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ON MUMFORD'S CONSTRUCTION OF DEGENERATING ABELIAN VARIETIES

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Abstract. For a one-dimensional family of abelian varieties equipped with principal theta divisors a canonical limit is constructed as a pair consisting of a reduced projective variety and a Cartier divisor on it. Properties of such pairs are established.

CONTENTS

Introduction. Assume that we are given a 1-parameter family of principally polarized abelian varieties with theta divisors. By this we will mean that we are in one of the following situations:

1. $\mathcal{R}$ is a complete discrete valuation ring (DVR, for short) with the fraction field K, $S = \operatorname{Spec}\mathcal{R}$, $\eta = \operatorname{Spec}K$ is the generic point, and we have an abelian variety G_η over K together with an effective ample divisor Θ_η defining a principal polarization; or
2. we have a projective family (G, Θ) over a small punctured disk D_ε^0.

In this paper we show that, possibly after a finite ramified base change, the family can be completed in a simple and absolutely canonical manner to a projective family (P, Θ) with a relatively ample Cartier divisor Θ over S, resp. D_ε. Moreover, this construction is stable under further finite base changes. We give a combinatorial description of this family and its central fiber (P_0, Θ_0) and study their basic properties. In particular, we prove that P_0 is reduced and Cohen-Macaulay and that $H^i(P_0, \mathcal{O}(d\Theta_0))$, $d \geq 0$ are the same as for an ordinary PPAV (principally polarized abelian variety).

Existence of such construction has profound consequences for the moduli theory. Indeed, with it one must expect that there exists a canonical compactification $\bar{A}_g$ of the moduli space A_g of PPAVs, similar to the Mumford-Deligne compactification of the moduli space of curves. Without it, one has to believe that there is no single "best" geometrically meaningful

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compactification of A_g and work with the infinitely many toroidal compactifications instead. The moduli implications of our construction are explored in [Ale].

Degenerations of abelian varieties have been studied exhaustively which makes our result somewhat surprising. There is a very complete description of degenerations of polarized abelian varieties of arbitrary degree of polarization over a complete Noetherian domain of arbitrary dimension, with or without an ample line bundle. This description is called Mumford's construction, which was first published in a beautiful short paper [Mum72] and later substantially expanded and improved by Faltings and Chai in [Fal85, Cha85, FC90] (we note a parallel construction of Raynaud which works in the context of rigid analytic geometry). Mumford's construction gives an equivalence of categories $\mathrm{DEG}_{\mathrm{pol}}$, resp. $\mathrm{DEG}_{\mathrm{ample}}$ of degenerations of polarized abelian varieties, resp. with a line bundle, and the categories $\mathrm{DD}_{\mathrm{pol}}$, resp. $\mathrm{DD}_{\mathrm{ample}}$ of the "degeneration data".

As an auxiliary tool, Mumford's construction uses *relatively complete models*. Mumford remarks that such a model "is neither unique nor canonical" and that "in fact, the non-uniqueness of $\tilde{P}$ gives one freedom to seek for the most elegant solutions in any particular case". What we show in this paper is that if one is willing to give up some of the properties of $\tilde{P}$ and concentrate on the others, then in fact there is a canonical choice! Here is what we do:

1. We only consider the case of a 1-dimensional base S. This certainly makes the problem easier but not significantly. In view of the moduli theory one shouldn't expect that a higher-dimensional family can be canonically completed, unless one is in a very special situation, such as for a "test family" over a special toric scheme.
2. We allow an additional finite ramified base change $S' \to S$, even after one already has the semiabelian reduction. This, again, is perfectly natural from the moduli point of view.
3. Most importantly, we do not care where in the central fiber the limit of the zero section of G_η ends up. Our relatively complete model contains a semiabelian group scheme in many different ways, but the closure of the zero section need not be contained in any of them. Hence, P_0 is a limit of G_η as an abelian torsor, not as an abelian variety.
4. Instead of a section, we pay a very special attention to the limit of the theta divisor Θ_η, something which was overlooked in the previous constructions.

For the most part of the paper we work in the algebraic situation, over a complete DVR. The complex-analytic case is entirely analogous, and we explain the differences in Section 5.

Shortly after [Mum72] appeared, a series of works of Namikawa and Nakamura [Nam76, Nam77, Nam79, Nam80, Nak75, Nak77] was published that dealt with the complex-analytic situation. They contain a toric construction for an extended 1-parameter family. This construction is very similar to Mumford's, and the main difference is a substitute for the relatively complete model. One unpleasant property of that substitute is that in dimension $g \geq 5$ the central fiber need not be reduced.

When restricted to the complex-analytic setting, our construction has a lot in common with the Namikawa-Nakamura construction as well. The main difference is again the fact that we use and pay special attention to the theta divisor. Our solution to the problems arising in dimension $g \geq 5$ is simple—a base change. To underline the degree of dependence on the previous work, we call our construction *simplified Mumford's construction* and we call the central fibers *stable quasiabelian varieties*, or SQAVs, following Namikawa. We call the pairs (P_0, Θ_0) *stable quasiabelian pairs*, or SQAP.

We note that Namikawa had constructed families $\mathcal{X}_g^{(2n)}$ over the Voronoi compactification $\bar{A}_{g,1,2n}^{\mathrm{VOR}}$ of the moduli space $A_{g,1,2n}$ of PPAVs with a principal level structure of level $2n$, $n \geq 3$. The boundary fibers in these families are different for different n, and some of them are non-reduced when $g \geq 5$.

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1. Delaunay and Voronoi polyhedral decompositions. The structure of the extended family will be described explicitly in terms of two polyhedral decompositions which we now introduce.

NOTATION 1.1. $X \simeq \boldsymbol{Z}^r$ will denote a lattice in a real vector space $X_{\boldsymbol{R}} = X \otimes \boldsymbol{R}$, for a fixed positive integer r. $B : X \times X \to \boldsymbol{R}$ will be a symmetric bilinear form assumed to be positive definite. We denote the norm $\sqrt{B(x,x)}$ by $\|x\|_B$ or simply by $\|x\|$.

DEFINITION 1.2. For an arbitrary $\alpha \in X_{\boldsymbol{R}}$ we say that a lattice element $x \in X$ is α-nearest if

$$\|x - \alpha\|_B = \min\{\|x' - \alpha\|_B \,\big|\, x' \in X\}.$$

We define a B-Delaunay cell σ (or simply a *Delaunay cell* if B is understood) to be the closed convex hull of all lattice elements which are α-nearest for some fixed $\alpha \in X_{\boldsymbol{R}}$. Note that for a given Delaunay cell σ the element α is uniquely defined only if σ has the maximal possible dimension, equal to r. In this case α is called the *hole* of σ, cf. Section 2.1.2 of the "encyclopedia of sphere packings and lattices" [CS93]. One should imagine a sphere around the α-closest lattice elements (which is known as "the empty sphere" because there are no other lattice elements in its interior) with α at the center.

Together all the Delaunay cells constitute a locally finite decomposition of $X_{\boldsymbol{R}}$ into infinitely many bounded convex polytopes which we call the *Delaunay cell decomposition* Del_B.

REMARK 1.3. It is clear from the definition that the Delaunay decomposition is invariant under translation by the lattice X and that the 0-dimensional cells are precisely the elements of X.

DEFINITION 1.4. For a given B-Delaunay cell σ consider all $\alpha \in X_{\boldsymbol{R}}$ that define σ. They themselves form a locally closed bounded convex polytope. We denote the closure of this polytope by $\hat{\sigma} = V(\sigma)$ and call it the B-Voronoi cell or simply the *Voronoi cell*. The Voronoi cells make up the *Voronoi cell decomposition* Vor_B of $X_{\boldsymbol{R}}$.

It is easy to see that the Delaunay and Voronoi cells are dual to each other in the following sense:

LEMMA 1.5. (i) *For a fixed form B there is a* 1*-to-*1 *correspondence between Delaunay and Voronoi cells given by* $\hat{\sigma} = V(\sigma), \sigma = D(\hat{\sigma})$.

(ii) $\dim \sigma + \dim \hat{\sigma} = r$.

(iii) $\sigma \subset \tau$ *if and only if* $\hat{\tau} \subset \hat{\sigma}$.

(iv) *For* $\sigma = x \in X$ *the corresponding Voronoi cell* $V(x)$ *has the maximal dimension.* $V(x)$ *is the set of points of* $X_{\boldsymbol{R}}$ *that are at least as close to* x *as to any other lattice element* x'.

(v) *For an arbitrary Delaunay cell* σ *the dual Voronoi cell* $\hat{\sigma}$ *is the polytope with vertices at holes* $\alpha(\tau)$ *where* τ *goes over all maximal-dimensional Delaunay cells containing* σ.

REMARK 1.6. With the definition this natural, it is no wonder that Voronoi and Delaunay cells have a myriad of applications in physics, chemistry and even geography ([OBS92], many more references in [CS93]) and go by many different names. Some alternative names for Voronoi cells are: Voronoi polytopes, nearest neighbor regions, Dirichlet regions, Brillouin zones, Wigner-Seitz cells. Delaunay cells had been called by various authors Delone cells, Delony cells, L-polytopes. Evidently, even the spelling of the last name of Boris Nikolaevich Delone is not agreed upon. A quick computer database search shows that the French variant "Delaunay" is preferred in 99% of all papers, so this is our choice too.

EXAMPLE 1.7. The Figure 1 and Figure 2 give the only two, up to the action of $SL(2, \boldsymbol{Z})$. Delaunay decompositions of $\boldsymbol{Z}^2$. The broken lines show the corresponding Voronoi decompositions. Note that, unlike Delaunay, the Voronoi decompositions may have some continuous moduli.

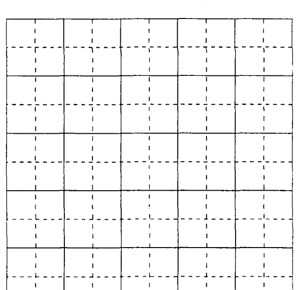

FIGURE 1.

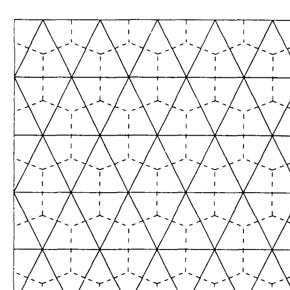

FIGURE 2.

Here is another way to understand the Delaunay decompositions.

LEMMA 1.8. *Consider an* $(r+1)$*-dimensional real vector space* $X_{\boldsymbol{R}} \oplus \boldsymbol{R}$ *with coordinates* (x, x_0) *and a paraboloid in it defined by the equation*

$$x_0 = A(x) = B(x,x)/2 + lx/2$$

for some $l \in X_{\boldsymbol{R}}^*$. Consider the convex hull Q of countably many points on this paraboloid with $x \in X$.
This object has a multifaceted shape and projections of the facets onto $X_{\boldsymbol{R}}$ are precisely the B-Delaunay cells. The equations that cut out the cone at the vertex $(c, A(c))$ are

$$x_0 - A(c) \geq B(\alpha(\sigma), x) + lx/2 = dA(\alpha(\sigma))(x),$$

where σ goes over all the maximal-dimensional Delaunay cells containing c. If σ is a maximal-dimensional Delaunay cell, then the interior normal in the dual space $X_{\boldsymbol{R}}^* \oplus \boldsymbol{R}$ to the corresponding facet is $(1, -dA(\alpha(\sigma)))$. The normal fan to the paraboloid consist of $\{0\}$ and the cones over the shifted Voronoi decomposition $(1, -dA(\mathrm{Vor}_B))$.

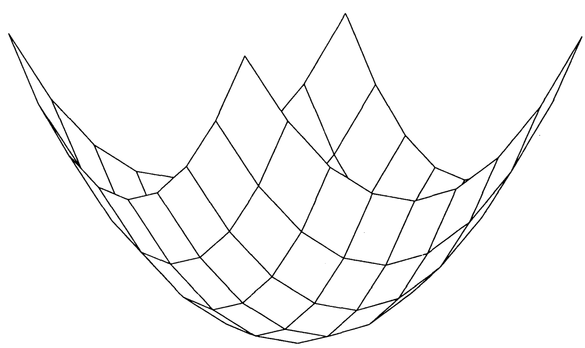

FIGURE 3. The paraboloid.

The inequality for x_0 does not depend on the cell $\sigma \ni x$ chosen because for any Delaunay vector $v \in \sigma_1 \cap \sigma_2$

$$B(\alpha(\sigma_1), v) = B(v, v)/2 = B(\alpha(\sigma_2), v)$$

(see Remark 1.9). We look at dA here as being a map from $X_{\boldsymbol{R}}$ to $X_{\boldsymbol{R}}^*$.

PROOF. We claim that the hyperplanes that cut out the facets at the origin are in 1-to-1 correspondence with the maximal-dimensional Delaunay cells containing 0. Let $\sigma \ni 0$ be one of such cells with a hole α. Then the points $(x, A(x))$ with $x \in \sigma$ lie on the hyperplane $x_0 = B(\alpha, x) + lx/2$ and those with $x \notin \sigma$ lie above it. Indeed,

$$A(x) - B(\alpha, x) - lx/2 = B(x, x)/2 - B(\alpha, x) = (\|\alpha - x\|^2 - \|\alpha - 0\|^2)/2 \geq 0$$

and the equality holds if and only if $x \in \sigma$. The other vertices are checked similarly. □

REMARK 1.9. As the proof shows, for a maximal-dimensional Delaunay cell the hole α is the unique solution of the system of linear equations

$$2B(\alpha, x) = B(x, x), \quad x \in \sigma \cap X.$$

DEFINITION 1.10. We will denote the minimum of the functions $dA(\alpha(\sigma))(x)$ in the above lemma by $\eta(x, c)$.

REMARK 1.11. Note that all $\eta(x, c)$ are integral-valued on X if and only if all $\alpha(\sigma)$ are integral, i.e., belong to X^*.

DEFINITION 1.12. The union of the Delaunay cells containing a lattice vertex c is called the *star of Delaunay cells* and denoted by $\mathrm{Star}(c)$. For a cell $\sigma \subset \mathrm{Star}(0)$ the nonzero lattice elements $v_1 \dots v_m \in \sigma$ are called the Delaunay vectors. In addition, we introduce

$$\mathrm{Cone}(0, \sigma) = \boldsymbol{R}_+ v_1 + \cdots + \boldsymbol{R}_+ v_m ,$$

a convex cone with the vertex at the origin. By translation we obtain cones $\mathrm{Cone}(c, \sigma)$ with vertices at other lattice elements. The function $\eta(x, c)$ is linear on each $\mathrm{Cone}(c, \sigma)$, $\sigma \subset \mathrm{Star}(c)$.

A lattice vector in a cone is called *primitive* if it cannot be written as a sum of two nonzero lattice vectors in this cone. We will denote by Prim the union of primitive vectors in Delaunay cells σ for all $\sigma \subset \mathrm{Star}(0)$.

We say that some elements $x_1 \dots x_m$ are *cellmates* if there exists a (maximal-dimensional) Delaunay cell $\tau \ni 0$ such that $x_1 \dots x_m \in \mathrm{Cone}(0, \tau)$.

We thank S. Zucker for suggesting the term "cellmates".

DEFINITION 1.13. We call the cell σ *generating* if its Delaunay vectors contain a basis of X and *totally generating* if, moreover, the integral combinations $\sum n_i v_i$ of Delaunay vectors with $n_i \geq 0$ give all lattice elements of the cone $\mathrm{Cone}(0, \sigma)$.

1.14. In dimensions $g \leq 4$ all Delaunay cells are totally generating as was proved by Voronoi in a classical series of papers [Vor09]. It was only recently shown that there exist non-generating cells in dimension 5 (see [ER87], p. 796). But the following example from [Erd92] of a non-generating cell is by far the easiest.

EXAMPLE 1.15. Take $B = E_8$, an even unimodular positive definite quadratic form given by a familiar 8×8 matrix with $+2$ on the main diagonal. By Remark 1.9 we have

$$E_8(\alpha, v_i) = E_8(v_i, v_i)/2 \in \boldsymbol{Z}, \quad v_i \in \sigma .$$

If the Delaunay vectors v_i contained a basis, then we would have $\alpha \in X$, since E_8 is unimodular. But the holes cannot belong to the lattice by their very definition.

In this example each maximal-dimensional Delaunay cell generates a sublattice of index 2 or 3.

DEFINITION 1.16. The *nilpotency* of a Delaunay cell σ is the minimal positive integer n such that the lattice generated by the vectors $v_1/n, \dots, v_m/n$ contains X. The nilpotency of the Delaunay decomposition is the least common multiple of nilpotencies of its cells.

REMARK 1.17. Existence of non-generating cells is responsible for many combinatorial complications that occur in dimension $g \geq 5$.

2. Degeneration data. The purpose of this section is to recall the Mumford-Faltings-Chai description [FC90, III] of the degenerations of abelian varieties. We will use this description in the next section. We refer the reader to [FC90] for basic definitions and facts about polarizations, semiabelian group schemes etc. For easier reference we use the same notation as in [FC90] where this is convenient.

NOTATION 2.1. $\mathcal{R}$ is a Noetherian normal integral domain complete with respect to an ideal $I = \sqrt{I}$, K is the fraction field and $k = \mathcal{R}/I$ is the residue ring, $S = \operatorname{Spec}\mathcal{R}$, $S_0 = \operatorname{Spec} k$, and $\eta = \operatorname{Spec} K$ is the generic point. $\mathcal{R}$ is assumed to be regular or to be the completion of a normal excellent domain.

In our application $\mathcal{R}$ will be a DVR, and k will be the residue field.

DEFINITION 2.2. The objects of the category $\mathrm{DEG}_{\mathrm{ample}}$ are pairs $(G, \mathcal{L})$, where G is a semiabelian group scheme over S with abelian G_η and $\mathcal{L}$ is an invertible sheaf on G with ample $\mathcal{L}_\eta$. The morphisms are group homomorphisms respecting $\mathcal{L}$'s.

The objects of the category $\mathrm{DEG}_{\mathrm{pol}}$ are pairs (G, λ_η) with G as above and with the polarization $\lambda_\eta : G_\eta \to G_\eta^t$, where G_η^t is the dual abelian variety. The morphisms are group homomorphisms respecting λ's.

Next, we recall the definition of the degeneration data. We will only need the split case.

DEFINITION 2.3. The degeneration data in the split case consists of the following:

1. An abelian scheme A/S of relative dimension a, a split torus T/S of relative dimension r, $g = a + r$, and a semiabelian group scheme $\tilde{G}/S$,

$$1 \to T \to \tilde{G} \xrightarrow{\pi} A \to 0 .$$

This extension is equivalent via negative of pushout to a homomorphism $c : \underline{X} \to A^t$. Here X is a rank r free commutative group, and $\underline{X} = X_S$ is a constant group scheme, the group of characters of T. A^t/S is the dual abelian scheme of A/S.

2. A rank r free commutative group Y and the constant group scheme $\underline{Y} = Y_S$.

3. A homomorphism $c^t : \underline{Y} \to A$. This is equivalent to giving an extension

$$1 \to T^t \to \tilde{G}^t \to A^t \to 0 ,$$

where T^t is a torus with the group of characters $\underline{Y}$.

4. An injective homomorphism $\phi : Y \to X$ with finite cokernel.

5. A homomorphism $\iota : Y_\eta \to \tilde{G}_\eta$ lying over c_η^t. This is equivalent to giving a bilinear section τ of $(c^t \times c)^* \mathcal{P}_{A,\eta}^{-1}$ on $Y \times X$, in other words a trivialization of the biextension $\tau : 1_{Y \times X} \to (c^t \times c)^* \mathcal{P}_{A,\eta}^{-1}$. Here, $\mathcal{P}_\eta$ is the Poincare sheaf on $A_\eta \times A_\eta^t$, which comes with a canonical biextension structure.

6. An ample sheaf $\mathcal{M}$ on A inducing a polarization $\lambda_A : A \to A^t$ of A/S such that $\lambda_A c^t = c\phi$. This is equivalent to giving a T-linearized sheaf $\tilde{\mathcal{L}} = \pi^* \mathcal{M}$ on $\tilde{G}$.

7. An action of Y on $\tilde{\mathcal{L}}_\eta$ compatible with ϕ. This is equivalent to a cubical section ψ of $(c^t)^* \mathcal{M}_\eta^{-1}$ on Y, in other words to a cubical trivialization $\psi : 1_Y \to (c^t)^* \mathcal{M}_\eta^{-1}$, which is compatible with $\tau \circ (\mathrm{id}_Y \times \phi)$. ψ is defined up to a shift by Y.

The trivialization τ is required to satisfy the following *positivity condition*: $\tau(y, \phi y)$ for all y extends to a section of $\mathcal{P}^{-1}$ on $A \underset{S}{\times} A^t$, and it is 0 modulo I if $y \neq 0$.

The objects of the category $\mathrm{DD}_{\mathrm{ample}}$ are the data above, and the morphisms are the homomorphisms of $\tilde{G}$'s respecting this data.

DEFINITION 2.4. Similarly, the objects of the category $\mathrm{DD}_{\mathrm{pol}}$ consist of the data as above minus the sheaves $\mathcal{M}$, $\tilde{\mathcal{L}}$ and the section ψ, with the positivity condition again. In addition, one requires the trivialization τ to be symmetric (in the previous case this was automatic). The morphisms are homomorphisms of $\tilde{G}$'s respecting this data.

THEOREM 2.5 (Faltings-Chai). *The categories* $\mathrm{DEG}_{\mathrm{ample}}$ *and* $\mathrm{DD}_{\mathrm{ample}}$, *resp.* $\mathrm{DEG}_{\mathrm{pol}}$ *and* $\mathrm{DD}_{\mathrm{pol}}$ *are equivalent.*

In the case where $A = 0$ and $\tilde{G} = T$ is a torus (or, more generally, when $c = c^t = 0$) the section τ is simply a bilinear function $b : Y \times X \to K^*$, and ψ is a function $a : Y \to K^*$ satisfying

1. $b(y_1, \phi y_2) = a(y_1 + y_2)a(y_1)^{-1}a(y_2)^{-1}$,
2. $b(y, \phi y) \in I$ for $y \neq 0$.

This also implies that

3. $b(y_1, \phi y_2) = b(y_2, \phi y_1)$,
4. $a(0) = 1$,
5. $a(y_1 + y_2 + y_3)a(y_2 + y_3)^{-1}a(y_3 + y_1)^{-1}a(y_1 + y_2)^{-1}a(y_1)a(y_2)a(y_3) = 1$.

We will use the following notation. $\tilde{G}$ is affine over A and one has $\tilde{G} = \mathrm{Spec}_A(\oplus_{x\in X}\mathcal{O}_x)$. Each $\mathcal{O}_x$ is an invertible sheaf on A, canonically rigidified along the zero section, and one has $\mathcal{O}_x \simeq c(x)$. The pushout of the T-torsor $\tilde{G}$ over A by $x \in X$ is $\mathcal{O}_{-x}$. The sheaf $\mathcal{M} \otimes \mathcal{O}_x$, rigidified along the zero section, is denoted by $\mathcal{M}_x$. The Y-action on $(\oplus_{x\in X}\mathcal{M}_x) \otimes_{\mathcal{R}} K$ defined by ψ is denoted by $S_y : T^*_{c^t(y)}\mathcal{M}_x \to \mathcal{M}_{x+\phi y,\eta}$.

3. Simplified Mumford's construction.

SETUP 3.1. In this section, $\mathcal{R}$ is a DVR, $I = (s)$, and k is the residue field. We will denote the point S_0 simply by 0. We start with an abelian variety A_η with an effective ample Cartier divisor Θ_η defining principal polarization, and $\mathcal{L}_\eta = \mathcal{O}(\Theta_\eta)$. Applying the stable reduction theorem ([SGA7.1, AW71]) after a finite base change $S' \to S$ we have a semiabelian group scheme G'/S' and an invertible sheaf $\mathcal{L}$ extending $(A'_\eta, \mathcal{L}'_\eta)$ such that the toric part T'_0 of the central fiber G'_0 is split. In order not to crowd notation, we will continue to denote the objects by S, G, $\mathcal{L}$ etc.

We have an object of $\mathrm{DEG}_{\mathrm{ample}}$, and, by the previous section, an object of $\mathrm{DD}_{\mathrm{ample}}$, i.e., the degeneration data. Since the polarization is principal, $\phi : Y \to X$ is an isomorphism and we can identify Y with X. Further, the sheaf $\mathcal{M}$ on A defines a principal polarization. We denote by θ_A a generator of $H^0(A, \mathcal{M})$.

Here is the main object of our study:

DEFINITION 3.2. Consider the graded algebra

$$\mathcal{S}_2 = \left(\sum_{d \geq 0} \left(\bigoplus_{x \in X} \mathcal{O}_x \right) \otimes \mathcal{M}^d \vartheta^d \right) \underset{R}{\otimes} K \,,$$

where ϑ is an indeterminate defining the grading. In this algebra consider the subalgebra ${}_1R$ generated in degree by the $\mathcal{M}_0 = \mathcal{M}$ and all its Y-translates, $S_y^*(\mathcal{M}_0)$. This is a locally free graded $\mathcal{O}_A$-algebra. Finally, the algebra R is the saturation of ${}_1R$ in an obvious sense which will be further explained below. We define the scheme $\tilde{P} = \mathrm{Proj}_A R$ and the sheaf $\tilde{\mathcal{L}}$ on it as $\mathcal{O}(1)$.

For each $x \in X = Y$ we have an element $S_x^*(\theta_A) \in H^0(A, \mathcal{M}_x)$ that will be denoted by ξ_x. We have a formal power series

$$\tilde{\theta} = \sum_{x \in X} \xi_x \,.$$

We will see that, possibly after another finite base change $S' \to S$, the scheme $\tilde{P}$ is a *relatively complete model* as defined in [FC90, III.3]. Via Mumford's construction, this gives a projective scheme P/S extending A_η. We will see that it naturally comes with a relative Cartier divisor Θ.

The subalgebra R defines a subalgebra R' in

$$\mathcal{S}_1 = \left(\sum_{d \geq 0} \left(\bigoplus_{x \in X} \mathcal{O}_x \right) \vartheta^d \right) \underset{R}{\otimes} K \,.$$

One has $\mathrm{Proj} R = \mathrm{Proj} R'$ and $\tilde{\mathcal{L}} \simeq \tilde{\mathcal{L}} \otimes \pi^* \mathcal{M}$.

A. *Case of maximal degeneration.* In this case, $A = 0$ and $\tilde{G} = T = \mathrm{Spec} R[w^x; x \in X]$. ${}_1R$ is the R-subalgebra of $K[\vartheta, w^x; x \in X]$ generated by $a(x)\vartheta$. We have

$$\tilde{\theta} = \sum_{x \in X} \xi_x = \sum_{x \in X} a(x) w^x \vartheta \,.$$

DEFINITION 3.3. We define the functions $a_0 : Y \to k^*$, $A : Y \to \mathbf{Z}$, $b_0 : Y \times X \to k^*$, $B : Y \times X \to \mathbf{Z}$ by setting

$$a(y) = a'(y) s^{A(y)} \,, \quad b(y,x) = b'(y,x) s^{B(y,x)}$$

with $a'(y), b'(y) \in R \setminus I$ and taking a_0, b_0 to be a', b' modulo I.

REMARK 3.4. We are using the letter A for two purposes now: to denote an abelian variety, and to denote the integral-valued function above. This should not lead to any confusion since their meanings are very different.

Through our identification $\phi : Y \xrightarrow{\sim} X$ the functions a, A, b and B become functions on X and $X \times X$. The functions a and A are quadratic non-homogeneous, the functions b, B are symmetric and they are the homogeneous parts of a^2, $2A$, respectively. We have

$$A(x) = B(x,x)/2 + lx/2$$

for some $l \in X^*$. The positivity condition implies that B is positive definite.

REMARK 3.5. The function $B : Y \times X \to \mathbf{Z}$ describes the monodromy of the family G and is called the monodromy pairing, cf. [SGA7.1, IX.10.4].

Since all $a'(y)$ are invertible in $\mathcal{R}$, the algebra ${}_1R$ is generated by monomials $\zeta_x = s^{A(x)} w^x \vartheta$, so it is a semigroup algebra.

DEFINITION 3.6. We introduce two lattices $M = X \oplus \mathbf{Z}_{e_0} \simeq \mathbf{Z}^{r+1}$ and its dual $N = X^* \oplus \mathbf{Z} f_0$.

3.7. Each ζ_x corresponds to a lattice element $(x, A(x)) \in M$. These are exactly the vertices of the multifaceted paraboloid Q in Figure 3 which we imagine lying in the hyperplane $(1, M)$ inside $\mathbf{Z} \oplus M$. The extra $\mathbf{Z}$ corresponds to the grading by ϑ. The saturation R of ${}_1R$ is generated by monomials corresponding to all lattice vectors lying inside $\mathrm{Cone}(Q)$.

THEOREM 3.8. (i) *$\tilde{P}$ is covered by the affine toric schemes $U(c) = \mathrm{Spec} R(c)$, $c \in X$, where $R(c)$ is the semigroup algebra corresponding to the cone at the vertex $c \in Q$ of lattice elements*

$$\{(x, x_0) \mid x_0 \geq \eta(x, c)\}$$

($\eta(x, c)$ is defined in 1.10*).*

(ii) *$R(c)$ is a free $\mathcal{R}$-module with the basis $\zeta_{x,c} = s^{\lceil \eta(x,c) \rceil} w^x$ (here $\lceil z \rceil$ denotes the least integer $\geq z$).*

(iii) *All the rings $R(c)$ are isomorphic to each other, and each is finitely generated over $\mathcal{R}$. The scheme $\tilde{P}$ is locally of finite type over $\mathcal{R}$.*

(iv) *$\mathrm{Spec} R(c)$ is the affine torus embedding over $S = \mathrm{Spec}\mathcal{R}$ corresponding to the cone $\Delta(c)$ over*

$$(1, -dA(\hat{c})) \subset (1, N_0) \subset N,$$

where $\hat{c}$ is the Voronoi cell dual to c.

(v) *$\tilde{P}$ is the torus embedding T_N $\mathrm{emb}\Delta$, where Δ is the fan in $N_{\mathbf{R}}$ consisting of $\{0\}$ and the cones over the shifted Voronoi decomposition $(1, -dA(\mathrm{Vor}_B))$. The morphism $\tilde{P} \to S$ is described by the map of fans from Δ to the half line $\mathbf{R}_{\geq 0} f_0$.*

(vi) *$\tilde{\mathcal{L}}$ is invertible and ample.*

(vii) *One has natural compatible actions of T on $\tilde{P}$ and of $T \times \mathbf{G}_m$ on $\tilde{\mathcal{L}}$.*

REMARK 3.9. The reference for torus embeddings over a DVR is [TE73, IV, §3]. Formally, all computations work the same way as for a toric variety with torus action of $k[s, 1/s, w^x; x \in X]$.

PROOF. The first part of (i) is simply the description of the standard cover of Proj by Spec's. The second part, as well as (iv) and (v) follow immediately from Lemma 1.8 (ii) and (iii) follow at once from (i).

The ring extension ${}_1R \subset R$ is integral, hence $\mathrm{Proj} R \to \mathrm{Proj}\,{}_1R$ is well-defined and is finite. The sheaf $\mathcal{O}(1)$ on $\mathrm{Proj}\,{}_1R$ is invertible and ample, and $\tilde{\mathcal{L}}$ is its pullback. This gives (vi).

The actions in (vii) are defined by the X-, resp. $(X \oplus \mathbf{Z})$-gradings. $\square$

As a consequence, we can apply the standard description in the theory of torus embeddings of the open cover, torus orbits and their closures:

THEOREM 3.10. (i) *For each Delaunay cell* $\sigma \in \mathrm{Del}_B$ *one has a ring* $R(\sigma)$ *corresponding to the cone over the dual Voronoi cell* $\tilde{\sigma}$. $U(\sigma)$ *is open in* $\tilde{P}$ *and* $U(\sigma_1) \cap U(\sigma_2) = U(D(\tilde{\sigma}_1 \cap \tilde{\sigma}_2))$.

(ii) $R(\sigma)$ *is the localization of* $R(c)$, $c \in \sigma$, *at* $\zeta_{d,c}$, $d \in X \cap \mathbf{R}(\sigma - c)$.

(iii) *In the central fiber* $\tilde{P}_0$, *the* T_0*-orbits are in* 1*-to-*1 *dimension-preserving correspondence with the Delaunay cells* σ. *In particular, the irreducible components of* $\tilde{P}$ *correspond to the maximal-dimensional cells.*

(iv) *The closure* $\tilde{V}(\sigma)$ *of* $\mathrm{orb}(\sigma)$ *together with the restriction of the line bundle* $\tilde{\mathcal{L}}$ *is a projective toric variety over* k *with a* T_0*-linearized ample line bundle corresponding to the lattice polytope* σ.

(v) $\tilde{V}(\sigma_1) \cap \tilde{V}(\sigma_2) = \tilde{V}(\sigma_1 \cap \sigma_2)$.

(vi) *For a maximal-dimensional cell* σ, *the multiplicity of* $\tilde{V}(\sigma)$ *in* $\tilde{P}_0$ *is the denominator of* $dA(\alpha(\sigma)) \in X^*_{\mathbf{Q}}$.

QUESTION 3.11. *For a maximal-dimensional cell* σ, *when is* $\tilde{P}$ *generically reduced at* $\tilde{V}(\sigma)$? *In other words, when is* $dA(\alpha(\sigma))$ *integral?*

LEMMA 3.12. $dA(\alpha(\sigma)) \in X^*$ *in any of the following cases:*

(i) σ *is generating.*

(ii) $A(x)/n \in \mathbf{Z}$ *for every* $x \in X$, *where* n *is the nilpotency of* σ.

PROOF. (i) is a particular case of (ii), so let us prove the second part.

$dA(\alpha) \in X^*$ if and only if $dA(\alpha)(x) \in \mathbf{Z}$ for every $x \in X$. Now let $v_1 \dots v_m$ be the Delaunay vectors of σ. By the definition of the nilpotency in Definition 1.16 we have $x = (1/n)\sum n_i v_i$ for some $n_i \in \mathbf{Z}$. Then

$$\begin{aligned} dA(\alpha)(x) &= B(\alpha, x) + lx/2 = \frac{1}{n}\sum n_i (B(\alpha, v_i) + lv_i/2) \\ &= \frac{1}{n}\sum n_i (B(v_i, v_i)/2 + lv_i/2) = \sum n_i A(v_i)/n \in \mathbf{Z} \end{aligned}$$

□

3.13. For the central fiber $\tilde{P}_0$ to be generically reduced, we need $A(x)$ to be divisible by the nilpotency of the lattice. This certainly holds after a totally ramified base change. Consider the polynomial $z^n - s \in K[z]$. It is irreducible by the Eisenstein criterion. The field extension $K \subset K' = K[z]/(z^n - s)$ has degree n and is totally ramified. The integral closure of R in K' is again a DVR, complete with respect to the maximal ideal $I' = (s')$ (see e.g., [Ser79, II.3]) and $\mathrm{val}_{s'}(s) = n$, so that $A'(x) = nA(x)$.

The following example, very similar to Example 1.15, shows that this base change is indeed sometimes necessary.

EXAMPLE 3.14. Consider the degeneration data $A(x)=E_8(x)/2$, $B(y,x)=E_8(y,x)$. Then $dA = E_8$ and for every hole $\alpha(\sigma)$ we have $dA(\alpha) \notin X^*$. Indeed, otherwise we would have $\alpha \in X$ since E_8 is unimodular, and this is impossible by the definition of a hole.

In this example every irreducible component of the central fiber $\tilde{P}_0$ has multiplicity 2 or 3.

ASSUMPTION 3.15. From now on, we assume that the necessary base change has been done, so $dA(\alpha)$ is integral for each hole α. This implies that all $\eta(x,c)$ are integral-valued on X.

3.16. Y-action on $\tilde{P}$. We are given a canonical Y-action on $K[\vartheta, w^x; x \in X]$ by construction. It is constant on K and sends each generator $\xi_x = a(x)w^x\vartheta$ to another generator $\xi_{x+y} = a(x+y)w^{x+y}\vartheta$. Precisely because $a(x)$ is quadratic, this action extends uniquely to the whole $K[\vartheta, w^x]$. Clearly, the subrings ${}_1R$ and R are Y-invariant, so we have the Y-action on R which will be denoted by S_y^*. We easily compute:

$$S_y^*\left(\frac{a'(x+c)}{a'(x)}\zeta_{x,c}\right) = \frac{a'(x+c+y)}{a'(x+y)}\zeta_{x,c+y}\,,$$
$$S_y^*(\zeta_{x,c}) = b'(x,y)\zeta_{x+y,c}\,.$$

This describes the action $S_y^* : R(c) \to R(c+y)$ and $S_y : \mathrm{Spec}R(c+y) \to \mathrm{Spec}R(c)$.

THEOREM 3.17. *$\tilde{P}_0$ is a scheme locally of finite type over k. It is covered by the affine schemes $\mathrm{Spec}R_0(\sigma)$ of finite type over k for Delaunay cells $\sigma \in \mathrm{Del}_B$, where $R_0(\sigma) = R(\sigma) \otimes_{\mathcal{R}} k$. The following holds:*

(i) *$R_0(c)$ is a k-vector space with basis $\{\bar{\zeta}_{x,c}; x \in X\}$, the multiplication being defined by*

$$\bar{\zeta}_{x_1,c} \cdots \bar{\zeta}_{x_m,c} = \bar{\zeta}_{x_1+\cdots+x_m,c}$$

if $x_1 \dots x_m$ are cellmates with respect to the B-Delaunay decomposition, and 0 *otherwise.*

(ii) *For $\sigma \ni c$ the ring $R_0(\sigma)$ is the localization of $R_0(c)$ at $\{\bar{\zeta}_{d,c}; d \in X \cap \mathbf{R}(\sigma - c)\}$.*

(iii) $U_0(\sigma_1) \cap U_0(\sigma_2) = U_0(D(\hat{\sigma}_1 \cap \hat{\sigma}_2))$.

(iv) *The group Y of periods acts on $\tilde{P}_0$ by sending* $\mathrm{Spec}R_0(c+\phi(y))$ *to* $\mathrm{Spec}R_0(c)$ *in the following way:*

$$S_y^*(\bar{\zeta}_{x,c}) = b_0(y,x)\bar{\zeta}_{x,c+\phi(y)}\,.$$

PROOF. This follows from Theorem 3.8, Theorem 3.10 and 3.16.

Here is a way to see (i) geometrically: each $\zeta_{x_i,c}$ corresponds to a point on a face of the cone of Q at c. The sum of several such points lie on a face if and only if they belong to a common face, i.e., if and only if $x_1 \dots x_m$ are cellmates. Otherwise, the product corresponds to a point in the interior of the cone and equals $s^n\zeta_{x_1+\cdots+x_m,c}$ for some $n > 0$. Therefore, it reduces to 0 modulo (s). □

COROLLARY 3.18. *$\tilde{P}_0$ is reduced and geometrically reduced.*

LEMMA 3.19. *For each n only finitely many of the elements* $\xi_{x+c}\xi_c^{-1}$ *in* $R_0(c)$ *are not zero modulo* $\Gamma^{n+1} = (s^{n+1})$. *For* $n = 0$ *the only ones not zero correspond to the lattice points* $x + c \in \mathrm{Star}(\mathrm{Del}_B, c)$, *i.e.*, $x \in \mathrm{Star}(\mathrm{Del}_B, 0)$.

PROOF. Indeed, $\xi_{x+c}\xi_c^{-1} = \zeta_{x,c}s^{B(x,x)/2-B(\alpha,x)}$, and B is positive definite. The second part was proved in Lemma 1.8. □

DEFINITION 3.20. We define the Cartier divisor $\tilde{\Theta}_0$ on $\tilde{P}_0$ by the system of compatible equations $\{\tilde{\theta}/\xi_c \in R_0(c)\}$. Explicitly,

$$\tilde{\theta}/\xi_c = \sum_{x\in\mathrm{Star}(\mathrm{Del}_B,0)} a_0(x+c)a_0^{-1}(c)\bar{\zeta}_{x,c}\,.$$

Clearly, $\tilde{\theta}$ defines a Y-invariant global section of $\tilde{\mathcal{L}}_0 = \tilde{\mathcal{L}}|_{\tilde{P}_0}$, so the divisor $\tilde{\Theta}_0$ is Y-invariant.

LEMMA 3.21. *Once the base change in Assumption* 3.15 *has been made, for any further finite base change* $S' \to S$, *one has* $\tilde{P}' \simeq \tilde{P}\times_S S'$. *In other words, our construction is stable under base change.*

PROOF. This statement is sufficient to check for the semigroup algebras $R(c)$, which is obvious using the basis in Theorem 3.8(ii). □

B. *Case of arbitrary abelian part.*

3.22. Most of the statements above transfer to the general case without any difficulty. The main difference is that $\tilde{P}$ is now fibered over A instead of a point, and each $U(\sigma)$, resp. $U_0(\sigma)$ is an affine scheme over A, resp. A_0. One easily sees that $\tilde{P}$ is isomorphic to the contracted product $\tilde{P}^r \overset{T}{\times} \tilde{G}$ of an r-dimensional scheme $\tilde{P}^r$ over S corresponding to the positive definite integral-valued bilinear form $B(x, y)$ on the r-dimensional lattice X with $\tilde{G}$. Recall that the contracted product is the quotient of $\tilde{P}^r \times_S \tilde{G}$ by the free action of T with the standard action on the first factor and the opposite action on the second factor. In the same way, $\tilde{P}_0 \simeq \tilde{P}_0^r \overset{T}{\times} \tilde{G}_0$. Moreover, the $\boldsymbol{G}_m$-torsor $\tilde{\boldsymbol{L}}'$ corresponding to the sheaf $\tilde{\mathcal{L}}' \simeq \tilde{\mathcal{L}} \otimes \mathcal{M}^{-1}$ is the contracted product of the $\boldsymbol{G}_m$-torsor $\tilde{\boldsymbol{L}}^r$ and $\tilde{G} \times_S \boldsymbol{G}_{m,S}$, and similarly for the central fiber.

The power series $\tilde{\theta} = \sum \xi_x$ defines a Y-invariant section of $\tilde{\mathcal{L}}$.

LEMMA 3.23. $\tilde{P}_0$ *is a disjoint union of semiabelian varieties* $\tilde{G}_0(\sigma)$ *which are in* 1-*to*-1 *correspondence with the Delaunay cells. One has*

$$1 \to \mathrm{orb}(\sigma) \to \tilde{G}_0(\sigma) \to A \to 0\,.$$

The closure $\tilde{V}(\sigma)$ *of* $\tilde{G}_0(\sigma)$ *is a projective variety* $\tilde{V}^r(\sigma) \overset{T}{\times} \tilde{G}_0$.

For a 0-*dimensional cell* $c \in X$ *the restrictin of* $\tilde{\theta}$ *to* $\tilde{V}(c) \simeq A$ *is* ξ_x *and* $(\xi_x) = T_{c^t(x)}(\Theta_A)$. *In particular,* $\tilde{\Theta} = (\tilde{\theta})$ *does not contain any of the strata entirely.*

PROOF. The first part follows at once from Theorem 3.10 by applying the contracted product. The second part is obvious because all the other ξ_x, $x \neq c$ are zero on $\tilde{V}(c)$. □

C. *Taking the quotient by Y.*

LEMMA 3.24. *$\tilde{P}$ is a relatively complete model as defined in* [FC90, III.3.1].

PROOF. We do not even recall the fairly long definition of a relatively complete model because most of it formalizes what we already have: a scheme $\tilde{P}$ locally of finite type over R with an ample sheaf $\tilde{\mathcal{L}}$, actions of Y and T etc.

There are two additional conditions which we have not described yet. The first one is the completeness condition. It is quite tricky but it is used in [Mum72, FC90] only to prove that every irreducible component of $\tilde{P}_0$ is proper over k. We already know this from Theorem 3.10.

The second condition is that we should have an embedding $\tilde{G} \hookrightarrow \tilde{P}$. It is sufficient to give such an embedding for the toric case, since then we simply apply the contracted product. In the toric language, T corresponds to a fan in $N_{\boldsymbol{R}} = X^*_{\boldsymbol{R}} \oplus \boldsymbol{R}$ consisting of the ray $\boldsymbol{R}_{\geq 0} f_0$. A map from this fan to the fan Δ, which sends f_0 to $(1, -dA(\alpha(\sigma)))$ for an arbitrary maximal-dimensional Delaunay cell σ, defines an embedding $T \hookrightarrow \tilde{P}^r$. We have used the fact that $dA(\alpha(\sigma))$ is integral here. □

REMARK 3.25. Note that the embedding $\tilde{G} \hookrightarrow \tilde{P}$ defines a section of $\tilde{P}$ which has absolutely nothing to do with the zero section z_η of A_η and its closure z. The embedding $z \hookrightarrow P$ is described by the embedding of fans $(\boldsymbol{Z}, \boldsymbol{R}_{\geq 0} f_0) \hookrightarrow (N, \Delta)$, $f_0 \mapsto f_0$. From this, we see that $z_0 \in \operatorname{orb}(\sigma)$, where σ is the "bottom" face of the hyperboloid Q. It need not be maximal-dimensional.

3.26. We can now apply Mumford's construction as described in [Mum72, FC90]. This consists of considering all fattenings $(\tilde{P}_n, \tilde{\mathcal{L}}_n) = (\tilde{P}, \tilde{\mathcal{L}}) \underset{R}{\times} R/I^{n+1}$, their quotients $(P_n, \mathcal{L}_n) = (\tilde{P}_n, \tilde{\mathcal{L}}_n)/Y$ and then algebraizing this system to a projective scheme $(P, \mathcal{L})/S$ such that the generic fiber P_η is abelian and $\mathcal{L}_\eta$ defines a principal polarization. By Theorem 2.5, $(A_\eta, \mathcal{O}(\Theta_\eta)) \simeq (P_\eta, \mathcal{L}_\eta)$, and since the polarization is principal, this isomorphism is uniquely defined. Thus, we have obtained the extended family.

To this construction we will add a theta divisor. By Lemma 3.19 for each n the power series $\tilde{\theta}$ defines a finite sum in each $R_n(c)$. Hence, we have a compatible system of Y-invariant sections of $\tilde{\mathcal{L}}_n$ that descend to compatible sections of $\mathcal{L}_n$ that algebraize to a section θ of $\mathcal{L}$.

From Lemma 3.23 we have:

COROLLARY 3.27. *P_0 is a disjoint union of semiabelian varieties $G_0(\bar{\sigma})$ which are in 1-to-1 correspondence with the classes of Delaunay cells modulo Y-translations. $\Theta_0 = (\theta_0)$ does not contain any of the strata entirely.*

DEFINITION 3.28. In the split case, a *stable quasiabelian pair*, SQAP for short, over a field k is a pair (P_0, Θ_0) from our construction. In general, a pair of a reduced projective variety and an ample Cartier divisor over k is called a stable quasiabelian pair if it becomes one after a field extension. We will call P_0 itself a stable quasiabelian variety, SQAV for short.

Let $\Gamma^2(X^*)$ be the lattice of integral-valued symmetric bilinear forms on $X \times X$. For each Delaunay decomposition Del, let $K(\mathrm{Del})$ be the subgroup of $\Gamma^2(X^*)$ generated by the positive-definite forms B with $\mathrm{Del}_B = \mathrm{Del}$, and set $N(\mathrm{Del}) = \Gamma^2(X^*)/K(\mathrm{Del})$.

THEOREM 3.29. *Each SQAP over an algebraically closed field k is uniquely defined by the following data:*

1. *a semiabelian variety* G_0, $1 \to T_0 \to G_0 \to A_0 \to 0$,
2. *a principal polarization* $\lambda_{A_0} : A_0 \to A_0^t$,
3. *a Delaunay decomposition* Del *on X, where* $X = Y$ *is the group of characters of* T_0,
4. *a class of bilinear symmetric sections* τ_0 *of* $(c_0^t \times c_0)^* \mathcal{P}_{A_0}^{-1}$ *on* $X \times X$ *modulo the action of the torus* $K(\mathrm{Del}) \otimes \boldsymbol{G}_{m,k}$.

PROOF. It is clear from the construction that an SQAP depends only on A_0, Θ_0 and A, B, ψ_0, τ_0. Since we do not care about the origin and the polarization is principal, giving the pair (A_0, Θ_0) is the same as giving the pair (A_0, λ_{A_0}). Replacing ψ by ψ_1 with the same homogeneous part does not change the isomorphism classes of subalgebras ${}_1R$ and R, since the relations between $S_y(\mathcal{M}_0)$ remain the same.

The only information we are getting from B is the Delaunay decomposition. Moreover, for a fixed B if we replace the uniformizing parameter s by μs, $\mu \in R \setminus I$, then the central fiber will not change, but $b(y, x)$ will change to $b(y, x)\mu_0^{B(y,x)}$. Therefore we only have the equivalence class by the $K(\mathrm{Del}) \otimes \boldsymbol{G}_{m,k}$-action. □

REMARK 3.30. We can further divide the data above by the finite group of automorphisms of (A_0, λ_{A_0}) extended by the finite subgroup of $GL(X)$ preserving Del. The quotient data exactly corresponds to the k-points of the Voronoi compactification of A_g. Hence, every k-point of $\bar{A}_g^{\mathrm{VOR}}$ defines a unique SQAP over k.

REMARK 3.31. In [Mum83] Mumford considered the first-order degenerations of abelian varieties over $\boldsymbol{C}$. These are exactly our pairs in the case where the toric part of $\tilde{G}$ is 1-dimensional, i.e., $r = 1$.

In conclusion, we would like to make the following obvious observation.

LEMMA 3.32. *The family* $P \to S$ *is flat. The family of theta divisors* $\Theta \to S$ *is also flat.*

PROOF. Indeed, S is integral and regular of dimension 1, and P is reduced and irreducible so the statement follows, e.g., by [Har77, III.9.7]. The family of divisors $\Theta \to S$ is flat because $\Theta \cdot P_t$ is defined at every point $t \in S$ ([Har77, III.9.8.5]). □

4. Further properties of SQAVs. All statements in this section are stable under field extensions. Therefore without loss of generality we may assume that k is algebraically closed.

LEMMA 4.1. *P_0 is Gorenstein.*

PROOF. A Noetherian local ring is Gorenstein if and only if its formal completion is such ([Mat89, 18.3]). Therefore, we can check this property on an étale cover $\tilde{P}_0$ of P_0.

Moreover, the purely toric case suffices, since $\tilde{P}_0$ is a fibration over a smooth variety A_0 with a fiber $\tilde{P}_0^r$, locally trivial in étale topology.

Recall that the scheme $\tilde{P}$, as any torus embedding, is Cohen-Macaulay.

$\tilde{P}_0 \subset \tilde{P}$ is the union of divisors $\tilde{V}(\sigma)$ corresponding to the 1-dimensional faces $\Delta(\sigma)$ of the fan Δ (i.e., to maximal-dimensional Delaunay cells σ). The following is a basic formula for the dualizing sheaf of a torus embedding:

$$\omega_{\tilde{P}} = \mathcal{O}\big(-\sum \tilde{V}(\sigma)\big) = \mathcal{O}(-\tilde{P}_0).$$

Since $\tilde{P}_0$ is Cartier, $\omega_{\tilde{P}}$ is locally free, so $\tilde{P}$ is Gorenstein. The scheme $\tilde{P}_0$ is then Gorenstein as a subscheme of a Gorenstein scheme that is defined by one regular element.

Alternatively, $\tilde{P}_0$ is Gorenstein as the complement of the main torus in a toric variety, see [Oda88, p. 126, Ishida's criterion]. □

LEMMA 4.2. $\omega_{P_0} \simeq \mathcal{O}_{P_0}$.

PROOF. As above, we have the canonical isomorphism

$$\omega_{\tilde{P}}(\tilde{P}_0) \simeq \mathcal{O}_{\tilde{P}},$$

which by adjunction gives $\omega_{\tilde{P}_0} \simeq \mathcal{O}_{\tilde{P}_0}$. Since both sides are invariant under the action of lattice Y, this isomorphism descends to P_0. □

THEOREM 4.3.

$$h^i(P_0, \mathcal{O}) = \binom{g}{i}.$$

PROOF. We want to exploit the fact that $\tilde{P}_0$ is built of "blocks" $\tilde{V}(\sigma)$ and that the cohomologies of each block are easily computable. Indeed, since the fibers of $\tilde{V}(\sigma) \to A_0$ are toric varieties $\tilde{V}^r(\sigma)$ and $H^i(\tilde{V}^r(\sigma), \mathcal{O}) = 0$ for $i > 0$, we have $R^i\pi_*\mathcal{O}_{\tilde{V}(\sigma)} = 0$ for $i > 0$ and $H^i(\tilde{V}(\sigma), \mathcal{O}) = H^i(A_0, \mathcal{O})$. It is well-known that for an abelian variety A_0 these groups have dimension $\binom{a}{i}$.

The following important sequence for a union of torus orbits is contained in [Oda88, p. 126], where it is called Ishida's complex:

$$0 \to \mathcal{O}_{\tilde{P}_0^r} \to \bigoplus_{\dim\sigma = g} \mathcal{O}_{\tilde{V}^r(\sigma)} \to \cdots \bigoplus_{\dim\sigma = 0} \mathcal{O}_{\tilde{V}^r(\sigma)} \to 0.$$

The morphisms in this sequence are the restrictions for all pairs $\sigma_1 \supset \sigma_2$, taken with $\pm$ depending on a chosen orientation of the cells.

By taking the contracted product, we obtain a similar resolution for the sheaf $\mathcal{O}_{\tilde{P}_0}$, with $\tilde{V}^r(\sigma)$ replaced by $\tilde{V}(\sigma)$. Finally, by dividing this resolution by the Y-action, we obtain a resolution of $\mathcal{O}_{P_0}$. In this resolution the morphism $\mathcal{O}_{\tilde{V}(\bar{\sigma}_1)} \to \mathcal{O}_{\tilde{V}(\bar{\sigma}_2)}$ is a linear combination of several restriction maps according to the ways representatives of $\bar{\sigma}_1$ contain representatives of σ_2. We can now compute $H^i(\mathcal{O}_{\tilde{P}_0})$ by using the hypercohomologies of the above complex.

First, consider the special case where $r = g$ and $a = 0$. In this case each $H^0(\tilde{V}(\sigma), \mathcal{O})$ is 1-dimensional and the higher cohomologies vanish. Therefore, $H^i(P_0, \mathcal{O})$ are the cohomologies of the complex

$$0 \to \bigoplus_{\dim \bar{\sigma} = g} H^0(\tilde{V}(\bar{\sigma}), \mathcal{O}) \to \cdots \bigoplus_{\dim \bar{\sigma} = 0} H^0(\tilde{V}(\bar{\sigma}), \mathcal{O}) \to 0 .$$

But this complex computes the cellular cohomologies of the cell complex Del_B/Y whose geometric representation is homeomorphic to $\boldsymbol{R}^r/\boldsymbol{Z}^r$. Hence $h^i = \binom{r}{i} = \binom{g}{i}$.

In general, we obtain the spectral sequence $E_1^{pq} = H^p(\boldsymbol{R}^r/\boldsymbol{Z}^r, H^q(A_0, \mathcal{O})) \Rightarrow H^{p+q}(P_0, \mathcal{O})$ degenerating in degree 1. Therefore,

$$h^i(P_0, \mathcal{O}) = \sum_{p+q=i} \binom{r}{p}\binom{a}{q} = \binom{r+a}{i} = \binom{g}{i} .$$

$\square$

THEOREM 4.4. *For every $d > 0$ and $i > 0$ one has $h^0(P_0, \mathcal{L}_0^d) = d^g$ and $h^i(P_0, \mathcal{L}_0^d) = 0$.*

PROOF. Twist the above resolution of $\mathcal{O}_{P_0}$ by $\mathcal{L}$. Once again, the cohomologies of each building block are easy to compute. Indeed, for a toric variety $\tilde{V}^r(\sigma)$ higher cohomologies of an ample sheaf vanish. Moreover, since by Theorem 3.10 the pair $(\tilde{V}^r(\sigma), \tilde{\mathcal{L}})$ is the toric variety with a linearized ample sheaf corresponding to the polytope $\sigma \subset X_{\boldsymbol{R}}$, $H^0(\tilde{V}^r(\sigma), \tilde{\mathcal{L}}^d)$ is canonically the direct sum of 1-dimensional eigenspaces, one for each point $z \in \sigma \cap X$, cf., e.g., [Oda88, Ch.2]. Taking into account that higher cohomologies of ample sheaves on abelian varieties vanish, we see that $H^i(\tilde{V}(\sigma), \tilde{\mathcal{L}}^d) = 0$ for $i > 0$ and $H^0(\tilde{V}(\sigma), \tilde{\mathcal{L}}^d) \simeq H^0(\tilde{V}^r(\sigma), \tilde{\mathcal{L}}^d) \otimes H^0(A_0, \mathcal{M}^d)$. It is well-known that for the abelian variety A_0 the latter cohomology space has dimension d^a.

The above decomposition into eigenspaces extends to the hypercohomologies and we obtain

$$H^i(P_0, \mathcal{L}^d) \simeq H^0(A_0, \mathcal{M}^d) \otimes \left(\bigoplus_{\bar{z} \in X/dX} W_{\bar{z}}^i \right) ,$$

where $W_{\bar{z}}^i$ is the $\bar{z}$-eigenspace of the i-th cohomology of the complex

$$0 \to \bigoplus_{\dim \bar{\sigma} = g} H^0(\tilde{V}^r(\bar{\sigma}), \tilde{\mathcal{L}}^d) \to \cdots \bigoplus_{\dim \bar{\sigma} = 0} H^0(\tilde{V}^r(\bar{\sigma}), \tilde{\mathcal{L}}^d) \to 0 .$$

Fix a representative $z \in X/n$ of $\bar{z}$. Let σ_0 be the minimal cell containing z. There is a 1-to-1 correspondence between the cells $\sigma \ni z$ and the faces of the dual Voronoi cell $\hat{\sigma}_0$. Since these are exactly the cells for which the z-eigenspace in $H^0(\tilde{V}(\sigma), \tilde{\mathcal{L}}^d)$ are nonzero (and 1-dimensional), we see that W_z^i computes the cellular cohomology $H^i(\hat{\sigma}_0, k)$. Since as a topological space $\hat{\sigma}_0$ is contractible, $\dim W_z^0 = 1$ and $W_z^i = 0$ for $i > 0$. Therefore, $h^i(P_0, \mathcal{L}^d) = 0$ and $h^0(P_0, \mathcal{L}^d) = h^0(A_0, \mathcal{M}^d) \cdot |X/dX| = d^a d^r = d^g$. $\square$

4.5. As a consequence of this theorem, we can write down explicitly a basis of $H^0(P_0, \mathcal{L}_0^d)$. First, let us do this for the toric case. For each $\bar{z} \in X/dX$ fix a representative $z \in X/d$

and choose $x_1, \dots, x_d$ with $x_1 + \dots + x_d = dz$. Consider a power series

$$\tilde{\theta}'(x_1, \dots, x_d) = \sum_{y \in Y = X} \xi'(x_1 + y, \dots, x_d + y),$$

$$\xi'(x_1 + y, \dots, x_d + y) := \prod_{i=1}^{d} \xi_{x_i + y}.$$

This power series is obviously invariant under the Y-action on $K[\vartheta, w^x \mid x \in X]$ and, under Assumption 3.15, there is a unique nonnegative integer $n = n(x_1, \dots, x_d)$ such that $\xi(x_1 + y, \dots, x_d + y) = s^{-n}\xi'(x_1 + y, \dots, x_d + y)$ defines a nonzero section of $H^0(\tilde{P}_0, \tilde{\mathcal{L}}_0^d)$, so that $\tilde{\theta}(x_1, \dots, x_d) = s^{-n}\tilde{\theta}'(x_1, \dots, x_d)$ defines a nonzero Y-invariant section of $H^0(\tilde{P}_0, \tilde{\mathcal{L}}_0^d)$ which descends to a nonzero section $\theta(x_1, \dots, x_d)$ of $H^0(P_0, \mathcal{L}_0^d)$. Choosing another representatives z and $x_1, \dots, x_d$ changes these sections by multiplicative constants. Therefore, in the cases where we do not care about these constants we will write simply ξ_z, $\tilde{\theta}_z$ and $\theta_{\bar{z}}$.

Geometrically, ξ_z is represented by a point on the surface of the multifaceted paraboloid Q of Figure 3 lying over z, and $\tilde{\theta}_z$ by the sum of countably many such points lying over all $z + y$, $y \in Y = X$.

In general, for any $e \in H^0(A_0, \mathcal{M}^d)$ we repeat the procedure taking instead of ξ_{x_i+y} sections $\xi_{x_i+y}(e) = S^*_{x_i+y}(e)$. In this way, after fixing a basis $\{e_1, \dots, e_{d^a}\}$ of $H^0(A_0, \mathcal{M}^d)$ we obtain a basis $\theta_{\bar{z}}(e_i)$ of $H^0(P_0, \mathcal{L}_0^d)$.

The following is an easy application of Theorem 3.17(i):

LEMMA 4.6. *For each $e \in H^0(A_0, \mathcal{M}^d)$ the following open sets coincide*:

$$\{\xi_z(e) \neq 0\} = \tilde{G}_0(\sigma) \cap \pi^{-1}\{S^*_{dz}(e) \neq 0\},$$

where σ is the Delaunay cell containing z in its interior σ^0.

THEOREM 4.7. *Let $(P_0, \mathcal{L}_0)$ be an SQAV of dimension g. Then the sheaf $\mathcal{O}(\mathcal{L}_0^d)$ is very ample if $d \geq 2g + 1$.*

PROOF. Consider the toric case first, i.e., assume $r = g$, $a = 0$. We need to prove:

(i) Sections $\theta_{\bar{z}}$, $z \in X/dZ$ separate the torus orbits $\mathrm{orb}(\bar{\sigma})$.

(ii) They embed every $\mathrm{orb}(\bar{\sigma})$.

(iii) This embedding is an immersion at every point.

Let σ be a maximal-dimensional cell and pick a point $z \in \sigma^0 \cap X/d$ in its interior, which exists by Lemma 4.8. By Lemma 4.6, ξ_z is nonzero exactly on $\mathrm{orb}(\sigma)$. Therefore, $\tilde{\theta}_z$ is nonzero exactly on the union of Y-translates of this orbit, and $\theta_{\bar{z}}$ is not zero precisely on $\mathrm{orb}(\bar{\sigma})$. Thus, we have separated the points of this orbit from all the others. Continuing by induction down the dimension, we get (i).

The restriction of each $\theta_{\bar{z}}$ to $\mathrm{orb}(\bar{\sigma})$ is a sum of degree d monomial corresponding to $\bar{z}$, provided $\bar{z} \in \bar{\sigma}$. If $\bar{z} \in \bar{\sigma}^0$, there is just one such monomial. Therefore, the condition that suffices for (ii) is that the differences of vectors in $\sigma^0 \cap X/d$ generate $\boldsymbol{R}\sigma \cap X/d$ as a group. This holds by Lemma 4.8(ii).

In suffices to prove the immersion condition for the 0-dimensional orbit p corresponding to $0 \in X$ only. Indeed, it then holds in an open neighborhood which intersects every other orbit, and due to the torus action everywhere. Moreover, we can work on the étale cover $\tilde{P}_0$. We have $\tilde{\theta}_0(p) \neq 0$, $\tilde{\theta}_z(p) = 0$ for $\bar{z} \neq 0$, and we want to show that $\tilde{\theta}_z/\tilde{\theta}_0$ generate $\mathfrak{m}/\mathfrak{m}^2$, where $\mathfrak{m}$ is the maximal ideal of $R_0(0)$. On the other hand, $\xi_0(p) \neq 0$ as well, so we can consider $\tilde{\theta}_z/\xi_0$ instead.

By Theorem 3.17(i), $\mathfrak{m}/\mathfrak{m}^2$ is generated by *primitive* lattice vectors (see Definition 1.12). As an element of $R_0(0)$, $\tilde{\theta}_z/\xi_0$ is the sum with nonzero coefficients of monomials $\bar{\zeta}_{dz'}$ with $z' \in \operatorname{Star}(0)$ and $\bar{z}' = \bar{z}$. Therefore, for (iii) it suffices to have $\operatorname{Prim}/d \subset \operatorname{Star}(0)$ and $(\operatorname{Prim} - \operatorname{Prim}) \cap dX = \{0\}$. This follows by Lemma 4.8 again.

Next, assume that the abelian part A_0 is nontrivial. To separate the orbits $\tilde{G}_0(\bar{\sigma})$ and the points in the orbit, and to see the injectivity, repeat the above arguments with $\xi_z(e_i)$ such that e_i provide an embedding of A_0.

The immersion is again sufficient to prove at the minimal dimensional stratum. For every $p \in A_0 \subset P_0$ for the tangent space we have $\boldsymbol{T}_{p,P_0} \simeq \boldsymbol{T}_{p,\tilde{P}_0^r} \oplus \boldsymbol{T}_{p,A_0}$. Hence, if $e_0(p) \neq 0$, then to generate $\mathfrak{m}/\mathfrak{m}^2$ it is sufficient to take $\theta_{\bar{z}}(e_z)/\theta_0(e_0)$ and $\theta_0(e_i)/\theta_0(e_0)$, where $e_z(T_{c^t(dz)}(p)) \neq 0$ and e_i/e_0 generate $\boldsymbol{T}^*_{p,A_0}$.

LEMMA 4.8. (i) $\operatorname{Prim} \subset r\operatorname{Star}(0)$.

(ii) *Assume $d \geq r+2$. Then for each Delaunay cell σ the differences of vectors in $\sigma^0 \cap X/d$ generate $\boldsymbol{R}\sigma \cap X/d$.*

(iii) $(\operatorname{Star}(0) - \operatorname{Star}(0)) \cap (2+\varepsilon)X = \{0\}$ *for any* $0 < \varepsilon \ll 1$.

PROOF. Since the restriction of the Delaunay decomposition to $\boldsymbol{R}\sigma$ is again a Delaunay decomposition, we can assume that σ is maximal-dimensional.

Let $\sigma \subset \operatorname{Star}(0)$ be a Delaunay cell and let $w \in \operatorname{Cone}(0,\sigma)$ be a primitive lattice vector. Choose arbitrary r linearly independent Delaunay vectors $v_1, \dots, v_r$ with $w \in \operatorname{Cone}(v_1, \dots, v_r)$, and write $w = \sum p_i v_i$ for some $p_i \in \boldsymbol{Q}$. Then obviously $p_i \leq 1$, and w/r belongs to the convex hull of $0, v_1, \dots, v_r$, which is a part of $\operatorname{Star}(0)$. This proves (i).

For (ii) note that the vectors $\sum_{i=1}^r v_i$, $(\sum v_i) + v_j$, $j = 1 \dots r$ and $(\sum v_i) - w$, with $w \neq v_i$ primitive all belong to $((r+2)\sigma)^0$.

For (iii), let $\sigma_1 \neq \sigma_2$ be two Delaunay cells in $\operatorname{Star}(0)$. Then for any $y \in X$ the intersection $\sigma_1 \cap T_y\sigma_2$ is either σ_1, or a proper face of σ_1 or empty. In the first case y must be a Delaunay vector, since both σ_i contain 0. Therefore, $y \notin 2X$. Consequently, for any $y \neq 0$, $\operatorname{Star}(0) \cap T_{2y}\operatorname{Star}(0)$ has no interior, and $\operatorname{Star}(0) \cap T_{(2+\varepsilon)y}\operatorname{Star}(0) = \emptyset$. □

REMARK 4.9. The bound above is certainly not optimal. However, it seems that a better bound would require going much deeper into the combinatorics of Delaunay cells. In the proof above the only properties we used were that Del_B is X-periodic and that a Delaunay cell does not contain lattice points except its vertices.

5. Additions. A. *Complex-analytic case.* Everything works in the same way as in the algebraic case, only easier. The ring $\mathcal{R}$, resp. K is replaced by the stalk of functions homomorphic, resp. meromorphic in a neighborhood of 0. A major simplification comes in the construction of the quotient $(\tilde{P}, \tilde{\mathcal{L}})/Y$. Considering the fattenings $(\tilde{P}_n, \tilde{\mathcal{L}}_n)/Y$ and then algebraizing is unnecessary, since the Y-action is properly discontinuous in classical topology over a small neighborhood of 0. Hence, one can take the quotient directly.

The combinatorial description of the family and the central fiber and the data for an SQAP remain the same.

B. *Higher degree of polarization.* The formulas in our construction are set up in such a way that we can repeat it for any degree of polarization. The outcome, after a finite base change, is a normal family with reduced central fiber and a relatively ample divisor. However, in this case there are several additional choices to make:

1. an embedding $\mathcal{M}_x \hookrightarrow \mathcal{M}_{x,\eta}$ for each nonzero representative of X/Y,
2. a section $\theta_{A,x} \in H^0(A, \mathcal{M}_x)$ for each representative $x \in X/Y$.

According to the description of $H^0(A_\eta, \mathcal{L}_\eta)$ in [FC90, II.5.1], this data is equivalent to providing a theta divisor on the generic fiber. This is why there are infinitely many relatively complete models in the case of higher polarization.

6. Examples. Below we list all the SQAPs in dimensions 1 and 2, for illustration purposes. They can already be found in [Nam76, Nam80] (over $\boldsymbol{C}$).

A. *Dimension* 1. In this case there is only one Delaunay decomposition of $\boldsymbol{Z} \subset \boldsymbol{R}$, so there is only one principally polarized stable quasiabelian pair besides the elliptic curves. The 0-dimensional Delaunay cells correspond to integers n, and 1-dimensional cells to intervals $[n, n+1]$. By Theorem 3.10 the corresponding toric varieties are projective lines $(\boldsymbol{P}^1, \mathcal{O}(1))$ intersecting at points, and the intersections are transversal by Theorem 3.17. The theta divisor restricted to each $\boldsymbol{P}^1$ has to be a section of $\mathcal{O}(1)$, i.e., a point. The quotient P is, obviously, a nodal rational curve.

B. *Dimension* 2. Let us look at the case of the maximal degeneration first. There are only two Delaunay decompositions shown on Figure 1 and Figure 2. In each case the irreducible components are the projective toric varieties described by the lattice polytopes σ, see Theorem 3.10.

In the first case we have a net of $(\boldsymbol{P}^1 \times \boldsymbol{P}^1, \mathcal{O}(1,1))$'s intersecting transversally at lines which in turn intersect in fours at points. The quotient by $\boldsymbol{Z}^2$ will have one irreducible component. It is obtained from $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ by gluing two pairs of zero and infinity sections. Modulo the action of $K(\mathrm{Del}) \otimes \boldsymbol{G}_m$ we have only one parameter $z = b(e_1, e_2)$ and the SQAPs with z and $1/z$ are isomorphic. Therefore, we have a family of SQAPs of this type parameterized by $k^*/\boldsymbol{Z}_2 = k$. The theta divisor is the image of a conic on $\boldsymbol{P}^1 \times \boldsymbol{P}^1$. It is reduced. It is irreducible unless $z = 1$, in which case it is a pair of lines.

In the second case Theorem 3.10 we get a net of projective planes $(\boldsymbol{P}^2, \mathcal{O}(1))$ meeting at lines which in turn meet at points. The quotient will have two irreducible component, since there are two non-equivalent maximal-dimensional Delaunay cells modulo the lattice. In this case $K(\mathrm{Del}) \otimes \boldsymbol{G}_m$ is 3-dimensional, so a variety $(P_0, \mathcal{L}_0)$ of this type is unique up to isomorphism. The theta divisor restricted to $\boldsymbol{P}^2$ has to be a section of $\mathcal{O}(1)$, i.e., a line. Therefore, the theta divisor on P is a union of two rational curves and it is easy to see that they intersect at 3 points, one for each $\boldsymbol{P}^1$. In other words, the theta divisor in this case is a "dollar curve".

REMARK 6.1. Note that the lattices in the above two examples are of types $A_1 \oplus A_1$ and A_2, respectively (see f.e. [CS93]). The number 4, resp. 6, of branches meeting at the 0-dimensional strata has an interesting interpretation in this case. For any lattice of the A, D, E-type this is what in the lattice theory called the "kissing number" (think of the billiard balls with centers at the lattice elements, each ball is "kissed" by 4, resp. 6, other balls).

There is only one case for a nontrivial abelian part (besides the smooth abelian surfaces): when A_0 is an elliptic curve. This is the simplest case of what Mumford called "the first order degenerations of abelian varieties" in [Mum83].

Before dividing by the lattice $Y = \boldsymbol{Z}$ we have a locally free fibration over an elliptic curve with a fiber which corresponds to the case of maximal degeneration of dimension 1, i.e., a chain of projective lines. The group Y acts on this scheme $\tilde{P}_0$ by cycling through the chain and at the same time shifting "sideways" with respect to the elliptic curve. The theta divisor $\tilde{\Theta}_0$ on $\tilde{P}_0$ is invariant under this shift, so it descends to a divisor on P_0.

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