding to a non-singular subdivision Δ' of Δ such that the proper transform X' of X on V' is Δ'-regular.*

(ii) *For one such morphism as in* (i) *the discrepancy of $K_V + X$ at every exceptional divisor on V' is positive.*

PROOF. Let $f : V' = T_N(\Delta') \to V$ be the morphism satisfying the conditions (i) and (ii) and let $g : V'' \to V$ be another morphism satisfying (i). Take a non-singular toric variety $\tilde{V}$ which dominates both V' and V''. Then by Lemma 1.7, $K_{V'} + X'$ is toric terminal. Therefore the discrepancy of $K_V + X$ at every exceptional divisor on $\tilde{V}$ is positive, which yields the positivity of it at every exceptional divisor on V''. □

LEMMA 1.9. *Let V be a toric variety defined by a simplicial fan Δ and X an irreducible divisor on V. If the divisor $K_V + X$ is toric terminal, then V has at worst terminal singularities.*

PROOF. This follows from the fact that the discrepancy of K_V at each exceptional divisor is greater than or equal to that of $K_V + X$. □

Here we summarize the results of Reid ([7]) which are used in this section.

PROPOSITION 1.10 ([7]). *Let V be the toric variety defined by a proper simplicial fan Δ.*

(i) *The cone of numerical effective* 1*-cycles* NE(V) *is represented as* $\sum_{i=1}^r \boldsymbol{R}_{\geq 0}[l_i]$, *where l_i's are* 1*-dimensional strata on V. Here each $\boldsymbol{R}_{\geq 0}[l_i]$ is called an extremal ray.*

(ii) *For every extremal ray R there exists a toric morphism $\varphi_R : V \to V'$ which is an elementary contraction in the sense of Mori theory*: $(\varphi_R)_*\mathcal{O}_V = \mathcal{O}_{V'}$, *and $\varphi_R C = pt$ if and only if $[C] \in R$. Let $A \subset V$ and $B \subset V'$ be the loci on which φ_R is not an isomorphism. Then $\varphi_R\big|_A : A \to B$ is a flat morphism all of whose fibers are weighted projective spaces of the same dimension.*

(iii) *If $\varphi_R : V \to V' = T_N(\Delta')$ is birational and not isomorphic in codimension one, then the exceptional set of φ_R is an irreducible divisor and Δ' is proper simplicial. This φ_R is called a divisorial contraction.*

(iv) *If $\varphi_R : V \to V' = T_N(\Delta')$ is isomorphic in codimension one, then there exists a commutative diagram*

$$\begin{array}{ccc} & \tilde{V} & \\ {}^{\psi}\swarrow & & \searrow^{\psi_1} \\ V & & V_1 = T_N(\Delta_1) \\ {}_{\varphi_R}\searrow & & \swarrow_{\varphi_1} \\ & V' & \end{array}$$

such that Δ_1 is proper simplicial with the set of 1*-dimensional cones $\Delta_1(1) = \Delta(1)$, that all morphisms are elementary contractions of extremal rays, that ψ and ψ_1 are birational morphisms with a common exceptional divisor D, that φ_R and φ_1 are birational morphisms with the exceptional sets $\psi(D)$ and $\psi_1(D)$ respectively, and that by identifying $N_1(V)$ with $N_1(V_1)$, $-R$ is an extremal ray in* $\mathrm{NE}(V_1)$ *and $\varphi_1 = \varphi_{-R}$. The birational map $\varphi_1^{-1} \circ \varphi_R : V \dashrightarrow V_1$ is called a flip.*

LEMMA 1.11. *Let V be a toric variety defined by a proper simplicial fan Δ and X an irreducible divisor such that $K_V + X$ is toric terminal. Let R be an extremal ray such that $(K_V + X)R < 0$. Then the following hold*:

(i) *If $\varphi_R : V \to V' = T_N(\Delta')$ is a divisorial contraction, then $K_{V'} + X'$ is toric terminal, where X' is the proper transform of X on V'*;

(ii) *Let $\varphi_R : V \to V'$ be isomorphic in codimension one*; *in the diagram*

$$\begin{array}{ccc} & \tilde{V} & \\ {}^{\psi}\swarrow & & \searrow^{\psi_1} \\ V & & V_1 = T_N(\Delta_1) \\ {}_{\varphi_R}\searrow & & \swarrow_{\varphi_1} \\ & V' & \end{array}$$

of Proposition 1.10 (iv), *let X_1 be the proper transform of X on V_1, D the common exceptional divisor of ψ and ψ_1, α the discrepancy of $K_V + X$ at D and α' the discrepancy of $K_{V_1} + X_1$ at D*; *then $\alpha < \alpha'$ and $K_{V_1} + X_1$ is toric terminal.*

PROOF. For the proof of (i), first one should remark that V' is $\boldsymbol{Q}$-factorial, because Δ' is simplicial. Let E be the exceptional divisor for φ_R.

CLAIM 1.12. $ER < 0$.

For the proof of the claim, take an irreducible divisor H on V' such that $H \supset \varphi_R(E)$. Then $\varphi^* H = [H] + aE$ with $a > 0$, where $[H]$ is the proper transform of H on V. Since $(\varphi_R^* H)R = 0$ and $[H]R > 0$, it follows that $aER < 0$, which completes the proof of the claim.

Denote $K_V + X$ by $\varphi_R^*(K_{V'} + X') + bE$. Then $b > 0$. Indeed, by $(K_V + X)R < 0$, $\varphi_R^*(K_{V'} + X')R = 0$ and $ER < 0$, it follows that $b > 0$. Let $\bar{\Delta}$ be a non-singular subdivision of Δ such that the proper transform $\bar{X}$ of X on $\bar{V} = T_N(\bar{\Delta})$ is $\bar{\Delta}$-regular. Since $K_V + X$ is toric terminal, the discrepancy of $K_V + X$ at every exceptional divisor for $\bar{V} \to V$ is positive. By this, and $b > 0$, it follows that the discrepancy of $K_{V'} + X'$ at every exceptional divisor for $\bar{V} \to V'$ is positive. For the proof of (ii), take a curve l on $\tilde{V}$ such that $\psi_1(l) =$pt and $\psi(l) \neq$pt. This is possible because if a curve contracted by both ψ and ψ_1 exists, then the extremal rays corresponding to ψ and ψ_1 coincide, which implies $V \simeq V_1$ and $\varphi_R = \varphi_1$, a contradiction to $\varphi_1 = \varphi_{-R}$ in (iv) of Proposition 1.10. For this l, one can prove that

$Dl < 0$ in the same way as in the claim above. Now as $\psi_*(l)$ is contracted to a point by φ_R, $[\psi_*(l)] \in R$, therefore $\psi^*(K_V + X)l = (K_V + X)\psi_*(l) < 0$. By intersecting l with $K_{\tilde{V}} + \tilde{X} = \psi^*(K_V + X) + \alpha D$, we obtain

$$(K_{\tilde{V}} + \tilde{X})l < \alpha Dl\,.$$

Here the left hand side is $\psi_1^*(K_{V_1} + X_1)l + \alpha' Dl$, and $\psi_1^*(K_{V_1} + X_1)l = 0$ because of the definition of l. This proves that $\alpha < \alpha'$. To prove the last statement, take a non-singular subdivision $\tilde{\tilde{\Delta}}$ of $\tilde{\Delta}$ such that the proper transform $\tilde{\tilde{X}}$ of $\tilde{X}$ is $\tilde{\tilde{\Delta}}$-regular. Let $\lambda : \tilde{\tilde{V}} = T_N(\tilde{\tilde{\Delta}}) \to \tilde{V}$ be the corresponding morphism. Then $K_{\tilde{\tilde{V}}} + \tilde{\tilde{X}} = \lambda^*\psi^*(K_V + X) + \sum_i \beta_i E_i$, where $\beta_i > 0$ for every exceptional divisor E_i, because $K_V + X$ is toric terminal. Now by substitution of $\psi^*(K_V + X) = \psi_1^*(K_{V_1} + X_1) + (\alpha' - \alpha)D$ into the equality above, the discrepancy of $K_{V_1} + X_1$ at every exceptional divisor on $\tilde{\tilde{V}}$ turns out to be positive. □

THEOREM 1.13. *Let V be a toric variety defined by a proper simplicial fan Δ and X an irreducible divisor on V such that $K_V + X$ is toric terminal. Then there exists a sequence of birational toric maps*

$$V = V_1 \overset{\varphi_1}{-\to} V_2 \overset{\varphi_2}{-\to} \cdots \overset{\varphi_{r-1}}{-\to} V_r\,,$$

where

(i) *each φ_i is either a divisorial contraction or a flip, in particular V_i is defined by a proper simplicial fan*;

(ii) *for the proper transform X_i of X on V_i $(i = 1, \dots, r)$, $K_{V_i} + X_i$ is toric terminal*;

(iii) *either $K_{V_r} + X_r$ is nef or there exists an extremal ray R on V_r such that $(K_{V_r} + X_r)R < 0$ and the elementary contraction $\varphi_R : V_r \to Z$ is a fibration to a lower dimensional variety Z.*

PROOF. If $K_V + X$ is nef, then the statement is obvious. If $K_V + X$ is not nef, then there is an extremal ray R such that $(K_V + X)R < 0$. Take the elementary contraction $\varphi_R : V \to V'$. If $\dim V' < \dim V$, then the statement holds. So assume that φ_R is birational. If φ_R is divisorial, then define $\varphi_1 := \varphi_R : V \to V' =: V_2$. If φ_R is not divisorial, then let $\varphi_1 : V - \to V_2$ be the flip. Then in both cases, $K_{V_2} + X_2$ is toric terminal by Lemma 1.11. Now if $K_{V_2} + X_2$ is nef, then the proof is completed. If it is not nef, follow the same procedure as above. By succession, one obtains a sequence of divisorial contractions and flips:

$$V = V_1 \overset{\varphi_1}{-\to} V_2 \overset{\varphi_2}{-\to} \cdots \overset{\varphi_{r-1}}{-\to} V_r \cdots\,.$$

It is sufficient to prove that the sequence terminates at a finite stage. Let us assume that there exists such a sequence of infinite length. Since the divisorial contraction makes the Picard number strictly less, the number of divisorial contractions in the sequence is finite. So we may assume that there is $m_0 \in N$ such that φ_m's are all flips for $m \geq m_0$. By (iv) of 1.10 the set of one-dimensional cones of the fan defining V_m $(m \geq m_0)$ are common. As the number of such fans is finite, there are numbers $m < m'$ such that $\varphi_{m'-1} \circ \cdots \circ \varphi_m : V_m - \to V_{m'}$ is the identity. For each flip φ_j $(j = m, \cdots, m'-1)$, take the dominating variety V'_j as in (iv) of Proposition 1.10:

$$\begin{array}{ccc} & V'_j & \\ {}^{\psi_j}\swarrow & & \searrow^{\psi'_j} \\ V_j & & V_{j+1}\,. \end{array}$$

Let D_j be the common exceptional divisor of ψ_j and ψ_{j+1}. Then take a proper toric variety $\tilde{V} = T_N(\tilde{\Delta})$ which dominates all V'_j, $j = m, \ldots, m'-1$ and on which the proper transform $\tilde{X}$ of X_j's is $\tilde{\Delta}$-regular. This is possible because $K_{V_j} + X_j$'s are toric terminal. Here one should note that the set of exceptional divisors on $\tilde{V}$ for all morphisms $\tilde{V} \to V_j$ $(j = m, \ldots, m'-1)$ are common. For every $j = m, \ldots, m'-1$, the discrepancy α of $K_{V_j} + X_j$ at D_j is less than the discrepancy α' of $K_{V_{j+1}} + X_{j+1}$ at D_j by Lemma 1.11. By this fact, for every exceptional divisor E on $\tilde{V}$, the discrepancy α_E of $K_{V_j} + X_j$ at E and the discrepancy α'_E of $K_{V_{j+1}} + X_{j+1}$ at E satisfy $\alpha_E \leq \alpha'_E$ with $\alpha_E < \alpha'_E$ for at least one exceptional divisor E. Therefore comparing $K_{V_m} + X_m$ and $K_{V_{m'}} + X_{m'}$, we conclude the existence of an exceptional divisor on $\tilde{V}$ at which the discrepancy of $K_{V_m} + X_m$ is less than that of $K_{V_{m'}} + X_{m'}$, a contradiction to the assumption that $V_m \to V_{m'}$ is the identity. □

To apply the theorem above to the minimal model problem for a divisor on a toric variety, one needs the following lemma.

LEMMA 1.14 ([2, Lemma 2.7]). *Let $Y \subset Z$ be an irreducible Weil divisor on a variety Z. Assume that Z admits at worst $\mathbf{Q}$-factorial log-terminal singularities. Let $\varphi : \tilde{Y} \to Y$ be a resolution of singularities on Y. Assume $K_{\tilde{Y}} = \varphi^*((K_Z + Y)|_Y) + \sum_i m_i E_i$ with $m_i > -1$ for all i, where E_i's are the exceptional divisors of φ. Then Y is normal.*

COROLLARY 1.15. *Let V be a toric variety defined by a proper fan Δ and X a Δ-regular divisor on V. If $\kappa(X) \geq 0$, then X has a minimal model with abundance. If $\kappa(X) = -\infty$, then X is birationally equivalent to a proper variety Y with at worst terminal singularities and a fibration $\varphi : Y \to Z$ to a lower dimensional variety Z with $-K_Y$ relatively ample.*

PROOF. Let V_1 be the toric variety defined by a non-singular subdivision Δ_1 of Δ and X_1 the proper transform of X on V_1. Then X_1 is Δ_1-regular and therefore $K_{V_1} + X_1$ is toric terminal by Lemma 1.7. Then one obtains a sequence

$$V_1 \overset{\varphi_1}{\dashrightarrow} V_2 \overset{\varphi_2}{\dashrightarrow} \cdots \overset{\varphi_{r-1}}{\dashrightarrow} V_r$$

as in Theorem 1.13. One can prove that for each $j = 1, \ldots, r$, X_j has at worst terminal singularities. Indeed, take a morphism $\varphi : \tilde{V} \to V_j$ corresponding to a non-singular subdivision $\tilde{\Delta}$ of the fan Δ_j of V_j such that the proper transform $\tilde{X}$ of X is $\tilde{\Delta}$-regular. Then, as $K_{V_j} + X_j$ is toric terminal, it follows that

$$(K_{\tilde{V}} + \tilde{X})|_{\tilde{X}} = \varphi^*((K_{V_j} + X_j)|_{X_j}) + \sum_i a_i E_i|_{\tilde{X}} \quad (a_i > 0 \text{ for all } i)\,.$$

Here the left hand side is the canonical divisor $K_{\tilde{X}}$ of a non-singular variety $\tilde{X}$. Therefore by Lemma 1.14, one sees that X_j is normal. As V_j has at worst terminal singularities by Lemma 1.9, it is non-singular in codimension 2. Therefore $K_{V_j} + X_j|_{X_j} = K_{X_j}$, which yields that X_j has at worst terminal singularities by $a_i > 0$. By (iii) of Theorem 1.13 there are two cases for V_r.

Case 1. $K_{V_r} + X_r$ is nef.

Then the linear system $|m(K_{V_r} + X_r)|$ is base point free for some $m \in \boldsymbol{N}$. This is proved by a slight modification of the proof of the toric Nakai criterion (2.18, [6]). Therefore $|mK_{X_r}|$ is base point free, which implies that X_r is a minimal model with abundance. In this case, $\kappa(X) = \kappa(X_r) \geq 0$.

Case 2. There exists an extremal ray R on V_r such that $(K_{V_r} + X_r)R < 0$ and the elementary contraction $\varphi_R : V_r \to Z$ is a fibration to a lower dimensional variety Z.

In this situation, consider the subcases:

Subcase. $\dim X_r > \dim \varphi_R(X_r)$.

Let F be a fiber of φ_R. Then by (ii) of Proposition 1.10, F is a weighted projective space and $(K_{V_r} + X_r)C < 0$ for every curve C in F, which implies that $-(K_{V_r} + X_r)$ is relatively ample over Z. Hence $-K_{X_r}$ is relatively ample over $\varphi_R(X_r)$. This yields the equality $\kappa(X) = \kappa(X_r) = -\infty$, and $\varphi_R|_{X_r} : X_r \to \varphi_R(X_r)$ is the desired fibration.

Subcase. $\dim X_r = \dim \varphi_R(X_r)$.

In this case $\dim Z = \dim V_r - 1$ and every fiber l of $\varphi_R : V_r \to Z$ is $\boldsymbol{P}^1$ by (ii) of Proposition 1.10. Therefore $K_{V_r}l = -2$. On the other hand, because $\varphi|_{X_r}$ is generically finite, $X_r l > 0$. Here, since V_r has at worst terminal singularities by Lemma 1.9, the singular locus has codimension greater than 2. Therefore the divisor X_r is a Cartier divisor along a general fiber l, which yields that $X_r l$ is an integer. By $(K_{V_r} + X_r)l < 0$, we have $X_r l = 1$ which implies that $\varphi_R|_{X_r} : X_r \to Z$ is a birational morphism. Therefore X_r is rational. So X and X_r are birationally equivalent to $\boldsymbol{P}^n$ which has ample anti-canonical divisor and $\kappa(X) = -\infty$. □

COROLLARY 1.16. *Let the ground field k be of characteristic zero. Let V be a proper toric variety, Λ a linear system without base point and X a general member of Λ. Then the consequences of Corollary* 1.15 *hold for X.*

PROOF. By Bertini's theorem, X is Δ-regular. □

COROLLARY 1.17. *Let V be a toric variety defined by a proper fan Δ and X a Δ-regular divisor on V. Assume $\kappa(X) \geq 0$. Then there exists a non-singular subdivision $\tilde{\Delta}$ of Δ such that the toric variety $\tilde{V} = T_N(\tilde{\Delta})$ and the proper transform $\tilde{X}$ of X on $\tilde{V}$ satisfy the following*:

$$\kappa(\tilde{V}, K_{\tilde{V}} + \tilde{X}) \geq 0\,.$$

PROOF. Use the notation of the proof of Corollary 1.15. Take a non-singular subdivision $\tilde{\Delta}$ of both Δ and Δ_r which is the fan of V_r. Then the proper transform $\tilde{X}$ of X on

$\tilde{V} = T_N(\tilde{\Delta})$ is $\tilde{\Delta}$-regular. Since $K_{V_r} + X_r$ is toric terminal and $|m(K_{V_r} + X_r)|$ is base point free for some $m \in \boldsymbol{N}$,

$$0 \neq \Gamma(V_r, m(K_{V_r} + X_r)) \subset \Gamma(\tilde{V}, m(K_{\tilde{V}} + \tilde{X}))\,.$$

$\square$

2. Divisors and Polytopes.

2.1. Here we summarize the basic notion of an invariant divisor of a toric variety and the corresponding polytope which will be used in the next section. In this paper, a polytope in an $\boldsymbol{R}$-vector space means the intersection of finite number of half-spaces $\{\boldsymbol{m} \,|\, f_i(\boldsymbol{m}) \geq a_i\}$ for linear functions f_i.

2.2. Let M be the free abelian group $\boldsymbol{Z}^n$ $(n \geq 3)$ and N be the dual $\mathrm{Hom}_{\boldsymbol{Z}}(M, \boldsymbol{Z})$. We denote $M \otimes_{\boldsymbol{Z}} \boldsymbol{R}$ and $N \otimes_{\boldsymbol{Z}} \boldsymbol{R}$ by $M_{\boldsymbol{R}}$ and $N_{\boldsymbol{R}}$, respectively. Define $M_{\boldsymbol{Q}}$ and $N_{\boldsymbol{Q}}$ in the same way. Then one has the canonical pairing $(\ ,\ ) : N \times M \to \boldsymbol{Z}$, which can be canonically extended to $(\ ,\ ) : N_{\boldsymbol{R}} \times M_{\boldsymbol{R}} \to \boldsymbol{R}$. For a fan Δ in $N_{\boldsymbol{R}}$, we construct the toric variety $T_N(\Delta)$. The fan Δ is always assumed to be proper, i.e., Δ is finite and the support $|\Delta| = N_{\boldsymbol{R}}$. Denote by $\Delta(k)$ the set of k-dimensional cones in Δ. Denote by $\Delta[1]$ the set of primitive vectors $\boldsymbol{q} = (q_1, \ldots, q_r) \in N$ whose rays $\boldsymbol{R}_{\geq 0}\boldsymbol{q}$ belong to $\Delta(1)$. For $\boldsymbol{q} \in \Delta[1]$, denote by $D_{\boldsymbol{q}}$ the corresponding divisor which is the closure of $\mathrm{orb}\,\boldsymbol{R}_{\geq 0}\boldsymbol{q}$ in [5]. Denote by U_σ the invariant affine open subset which contains $\mathrm{orb}(\sigma)$ as a unique closed orbit.

DEFINITION 2.3. For $\boldsymbol{p} \in N_{\boldsymbol{R}}$ and a subset $K \subset M_{\boldsymbol{R}}$, define

$$\boldsymbol{p}(K) := \inf_{\boldsymbol{m} \in K} (\boldsymbol{p}, \boldsymbol{m})\,.$$

DEFINITION 2.4. Let Δ be a proper fan in $N_{\boldsymbol{R}}$. A continuous function $h : N_{\boldsymbol{R}} \to \boldsymbol{R}$ is called a Δ-support function, if

(1) $h\big|_\sigma$ is $\boldsymbol{R}$-linear for every cone $\sigma \in \Delta$ and

(2) h is $\boldsymbol{Q}$-valued on $N_{\boldsymbol{Q}}$.

A Δ-support function h is called integral if

(2′) h is $\boldsymbol{Z}$-valued on N.

PROPOSITION 2.5. *For a Δ-support function h, define $D_h = -\sum_{\boldsymbol{p} \in \Delta[1]} h(\boldsymbol{p}) D_{\boldsymbol{p}}$. Then the correspondence $h \mapsto D_h$ gives a bijective map*:

$\{\Delta$-support functions$\} \simeq \{$invariant $\boldsymbol{Q}$-Cartier divisors on $T_N(\Delta)\}$.

Here D_h is a Cartier divisor if and only if h is integral.

DEFINITION 2.6. For a Δ-support function h, define

$$\square_h := \{m \in M_{\boldsymbol{R}} \,|\, (\boldsymbol{p}, \boldsymbol{m}) \geq h(\boldsymbol{p}) \ \text{ for all } \boldsymbol{p} \in N_{\boldsymbol{R}}\}\,,$$

and call it the polytope associated with h or with D_h. Actually it is a polytope by Lemma 2.10 below and compact since the fan Δ is finite and proper.

PROPOSITION 2.7 (see [6]). *For an integral Δ-support function h, the following are equivalent*:

(i) *the linear system $|D_h|$ is base point free*;

(ii) *h is upper convex; i.e.,* $h(\boldsymbol{n}) + h(\boldsymbol{n}') \leq h(\boldsymbol{n} + \boldsymbol{n}')$ *for arbitrary* $\boldsymbol{n}, \boldsymbol{n}' \in N_{\boldsymbol{R}}$;

(iii) $\square_h$ *coincides with the convex hull of* $\{h_\sigma \big| \sigma \in \Delta(n)\}$, *where* h_σ *is a point of* M *which gives the linear function* $h\big|_\sigma$ *for* $\sigma \in \Delta(n)$.

PROPOSITION 2.8 (see [6]). *For a Δ-support function h, the following are equivalent:*

(i) *the $\boldsymbol{Q}$-Cartier divisor D_h is ample;*

(ii) *h is strictly upper convex; i.e., h is upper convex and* $h(\boldsymbol{n}) + h(\boldsymbol{n}') < h(\boldsymbol{n} + \boldsymbol{n}')$, *if there is no cone σ such that* $\boldsymbol{n}, \boldsymbol{n}' \in \sigma$;

(iii) $\square_h$ *is of dimension n and the correspondence* $\sigma \mapsto h_\sigma$ *gives a bijective map* $\Delta(n) \simeq \{$*the vertices of* $\square_h\}$, *where h_σ is the point of $M_{\boldsymbol{Q}}$ which gives the linear function* $h\big|_\sigma$ *for* $\sigma \in \Delta(n)$.

In the next section we will use the following simple lemmas.

LEMMA 2.9. *Let h be a Δ-support function. If* $h_\sigma \in \square_h$ *for every* $\sigma \in \Delta(n)$, *then* $h(\boldsymbol{p}) = \boldsymbol{p}(\square_h)$ *for every* $\boldsymbol{p} \in N_{\boldsymbol{R}}$, *and the polytope $\square_h$ is the convex hull of the set* $\{h_\sigma \big| \sigma \in \Delta(n)\}$.

LEMMA 2.10. *Let* $D_h = \sum_{\boldsymbol{p} \in \Delta[1]} m_{\boldsymbol{p}} D_{\boldsymbol{p}}$ *be an invariant divisor. Then*

$$\square_h = \bigcap_{\boldsymbol{p} \in \Delta[1]} \{\boldsymbol{m} \in M_{\boldsymbol{R}} \big| (\boldsymbol{p}, \boldsymbol{m}) \geq -m_{\boldsymbol{p}}\}.$$

DEFINITION 2.11. Let $\square$ be a polytope in $M_{\boldsymbol{R}}$ defined by $\bigcap_{i=1}^{r} H_i$, where $H_i = \{\boldsymbol{m} \in M_{\boldsymbol{R}} \big| (\boldsymbol{p}_i, \boldsymbol{m}) \geq a_i\}$. We say that H_i contributes to $\square$, if $\square \cap \{\boldsymbol{m} \in M_{\boldsymbol{R}} \big| (\boldsymbol{p}_i, \boldsymbol{m}) = a_i\} \neq \emptyset$. We say that H_i contributes properly to $\square$ if $\bigcap_{j \neq i} H_j \neq \square$.

DEFINITION 2.12. Let $\square$ be an n-dimensional compact polytope in $M_{\boldsymbol{R}}$. Define the dual fan $\Gamma_\square$ of $\square$ as follows: $\Gamma_\square = \{\gamma^*\}$, where γ is a face of $\square$ and $\gamma^* := \{\boldsymbol{n} \in N_{\boldsymbol{R}} \big|$ the function $\boldsymbol{n}\big|_\square$ attains its minimal value at all points of $\gamma\}$. Then $\Gamma_\square$ turns out to be a proper fan.

2.13. If Δ is the dual fan of the polytope $\square_h$ corresponding to a Δ-support function h, then by Proposition 2.8 D_h is ample, hence the variety $T_N(\Delta)$ is a projective variety.

3. The construction of a minimal model.

3.1. In this section we concretely construct a projective minimal model with abundance for a Δ-regular divisor X with $\kappa(X) \geq 0$ on a toric variety $T_N(\Delta)$ by means of the polytope of the adjoint divisor. Let V be a toric variety defined by a proper fan Δ and X a Δ-regular divisor with $\kappa(X) \geq 0$. To construct a minimal model of X we may assume that V is non-singular and $\kappa(V, K_V + X) \geq 0$, by Corollary 1.17.

3.2. Pursuing elementary contractions and flips is like groping for a minimal model in the dark. The reason why the discussion of this section goes well without contractions or flips is because in toric geometry every exceptional divisor is visible as a vector in the space N. Then one can prepare so that every discrepancy of the adjoint divisor is positive (cf. 3.7), which makes the singularities terminal. In the discussion one puffed up the polytope of the adjoint divisor and took its dual fan Σ. This implies that in $T_N(\Sigma)$ the adjoint divisor is the limit of a sequence of ample divisors (cf. 3.5), which makes the adjoint divisor nef; or equivalently semi-ample. Forgetting non-contributing half space in 3.2 corresponds to divisorial contractions. The other procedures correspond to putting together of flips.

3.3. The construction. Let h be a Δ-support function such that $K_{T_N(\Delta)} + X \sim D_h$. Then, by $\kappa(T_N(\Delta), K_{T_N(\Delta)} + X) \geq 0$, it follows that $\square_h \neq \emptyset$. Let $\Delta[1] = \{\boldsymbol{p}_1, \dots, \boldsymbol{p}_s\}$ and $H_i = \{\boldsymbol{m} \in M_{\boldsymbol{R}} \big| (\boldsymbol{p}_i, \boldsymbol{m}) \geq h(\boldsymbol{p}_i)\}$. Then by Lemma 2.10 $\square_h = \bigcap_{i=1}^s H_i$. Assume that $H_1, \dots, H_r$ $(r \leq s)$ are all that contribute to $\square_h$. For $\varepsilon_i > 0$, $i = 1, \dots, r$, define $H_{i,\varepsilon_i} := \{\boldsymbol{m} \in M_{\boldsymbol{R}} \big| (\boldsymbol{p}_i, \boldsymbol{m}) \geq h(\boldsymbol{p}_i) - \varepsilon_i\}$, $\partial H_{i,\varepsilon_i} := \{\boldsymbol{m} \in M_{\boldsymbol{R}} | (\boldsymbol{p}_i, \boldsymbol{m}) = h(\boldsymbol{p}_i) - \varepsilon_i\}$ and $\square(\varepsilon) := \bigcap_{i=1}^r H_{i,\varepsilon_i}$, where $\varepsilon = (\varepsilon_1, \dots, \varepsilon_r)$. Here one should note that the polytope $\square_h$ may not be of the maximal dimension. By "puffing this up", one gets a polytope $\square(\varepsilon)$ of the maximal dimension. The subset $Z = \{\varepsilon \in \boldsymbol{R}^r_{>0} \big| \bigcup \partial H_{i,\varepsilon_i}$ is not with normal crossings$\}$ is Zariski closed and the complement $\boldsymbol{R}^r_{>0} \setminus Z$ is divided into finite number of chambers. Take a chamber W such that:

(3.2.1) $0 \in \bar{W}$;

(3.2.2) every H_{i,ε_i} $(i = 1, \dots, r)$ contributes properly to $\square(\varepsilon)$ for $\varepsilon \in W$.

Then the dual fan Σ of $\square(\varepsilon)$ is common for every $\varepsilon \in W$ and it is simplicial, because $\bigcup \partial H_{i,\varepsilon_i}$ is with normal crossings. Let $X(\Sigma)$ be the proper transform of X in $T_N(\Sigma)$. We claim that $X(\Sigma)$ is a minimal model of X with abundance. One can see that $T_N(\Sigma)$ is projective, because the invariant $\boldsymbol{Q}$-Cartier divisor $\sum_{\boldsymbol{p}_i \in \Sigma[1]} (h(\boldsymbol{p}_i) - \varepsilon_i) D_{\boldsymbol{p}_i}$ with all ε_i rational and $\varepsilon \in W$ is ample since Σ is the dual fan of the polytope corresponding to this divisor (cf. 2.13). Hence the projectivity of $X(\Sigma)$ follows automatically.

3.4. Now we are going to prove that $X(\Sigma)$ satisfies the desired conditions for a minimal model. First note that $\Sigma[1] = \{\boldsymbol{p}_1, \dots, \boldsymbol{p}_r\}$, by 3.2.2. Next note that every $\boldsymbol{Q}$-Weil divisor on $T_N(\Sigma)$ is a $\boldsymbol{Q}$-Cartier divisor, because Σ is simplicial and therefore $T_N(\Sigma)$ has at worst quotient singularities.

CLAIM 3.5. *The divisor $K_{T_N(\Sigma)} + X(\Sigma)$ is linearly equivalent to an invariant divisor $-\sum_{i=1}^r h(\boldsymbol{p}_i) D_{\boldsymbol{p}_i}$. Let k be the Σ-support function corresponding to this divisor. Then $h(\boldsymbol{p}_i) = k(\boldsymbol{p}_i)$ for $i = 1, \dots, r$ and $\square_h = \square_k$.*

PROOF. The first assertion follows from the fact that the divisor $K_{T_N(\Sigma)} + X(\Sigma)$ is the proper transform of $K_{T_N(\Delta)} + X \sim -\sum_{i=1}^s h(\boldsymbol{p}_i) D_{\boldsymbol{p}_i}$. The second assertion is obvious and the last assertion follows from Lemma 2.10 and the fact that $H_1, \dots, H_r$ are all that contribute to $\square_h$. □

CLAIM 3.6. *$k_\sigma \in \square_k$, for all $\sigma \in \Sigma(n)$.*

PROOF. Let $\{\varepsilon^{(m)}\}_m$ be a series of rational points in W which converges to 0. Let $k^{(m)}$ be the Σ-support function corresponding to a $\boldsymbol{Q}$-Cartier divisor $\sum_{i=1}^r (-h(\boldsymbol{p}_i) + \varepsilon^{(m)}{}_i) D_{\boldsymbol{p}_i}$. Then by Lemma 2.10 it follows that $\square_{k^{(m)}} = \square(\varepsilon^{(m)})$, and therefore by 2.13 the divisor is ample. Replacing $\{\varepsilon^{(m)}\}_m$ by a suitable subsequence, one can assume that there exists $\lim_{m \to \infty} k^{(m)}{}_\sigma$ for every $\sigma \in \Sigma(n)$. Indeed, replacing it by a suitable subsequence, one may assume that $\varepsilon^{(m)}{}_i \geq \varepsilon^{(m+1)}{}_i$ for every i, hence $\square_{\varepsilon^{(m)}} \supset \square_{\varepsilon^{(m+1)}} \supset \cdots$; therefore for every $\sigma \in \Sigma(n)$ and m it follows that $k^{(m)}{}_\sigma \in \square_{\varepsilon^{(1)}}$ which is compact; so $\{k^{(m)}{}_\sigma\}$ has an accumulating point. The point $k'_\sigma := \lim_{m\to\infty} k^{(m)}{}_\sigma$ belongs to $\Box_k$, because the ampleness of $D_{k^{(m)}}$ yields $k^{(m)}{}_\sigma \in \Box(\varepsilon^{(m)})$. The collection $\{k'_\sigma\}_{\sigma\in\Sigma(n)}$ defines a function k' on $N_{\boldsymbol{R}}$. Indeed, for every m, $k^{(m)}{}_\sigma = k^{(m)}{}_\tau$ as a function on $\sigma\cap\tau$, which yields that $k'_\sigma = k'_\tau$ as a function on $\sigma\cap\tau$. Now one obtains that $k' = k$. This is proved as follows: for every $\boldsymbol{p}_i \in \Sigma[1]$ take $\sigma \in \Sigma(n)$ such that $\boldsymbol{p}_i \in \sigma$; $k'(\boldsymbol{p}_i) = (\boldsymbol{p}_i, k'_\sigma) = \lim_{m\to\infty}(\boldsymbol{p}_i, k^{(m)}{}_\sigma) = \lim_{m\to\infty}(h(\boldsymbol{p}_i) - \varepsilon^{(m)}{}_i) = h(\boldsymbol{p}_i) = k(\boldsymbol{p}_i)$, since $k^{(m)}{}_\sigma$ is on the hyperplane $(\boldsymbol{p}_i, \boldsymbol{m}) = h(\boldsymbol{p}_i) - \varepsilon^{(m)}{}_i$. Hence it follows that $k = k'$ and therefore $k_\sigma = k'_\sigma$ for every $\sigma \in \Sigma(n)$, which shows that $k_\sigma \in \Box_k$. $\Box$

Now by Lemma 2.9 and Proposition 2.7 the linear system $|mD_k| = |m(K_{T_N(\Sigma)} + X(\Sigma))|$ has no base point for such m that mD_k is a Cartier divisor.

3.7. Let $\tilde{\Sigma}$ be a non-singular subdivision of both Σ and Δ. Let

$$\begin{array}{ccc} & & T_N(\Delta) \\ & \overset{\psi}{\nearrow} & \\ T_N(\tilde{\Sigma}) & & \\ & \underset{\varphi}{\searrow} & \\ & & T_N(\Sigma) \end{array}$$

be the corresponding morphisms and $X(\tilde{\Sigma})$ the proper transform of X in $T_N(\tilde{\Sigma})$. Since $X(\tilde{\Sigma})$ is $\tilde{\Sigma}$-regular, it is non-singular and $\varphi\big|_{X(\tilde{\Sigma})}$ is birational.

CLAIM 3.8. *It follows that*

$$K_{T_N(\tilde{\Sigma})} + X(\tilde{\Sigma}) = \varphi^*(K_{T_N(\Sigma)} + X(\Sigma)) + \sum_{\boldsymbol{p}\in\tilde{\Sigma}[1]\setminus\Sigma[1]} m_{\boldsymbol{p}} D_{\boldsymbol{p}}\,,$$

where $m_{\boldsymbol{p}} > 0$ *for* $\boldsymbol{p} \in \tilde{\Sigma}[1] \setminus \Sigma[1]$.

PROOF. Denote

$$K_{T_N(\tilde{\Sigma})} + X(\tilde{\Sigma}) = \psi^*(K_{T_N(\Delta)} + X) + \sum_{\boldsymbol{p}\in\tilde{\Sigma}[1]\setminus\Delta[1]} \alpha_{\boldsymbol{p}} D_{\boldsymbol{p}}\,.$$

Then $\alpha_{\boldsymbol{p}} > 0$ for $\boldsymbol{p} \in \tilde{\Sigma}[1] \setminus \Delta[1]$, since X is non-singular. Putting $\alpha_{\boldsymbol{p}} = 0$ for $\boldsymbol{p} \in \Delta[1]$, one obtains that $K_{T_N(\tilde{\Sigma})} + X(\tilde{\Sigma}) \sim \sum_{\boldsymbol{p}\in\tilde{\Sigma}[1]}(-h(\boldsymbol{p}) + \alpha_{\boldsymbol{p}})D_{\boldsymbol{p}}$, as $K_{T_N(\Delta)} + X \sim D_h$. On the other hand,

$$K_{T_N(\tilde{\Sigma})} + X(\tilde{\Sigma}) = \varphi^*(K_{T_N(\Sigma)} + X(\Sigma)) + \sum_{\boldsymbol{p}\in\tilde{\Sigma}[1]\setminus\Sigma[1]} m_{\boldsymbol{p}} D_{\boldsymbol{p}}\,.$$

Letting $m_{\boldsymbol{p}} = 0$ for $\boldsymbol{p} \in \Sigma[1]$, one obtains that $K_{T_N(\tilde{\Sigma})} + X(\tilde{\Sigma}) \sim \sum_{\boldsymbol{p}\in\tilde{\Sigma}[1]}(-k(\boldsymbol{p}) + m_{\boldsymbol{p}})D_{\boldsymbol{p}}$, as $K_{T_N(\Sigma)} + X(\Sigma) \sim D_k$.

Therefore $\sum_{\boldsymbol{p}\in\tilde{\Sigma}[1]}(-h(\boldsymbol{p}) + \alpha_{\boldsymbol{p}})D_{\boldsymbol{p}} \sim \sum_{\boldsymbol{p}\in\tilde{\Sigma}[1]}(-k(\boldsymbol{p}) + m_{\boldsymbol{p}})D_{\boldsymbol{p}}$. As $h(\boldsymbol{p}) = k(\boldsymbol{p})$ and $\alpha_{\boldsymbol{p}} = m_{\boldsymbol{p}} = 0$ for $\boldsymbol{p} \in \Sigma[1]$, one obtains that

$$\sum_{\boldsymbol{p}\in\tilde{\Sigma}[1]\setminus\Sigma[1]} ((-h(\boldsymbol{p}) + \alpha_{\boldsymbol{p}}) - (-k(\boldsymbol{p}) + m_{\boldsymbol{p}}))D_{\boldsymbol{p}} \sim 0\,.$$

Here $D_{\boldsymbol{p}}$ ($\boldsymbol{p} \in \tilde{\Sigma}[1] \setminus \Sigma[1]$) are all exceptional for φ. Then the divisor above is not only linearly equivalent to 0 but also equal to 0. Therefore $(-h(\boldsymbol{p}) + \alpha_{\boldsymbol{p}}) - (-k(\boldsymbol{p}) + m_{\boldsymbol{p}}) = 0$

for every $p \in \tilde{\Sigma}[1] \setminus \Sigma[1]$, where $k(p) = p(\Box_h)$ by $k_\sigma \in \Box_k = \Box_h$ and Lemma 2.9. For $p \in \tilde{\Sigma}[1] \setminus \Delta[1]$, $m_p = p(\Box_h) - h(p) + \alpha_p \geq \alpha_p > 0$. For $p \in \Delta[1] \setminus \Sigma[1]$, it follows that $m_p = p(\Box_h) - h(p) > 0$, because $\{m | (p, m) \geq h(p)\}$ does not contribute to $\Box_h$ by the definition of Σ (cf. 3.2). $\Box$

3.9. Since $T_N(\Sigma)$ has at worst quotient singularities, one can apply Lemma 1.14 to our situation and obtain that $X(\Sigma)$ is normal. By Claim 3.8 and Lemma 1.9, $T_N(\Sigma)$ has at worst terminal singularities. Therefore $K_{T_N(\Sigma)} + X(\Sigma)\big|_{X(\Sigma)} = K_{X(\Sigma)}$. Then by restricting the equality 3.8 onto $X(\tilde{\Sigma})$, one obtains that $X(\Sigma)$ has at worst terminal singularities. The linear system of $mK_{X(\Sigma)} = m(K_{T_N(\Sigma)} + X(\Sigma))\big|_{X(\Sigma)}$ $(m \gg 0)$ has no base point, because $|m(K_{T_N(\Sigma)} + X(\Sigma))|$ is base point free as is noted after the proof of 3.5. This completes the proof that $X(\Sigma)$ is a projective minimal model with abundance.

4. Examples. In this section the base field k is always assumed to be of characteristic zero. Let M be $\boldsymbol{Z}^3$ and N be its dual.

EXAMPLE 4.1. Let p_i $(i = 1, \dots, 6)$ and q_j $(j = 1, \dots, 8)$ be points in N as follows: $p_1 = (1,0,0)$, $p_2 = (-1,0,0)$, $p_3 = (0,1,0)$, $p_4 = (0,-1,0)$, $p_5 = (0,0,1)$, $p_6 = (0,0,-1)$, $q_1 = (1,1,1)$, $q_2 = (-1,-1,-1)$, $q_3 = (1,1,-1)$, $q_4 = (-1,-1,1)$, $q_5 = (1,-1,1)$, $q_6 = (-1,1,-1)$, $q_7 = (-1,1,1)$, $q_8 = (1,-1,-1)$. Let them generate one-dimensional cones $\boldsymbol{R}_{\geq 0}p_i, \boldsymbol{R}_{\geq 0}q_j$ and construct a fan Δ with these cones as in Figure 1. Here note that Figure 1 is the picture of the fan which is cut by a hypersphere with the center at the origin and unfolded onto the plane. This fan is the dual fan of the polytope in Figure 2 and it is easy to check that it is non-singular. Let X be a general member of a base point free linear system $|\sum_{i=1}^{6} D_{p_i} + 2\sum_{j=1}^{8} D_{q_j}|$. Let h be the Δ-support function such that $K_{T_N(\Delta)} + X \sim D_h$. Then the polytope $\Box_h$ is one point $\bigcap_{i=1}^{6}\{m | (p_i, m) \geq 0\}$ and the half spaces contributing

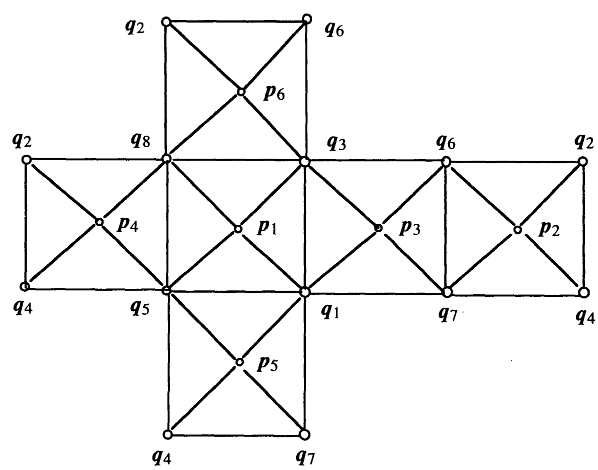

FIGURE 1.

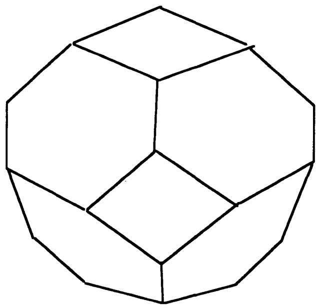

FIGURE 2.

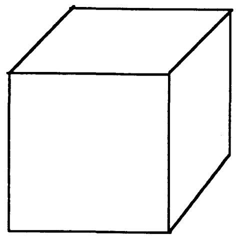

FIGURE 3.

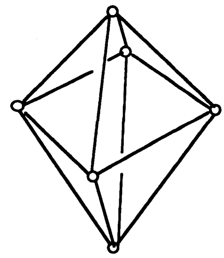

FIGURE 4.

to this polytope are $\{\boldsymbol{m} | (\boldsymbol{p}_i, \boldsymbol{m}) \geq 0\}$, $i = 1, \ldots, 6$, because $K_{T_N(\Delta)} + X \sim \sum D_{\boldsymbol{q}_j}$. Therefore, for a sufficiently small general ε, the polytope $\square(\varepsilon) = \bigcap_i \{\boldsymbol{m} \in M_{\boldsymbol{R}} | (\boldsymbol{p}_i, \boldsymbol{m}) \geq -\varepsilon_i\}$ is a hexahedron whose picture is as in Figure 3. The dual fan Σ of $\square(\varepsilon)$ (Figure 4) gives a minimal model $X(\Sigma)$ of X. Since $K_{T_N(\Sigma)} + X(\Sigma) \sim 0$, it follows that $\kappa(X) = 0$.

EXAMPLE 4.2. Let $\boldsymbol{p}_i$ and $\boldsymbol{q}_j$ be as in Example 4.1 and Δ the fan with the cones generated by these vectors as in Figure 5. This fan is the dual fan of the polytope in Figure 6 and is easily checked to be non-singular. Let X be a general member of a base point free linear system $|2D_{\boldsymbol{p}_1} + 2D_{\boldsymbol{p}_2} + \sum_{i=3}^{6} D_{\boldsymbol{p}_i} + 3\sum_{j=1}^{8} D_{\boldsymbol{q}_j}|$. Let h be the Δ-support function such that $K_{T_N(\Delta)} + X \sim D_h$. Then the polytope $\square_h$ is a segment $\bigcap_{i=1}^{2} \{\boldsymbol{m} | (\boldsymbol{p}_i, \boldsymbol{m}) \geq -1\} \cap$

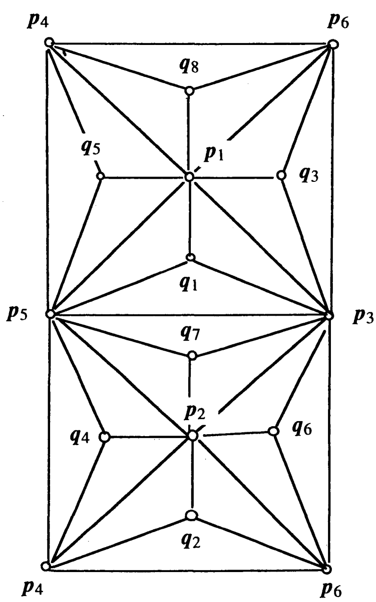

FIGURE 5.

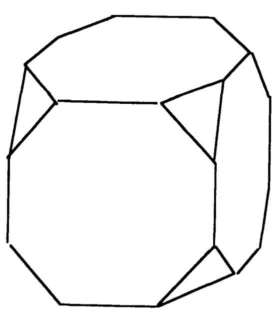

FIGURE 6.

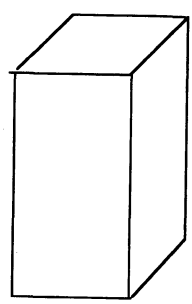

FIGURE 7.

$\bigcap_{i=3}^{6}\{\boldsymbol{m}|(\boldsymbol{p}_i,\boldsymbol{m})\geq 0\}$ and the half spaces contributing to this polytope are $\{\boldsymbol{m}|(\boldsymbol{p}_i,\boldsymbol{m})\geq -1\}$ $(i = 1,2)$ and $\{\boldsymbol{m}|(\boldsymbol{p}_i,\boldsymbol{m})\geq 0\}$ $(i = 3,\dots,6)$, because $K_{T_N(\Delta)}+X \sim D_{\boldsymbol{p}_1}+D_{\boldsymbol{p}_2}+2\sum D_{\boldsymbol{q}_j}$. Therefore, for a sufficiently small general ε, the polytope $\square(\varepsilon) = (\bigcap_{i=1}^{2}\{\boldsymbol{m}\in M_{\boldsymbol{R}}|(\boldsymbol{p}_i,\boldsymbol{m})\geq -1-\varepsilon_i\})\cap(\bigcap_{i=3}^{6}\{\boldsymbol{m}\in M_{\boldsymbol{R}}|(\boldsymbol{p}_i,\boldsymbol{m})\geq -\varepsilon_i\})$ is a hexahedron whose picture is as in Figure 7. The dual fan Σ of $\square(\varepsilon)$ (Figure 4) gives a minimal model $X(\Sigma)$ of X. Since $\square_h$ is of dimension one, $\dim \Gamma(T_N(\Sigma), m(K_{T_N(\Sigma)}+X(\Sigma)))$ grows in order 1, and therefore $\dim \Phi_{|m(K_{T_N(\Sigma)}+X(\Sigma))|}(T_N(\Sigma)) = 1$. This shows that $\dim \Phi_{|mK_{X(\Sigma)}|}(X(\Sigma)) \leq 1$. As the dual fan of the polytope of $X(\Sigma)\sim 2D_{\boldsymbol{p}_1}+2D_{\boldsymbol{p}_2}+\sum_{i=3}^{6}D_{\boldsymbol{p}_i}$ is Σ, $X(\Sigma)$ is ample by 2.13. Hence $X(\Sigma)$ intersects all fibers of $\Phi_{|m(K_{T_N(\Sigma)}+X(\Sigma))|}$, which shows that $\kappa(X) = 1$.

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