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MODULAR INEQUALITIES FOR THE CALDERÓN OPERATOR

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Abstract. If $P, Q : [0, \infty) \to$ are increasing functions and T is the Calderón operator defined on positive or decreasing functions, then optimal modular inequalities $\int P(Tf) \leq C \int Q(f)$ are proved. If $P = Q$, the condition on P is both necessary and sufficient for the modular inequality. In addition, we establish general interpolation theorems for modular spaces.

1. Introduction. Let $(\mathcal{M}, \mu)$ and $(\mathcal{N}, \nu)$ be two σ-finite measure spaces, and let $L_0(\mu)$ and $L_0(\nu)$ be the sets of measurable functions defined on $\mathcal{M}$ and $\mathcal{N}$, respectively. An operator $T : L_0(\nu) \to L_0(\mu)$ is called quasilinear if $|T(\lambda f)(x)| = |\lambda||Tf(x)|$ and if there exists a constant $K > 0$ independent of f and g such that $|T(f + g)(x)| \leq K(|Tf(x)| + |Tg(x)|)$. If $K = 1$, T is said to be sublinear.

A function $Q : [0, \infty) \to [0, \infty)$ is called a modular function if Q is an increasing (non-decreasing) function and $Q(0+) = 0$. If, in addition, Q satisfies the Δ_2-condition

$$Q(2t) \leq CQ(t)$$

for any $t > 0$, then Q is called a Δ_2-modular function and we write $Q \in \Delta_2$.

Let Q be a modular function and set

$$L_Q(\mu) = L_Q = \left\{ f \in L_0(\mu);\ \|f\|_Q = \int_{\mathcal{M}} Q(|f(x)|)d\mu(x) < \infty \right\}.$$

Then, we want to study mapping properties for which $T : L_Q(\mu) \to L_P(\nu)$ is bounded, for certain operators T.

Modular inequalities have been studied previously by several authors (cf. [KK], [C], [L]) in connection with weight characterizations. However, unlike the case treated here, the functions P and Q are typically Young's or N-functions, and the optimality of P and Q is not in general considered.

Recall that if T is an operator of weak type (a, a) and (b, b), $0 < a < b < \infty$; that is, $\nu(\{x \in \mathcal{N}; |Tf(x)| > y\}) \leq (C\|f\|_{\alpha,\mu}/y)^{\alpha}$ where $\alpha = a$ and $\alpha = b$, then

$$\int_{\mathcal{N}} P(|Tf(x)|)d\nu(x) \leq C \int_{\mathcal{M}} Q(|f(x)|)d\mu(x) \tag{1}$$

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is satisfied for $P(x) = Q(x) = |x|^p$ and $a < p < b$. Moreover, such operators satisfy the rearrangement inequality

$$(2) \qquad (Tf)^*_\nu(t) \le C\left(\frac{1}{t^{1/a}}\int_0^t f^*_\mu(s)s^{1/a-1}ds + \frac{1}{t^{1/b}}\int_t^\infty f^*_\mu(s)s^{1/b-1}ds\right),$$

where $f^*_\mu(t) \inf\{s \ge 0; \lambda^\mu_f(s) \le t\}$ is the rearrangement decreasing function of f and $\lambda^\mu_f(y) = \mu(\{x; |f(x)| > y\})$ is the distribution function of f. Similarly it is understood for $(Tf)^*_\nu$. The term in parenthesis on the right of (2) is called the Calderón operator.

In order to prove (1) for general modular functions, observe that for Q modular, an elementary argument shows that

$$\int_{\mathcal{M}} Q(|f(x)|)d\mu(x) = \int_0^\infty Q(f^*_\mu(t))dt = \int_0^\infty \lambda^\mu_f(y)dQ(y),$$

such that a general (P, Q) modular inequality will follows if

$$\int_0^\infty P[C((S_a f^*_\mu)(t) + (\tilde{S}_b f^*_\mu)(t))]dt \le C_1 \int_0^\infty Q(f^*_\mu(t))dt$$

holds, where

$$S_a f(t) = \frac{1}{t^{1/a}}\int_0^t f(s)s^{1/a-1}ds$$

and

$$\tilde{S}_b f(t) = \frac{1}{t^{1/b}}\int_t^\infty f(s)s^{1/b-1}ds\,.$$

Note that $S_1 = S$ is the Hardy averaging operator and $\tilde{S}_\infty = \tilde{S}$ is the conjugate Hardy operator.

The purpose of this paper is to provide optimal conditions characterizing modular pairs P and Q, for which (P, Q) (and in case $Q = P$, (P)) modular inequalities

$$\int_0^\infty P(S_a f(t))dt \le C\int_0^\infty Q(f(t))dt$$

and

$$\int_0^\infty P(\tilde{S}_b f(t))dt \le C\int_0^\infty Q(f(t))dt$$

are satisfied for $0 < a, b < \infty$. The case where $b = \infty$ and T is bounded on L^∞ is also considered. These results yield sharper estimates and interpolation theorems for several classical operators.

The paper is organized as follows: In Section 2, we characterize (P, Q) modular inequalities for S_a, $0 < a \le 1$ (Theorem 2.3) and give a corresponding characterization in the case when $a = 1$ and f is decreasing for a reverse Hardy modular inequality (Theorem 2.1). In order to prove corresponding (P, Q) modular inequalities for S_a, $a > 1$ and $\tilde{S}_b$, $0 < b < \infty$, some general modular results are required. These are proved in Section 3 and yield general modular interpolation theorems (Corollary 3.6). Finally the last section contains the (P, Q) and (P) modular inequalities for S_a, $a > 1$ and $\tilde{S}_b$, $0 < b \le \infty$. A characterization of P, Q modular functions for which a (P, Q) modular inequality for the Hilbert transform holds and a short proof of an interpolation theorem of Miyamoto ([M]) for modular functions are also given.

The notation used in this paper is standard: If f/g is bounded above and below by positive constants, we write $f \approx g$ and say that f and g are equivalent functions. Constants denoted by C, sometimes with subscripts, are assumed to be positive and independent of the functions involved, and may differ at different places. If $0 \leq g$ is decreasing, we write $g^{**}(x) = (1/x)\int_0^x g$, where the measure under which the rearrangement occurs is deleted when there is no ambiguity. χ_E is the characteristic function of the set E and its Lebesgue measure is denoted by $|E|$.

Finally, inequalities, such as (1), are interpreted in the sense that if the right side is finite, so is the left side and the inequality holds. Unless indicated to the contrary, we assume that P and Q are modular functions or are equivalent to modular functions.

2. Modular inequalities for the Hardy operator. We begin this section by proving (P, Q) modular inequalities for the Hardy averaging operator.

THEOREM 2.1. (i) *There exist two constants $C_1 > 0$ and $C_2 > 0$ such that*

$$\int_0^\infty P\left(\frac{\int_0^t f}{t}\right) dt \leq C_1 \int_0^\infty Q(C_2 f(t))dt \tag{3}$$

is satisfied for every decreasing nonnegative function f if and only if there exist constants $C_3 > 0$ and $C_4 > 0$ such that, for every $t > 0$,

$$P(t) + t\int_0^t \frac{P(y)}{y^2}dy \leq C_3 Q(C_4 t). \tag{4}$$

(ii) *The inequality* (3) *is reversed for every decreasing nonnegative function f if and only if the inequality* (4) *is reversed.*

(iii) *There exist constants $C_1 > 0$ and $C_2 > 0$ such that*

$$\sup_{t>0} tP\left(\frac{\int_0^t f}{t}\right) \leq C_1 \int_0^\infty Q(C_2 f(t))dt$$

is satisfied for every decreasing nonnegative function f if and only if there exist constants $C_3 > 0$ and $C_4 > 0$ such that, for every $r > 0$,

$$\sup_{u \leq r} \frac{P(u)}{u} \leq C_3 \frac{Q(C_4 r)}{r}.$$

We thank J. Soria for pointing out that the argument in proving (i) applies also to the proof of (iii).

PROOF. (i) To show the necessary condition, let us take $f(s) = t\chi_{[0,r)}(s)$. Then, we have that

$$\int_0^\infty P\left(\frac{t}{x}\min(r, x)\right) dx \leq C_1 r Q(C_2 t);$$

that is,

$$\int_0^r P(t)dx + \int_r^\infty P\left(\frac{tr}{x}\right)dx = rP(t) + rt\int_0^t \frac{P(y)}{y^2}dy \le C_1 r Q(C_2 t)\,,$$

from which the result follows with $C_3 = C_1$ and $C_4 = C_2$.

Conversely, if (4) holds, then we may assume that for small $t > 0$, $\int_0^t P(y)/y^2 dy < \infty$, and from this it follows that $P(y)/y \to 0$ as $y \to 0$.

Now, writing $f^{**}(t) = (1/t)\int_0^t f(s)ds$, we have

$$\int_0^\infty P(f^{**}(t))dt = \int_0^\infty \lambda_{f^{**}}(z)dP(z)\,,$$

where the distribution function of f^{**} satisfies (see [CS])

$$\frac{1}{2s}\lambda_f^f(s) \le \lambda_{f^{**}}(s) \le \frac{2}{s}\lambda_f^f(s/2)\,, \tag{5}$$

and hence

$$\begin{aligned}\int_0^\infty \lambda_{f^{**}}(z)dP(z) &\le \int_0^\infty \frac{2}{z}\left(\int_{\{x;\,f(x)>z/2\}} f(x)dx\right)dP(z)\\ &= 2\int_0^\infty f(x)\left(\int_0^{2f(x)} \frac{dP(z)}{z}\right)dx\\ &= 2\int_0^\infty f(x)\left(\frac{P(2f(x))}{f(x)} + \int_0^{2f(x)} \frac{P(z)}{z^2}dz\right)dx\\ &\le 2C_3\int_0^\infty Q(2C_4 f(x))dx\,.\end{aligned}$$

That is, (3) holds with $C_1 = 2C_3$ and $C_2 = 2C_4$.

(ii) The proof follows as in (i), but now the first inequality of (5) is applied.

(iii) The weak type characterization follows analogously. □

REMARK 2.2. (i) If either P or Q is equivalent to a Δ_2-modular function, then the estimate of the theorem holds with $C_2 = C_4 = 1$. In this case, under strong additional conditions on P, the result was proved in [HL, Prop. 2].

(ii) Since $\int_0^\infty Q(f(t))dt = \int_0^\infty Q(f^*(t))dt$ and $\int_0^t f \le \int_0^t f^*$, one can easily see that (4) is also a necessary and sufficient condition for (3) to hold for every measurable positive function.

(iii) If M is the Hardy-Littlewood maximal operator, then $(Mf)^* \approx Mf^* \approx f^{**} = (1/x)\int_0^x f^*$ ([BS, Theorem 3.8]). Therefore, it follows from Theorem 2.1(i) that (4) is necessary and sufficient for the modular inequality

$$\int_{\mathbf{R}^n} P(Mf(x))dx \le C\int_{\mathbf{R}^n} Q(|f(x)|)dx$$

to be satisfied.

For S_a, we have the following result:

THEOREM 2.3. *Let $0 < a < 1$. Then, there exist constants $C_1 > 0$ and $C_2 > 0$ such that*

$$\int_0^\infty P\left(\frac{1}{t^{1/a}}\int_0^t f(s)s^{1/a-1}ds\right)dt \le C_1\int_0^\infty Q(C_2 f(t))dt \tag{6}$$

is satisfied for every decreasing nonnegative function f if and only if there exist constants $C_3 > 0$ and $C_4 > 0$ such that, for every $t > 0$,

$$P(t) + t^a\int_0^t \frac{P(y)}{y^{a+1}}dy \le C_3 Q(C_4 t)\,. \tag{7}$$

PROOF. Let $g(s) = af(s^a)$. Then obvious change of variables shows that (6) is equivalent to

$$\int_0^\infty P\left(\frac{1}{t}\int_0^t g(s)ds\right)t^{a-1}dt \le C_1\int_0^\infty Q\left(\frac{C_2}{a}g(t)\right)t^{a-1}dt\,. \tag{8}$$

For the necessary condition, it is enough to apply the hypothesis to the functions $f(s) = t\chi_{(0,r)}(s)$. Then (7) follows with $C_3 = C_1$ and $C_4 = C_2/a$.

For the converse, first observe that we can assume $\int_0^t P(y)/y^{a+1}dy < \infty$ for small t, since otherwise the result is trivial. Also, in this case, $\lim_{y\to 0} P(y)/y^a = 0$.

To show that (7) implies (8), note that, interchanging the order of integration and applying (5), we obtain

$$\begin{aligned}
\int_0^\infty P\left(\frac{1}{x}\int_0^x g\right)x^{a-1}dx &= \frac{1}{a}\int_0^\infty [\lambda_{g^{**}}(z)]^a dP(z)\\
&\le \frac{2^a}{a}\int_0^\infty \left(\frac{1}{z}\lambda_g^g(z/2)\right)^a dP(z)\\
&= \frac{2^a}{a}\int_0^\infty \left(\int_{\{g(x)>z/2\}} g(x)dx\right)^a \frac{dP(z)}{z^a}\\
&= \frac{2^a}{a}\int_0^\infty \left(\int_0^{\lambda_g(z/2)} g(x)dx\right)^a \frac{dP(z)}{z^a}\\
&= 2^a\int_0^\infty \left(\int_0^{\lambda_g(z/2)}\left(\int_0^x g\right)^{a-1} g(x)dx\right)\frac{dP(z)}{z^a}\,.
\end{aligned}$$

But, since g is decreasing and $0 < a < 1$, it follows that $(\int_0^x g)^{a-1} \le (xg(x))^{a-1}$ and hence

$$\begin{aligned}\int_0^\infty P(g^{**}(x))x^{a-1}dx &\le 2^a \int_0^\infty \left(\int_0^{\lambda_g(z/2)} x^{a-1}g^a(x)dx\right)\frac{dP(z)}{z^a} \\ &= 2^a \int_0^\infty x^{a-1}g^a(x)\left(\int_0^{2g(x)} \frac{dP(z)}{z^a}\right)dx \\ &= 2^a \int_0^\infty x^{a-1}g^a(x)\left(\frac{P(2g(x))}{g(x)^a} + a\int_0^{2g(x)} \frac{P(z)}{z^{a+1}}dz\right)dx \\ &\le C_3 \int_0^\infty x^{a-1}Q(2C_4g(x))dx\,,\end{aligned}$$

where the last inequality follows from (7) with $t = 2g(x)$. Hence, (8) holds with $C_1 = C_3$ and $C_2 = 2C_4a$. □

3. General results. Clearly the arguments in proving Theorem 2.3 do not apply to obtain (P, Q) modular inequalities for S_a with $a > 1$. In order to obtain such estimates for S_a and $\tilde{S}_b$, $0 < b \le \infty$, we need some general results for quasilinear operators and the notion of admissible functions. As a consequence, we obtain a number of weak type estimates and general interpolation theorems.

Our first result shows that, under a simple condition on T, a (P) modular inequality implies $P \in \Delta_2$.

Let $L \subset L_0(\mu)$ be a set such that $\boldsymbol{R}^+L \subset L$. For us, L will be either $L_0(\mu)$ or the set of measurable decreasing functions on $\boldsymbol{R}^+$.

PROPOSITION 3.1. *Suppose that T satisfies a (P) modular inequality for every function in L. If there exist a measurable set E such that $\chi_E \subset L$ and $\mu(E) < \infty$ and a constant $d > 1$ such that*

$$\nu(\{x; |T\chi_E(x)| > d\}) \ne 0\,,$$

then $P \in \Delta_2$.

PROOF. Take $\lambda > 0$ and $f(x) = \lambda\chi_E(x)$. Then, since

$$P(y)\lambda^\nu_{Tf}(y) \le C\int_{\mathcal{M}} P(|f(x)|)d\mu(x)\,,$$

we get

$$P(y)\nu(\{x; |\lambda||T\chi_E(x)| > y\}) \le CP(\lambda)\mu(E)\,.$$

Choose now $y = d\lambda$. Then we get

$$P(d\lambda) \le \frac{C\mu(E)}{\nu(\{x; |T\chi_E(x)| > d\})}P(\lambda)\,,$$

from which the Δ_2 condition for P follows. □

Now, for our next purpose, we need to give the following definition:

DEFINITION 3.2. We say that a function $A : [0,\infty) \to [0,\infty)$ with $A(0) = 0$ is admissible for T and L if, for every function $f \in L$,

$$\lambda^{\nu}_{Tf}(1) \leq \int_{\mathcal{M}} A(|f(x)|)d\mu(x) .$$

REMARK 3.3. (i) In terms of the decreasing rearrangement the above inequality is

$$(Tf)^*_\nu \left[\int_{\mathcal{M}} A(|f(x)|)d\mu(x) \right] \leq 1 .$$

Since we are assuming $\boldsymbol{R}^+ L \subset L$, for every admissible function A for T and L and every $y > 0$, it holds that for any $f \in L$

$$\lambda^{\nu}_{Tf}(y) \leq \int_{\mathcal{M}} A\left(\frac{|f(x)|}{y}\right) d\mu(x) . \tag{9}$$

(ii) If B is a modular function such that $B(x) > 1$ for every $x > 1$ and, for every $f \in L$,

$$\int_{\mathcal{N}} B(|Tf(x)|)d\nu(x) \leq \int_{\mathcal{M}} A(|f(x)|)d\mu(x) ,$$

then

$$\begin{aligned}\sup_{y>0} yB[(Tf)^*_\nu(y)] &\leq \sup_{y>0} \int_0^y B[(Tf)^*_\nu(t)]dt \\ &= \int_{\mathcal{N}} B(|Tf(x)|)d\nu(x) \leq \int_{\mathcal{M}} A(|f(x)|)d\mu(x) .\end{aligned}$$

In particular, if $y = \int_{\mathcal{M}} A(|f(x)|)d\mu(x)$, then $B[(Tf)^*_\nu(\int_{\mathcal{M}} A(|f(x)|)d\mu(x))] \leq 1$. Then, by the hypothesis of B, this implies $(Tf)^*_\nu(\int_{\mathcal{M}} A(|f(x)|)d\mu(x)) \leq 1$, and hence A is admissible for T and L.

(iii) If T is of weak type (p, p) with $p > 0$, then $A(t) = \|T\|_{(p,p)} t^p$ is an admissible function for T and $L_0(\mu)$.

Observe that, for $0 < a < \infty$, $\|S_a f\|_\infty \leq a\|f\|_\infty$ and that if f is decreasing, then for $0 \leq b \leq \infty$, $\operatorname{supp}(\tilde{S}_b f) \subset \operatorname{supp} f$. For operators which satisfy conditions of this type we have the following result:

LEMMA 3.4. *Let L be a set as above and T a quasilinear operator defined on L.*

(i) *Let $\tilde{L} = \{g = f\chi_{\{|f|>y\}};\ f \in L, y > 0\}$ and A an admissible function for T and $\tilde{L}$. Suppose that $T : L^\infty(\mu) \to L^\infty(\nu)$ is bounded with (operator) norm less than or equal to M. Then, for every $f \in L$ and every $y > 0$,*

$$\lambda^{\nu}_{Tf}(y) \leq \int_{\{|f(x)|>y/(2MK)\}} A\left(\frac{2K|f(x)|}{y}\right) d\mu(x) , \tag{10}$$

where K is the constant arising from the quasilinearity of T.

(ii) *Let $\tilde{\tilde{L}} = \{g = f\chi_{\{|f|\leq y\}};\ f \in L, y > 0\}$ and A an admissible function for T and $\tilde{\tilde{L}}$. If there exists a constant $C > 0$ such that $\nu(\operatorname{supp} Tf) \leq C\mu(\operatorname{supp} f)$, then, for every $\varepsilon > 0$,* *every* $y > 0$ *and every* $f \in L$,

$$\lambda^{\nu}_{Tf}(y) \le \int_{\{|f(x)|\le y\}} A\left(\frac{(1+\varepsilon)K|f(x)|}{y}\right) d\mu(x) + C\lambda^{\mu}_{f}(y)\,. \tag{11}$$

PROOF. (i) Fix $y > 0$ and write $f = f_1 + f_2$, where $f_1(x) = f(x)$ if $|f(x)| > y/(2MK)$ and zero otherwise. Then,

$$\nu(\{x;\, |Tf(x)| > y\}) \le \nu\left(\left\{x;\, |Tf_1(x)| > \frac{y}{2K}\right\}\right) + \nu\left(\left\{x;\, |Tf_2(x)| > \frac{y}{2K}\right\}\right).$$

But, since $\|Tf_2\|_{\infty,\nu} \le M\|f_2\|_{\infty,\mu} \le y/(2K)$, the second term is zero, and hence, since $f_1 \in \tilde{L}$, we obtain by (9) that

$$\lambda^{\nu}_{Tf}(y) \le \int_{\mathcal{M}} A\left(\frac{2K|f_1(x)|}{y}\right) d\mu(x) = \int_{\{|f(x)|>y/(2MK)\}} A\left(\frac{2K|f(x)|}{y}\right) d\mu(x)\,.$$

To show (ii), fix $y > 0$ and write $f = f_1 + f_2$, where $f_1(x) = f(x)$ if $|f(x)| > y$ and zero otherwise. Then, for every $\varepsilon > 0$,

$$\begin{aligned}
\lambda^{\nu}_{Tf}(y) &\le \nu\left(\left\{x;\, |Tf_1(x)| \ge \frac{\varepsilon y}{K(1+\varepsilon)}\right\}\right) + \nu\left(\left\{x;\, |Tf_2(x)| \ge \frac{y}{K(1+\varepsilon)}\right\}\right)\\
&\le \nu(\{x;\, |Tf_1(x)| > 0\}) + \nu(\{x;\, K(1+\varepsilon)|Tf_2(x)| > y\})\\
&\le C\mu(\{x;\, |f(x)| > y\}) + \int_{\mathcal{M}} A\left(\frac{(1+\varepsilon)K|f_2(x)|}{y}\right) d\mu(x)\\
&= C\lambda^{\mu}_{f}(y) + \int_{\{|f(x)|\le y\}} A\left(\frac{(1+\varepsilon)K|f(x)|}{y}\right) d\mu(x)\,. \qquad \square
\end{aligned}$$

The lemma implies now the following (P, Q) interpolation theorem:

THEOREM 3.5. (i) *Let* $\tilde{L}$ *be as in Lemma* 3.4 (i). *Let T be a quasilinear operator such that* $T : L^{\infty}(\mu) \to L^{\infty}(\nu)$ *is bounded with norm M. If there exist a constant C and an admissible function A for T and* $\tilde{L}$ *such that, for every* $t > 0$,

$$\int_0^{2MKt} A\left(\frac{2Kt}{y}\right) dP(y) \le CQ(t)\,, \tag{12}$$

then T satisfies a (P, Q) *modular inequality for every function in L.*

(ii) *Let* $\tilde{\tilde{L}}$ *be as in Lemma* 3.4 (ii). *Suppose that for every* $f \in L$, $\nu(\operatorname{supp} Tf) \le C\mu(\operatorname{supp} f)$ *for some constant C independent of f. If there exist a constant C and an admissible functin A for T and* $\tilde{\tilde{L}}$ *such that, for every* $t > 0$, $\lim_{z\to 0} P(t/z)A(z) = 0$ *and, for some* ε,

$$P(t) + \int_t^{\infty} A\left(\frac{(1+\varepsilon)t}{z}\right) dP(z) \le CQ(t)\,, \tag{13}$$

then T satisfies a (P, Q) *modular inequality for every function in L.*

PROOF. (i) By (10) and (12),

$$\begin{aligned}\int_{\mathcal{N}} P(|Tf(x)|)d\nu(x) &= \int_0^\infty \lambda_{Tf}^{\nu}(y)dP(y)\\ &\le \int_0^\infty \left[\int_{\{|f(x)|>y/(2MK)\}} A\left(\frac{2K|f(x)|}{y}\right)d\mu(x)\right]dP(y)\\ &= \int_{\mathcal{M}}\left[\int_0^{2MK|f(x)|} A\left(\frac{2K|f(x)|}{y}\right)dP(y)\right]d\mu(x)\\ &\le C\int_{\mathcal{M}} Q(|f(x)|)d\mu(x)\,,\end{aligned}$$

which proves (i).

The proof of (ii) follows in the same way, using now (11) and (13). □

Note that if $S(x) = \chi_{(1,\infty)}(x)$, then $L_S(\nu) = \{f;\ \lambda_f^{\nu}(1) < \infty\}$. Similarly, if one defines $L^0(\mu)$ by $L^0(\mu) = \{f;\ \mu(\operatorname{supp} f) < \infty\}$, then Theorem 3.5 has the following formulation:

COROLLARY 3.6. *Suppose that $T : L_A(\mu) \to L_S(\nu)$ is bounded.*

(i) *If $\tilde{L}$ is as in Lemma* 3.4 (i), *$T : L^\infty(\mu) \to L^\infty(\nu)$ is bounded with norm M and A is an admissible function for T and $\tilde{L}$, then $T : L_Q(\mu) \to L_P(\nu)$ is bounded for (P, Q) satisfying* (12).

(ii) *If $\tilde{\tilde{L}}$ is as in Lemma* 3.4 (ii), *$T : L^0(\mu) \to L^0(\nu)$ is bounded and A is an admissible function for T and $\tilde{\tilde{L}}$, then $T : L_Q(\mu) \to L_P(\nu)$ is bounded for (P, Q) satisfying* (13).

4. The Calderón operator. We now derive (P) and (P, Q) modular inequalities for S_a with $a > 1$ and $\tilde{S}_b$, $0 < b < \infty$, as well as for $\tilde{S}$. In addition, we give a short proof and an extension of an interpolation theorem of Miyamoto [M].

PROPOSITION 4.1. *Assume that $a > 1$ and $\chi_{(0,1)} \in L$. Then, if S_a satisfies a (P) modular inequality for L, $P \in \Delta_2$.*

PROOF. Let $E = (0, 1)$ in Proposition 3.1. Then, it suffices to show that, for some $d > 1$, $|\{x;\ |S_a\chi_{(0,1)}(x)| > d\}| \neq 0$. But, since $a > 1$, we can choose $a > d > 1$, and hence, since

$$S_a\chi_{(0,1)}(x) = \begin{cases} a & \text{if } x < 1 \\ a/x^{1/a} & \text{if } x \ge 1\,,\end{cases}$$

we get

$$|\{x;\ |S_a\chi_{(0,1)}(x)| > d\}| = \left(\frac{a}{d}\right)^a \neq 0\,.$$ □

The main result for S_a is the following:

THEOREM 4.2. *Let $a > 1$ and assume that, for every $r > 0$, $\chi_{(0,r)} \in L$. Then the following hold.*

(i) *If S_a satisfies a (P, Q) modular inequality for L, there exists a constant C such that, for every $t > 0$,*

$$t^a \int_0^t \frac{P(y)}{y^{a+1}} dy \leq CQ(t) . \tag{14}$$

(ii) *If there exists an $\varepsilon > 0$ such that*

$$P(2t) + t^{a+\varepsilon} \int_0^{2t} \frac{P(y)}{y^{a+\varepsilon+1}} dy \leq CQ(t) , \tag{15}$$

then S_a satisfies a (P, Q) modular inequality for L.

(iii) *S_a satisfies a (P) modular inequality for L if and only if $P \in \Delta_2$ and there exists a constant C such that, for every $t > 0$,*

$$t^a \int_0^t \frac{P(y)}{y^{a+1}} dy \leq CP(t) . \tag{16}$$

PROOF. (i) It is enough to check the hypothesis on the functions $f = t\chi_{(0,r)}$.

(ii) Clearly, $S_a : L^\infty \to L^\infty$ is bounded with norm a and $S_a : L^{a,1} \to L^{a,\infty}$ is bounded. Therefore, by interpolation, $S_a : L^{a+\varepsilon} \to L^{a+\varepsilon}$ is bounded for every $\varepsilon > 0$, and hence, for some constant C, $A(t) = Ct^{a+\varepsilon}$ is an admissible function for S_a and every subset of $L_0(\boldsymbol{R}^+)$.

Now, by Theorem 3.5 with $M = a$ and $A(t) = Ct^{a+\varepsilon}$, the linear operator S_a satisfies the (P, Q) modular inequality provided that

$$(2t)^{a+\varepsilon} \int_0^{2at} \frac{dP(y)}{y^{a+\varepsilon}} \leq CQ(t) . \tag{17}$$

But since (15) is satisfied, it follows that $\lim_{y\to 0+} P(y)/y^{a+\varepsilon} = 0$, and hence an integration by parts shows that (17) is equivalent to (15).

(iii) If S_a satisfies a (P) modular inequality, then by Proposition 4.1, $P \in \Delta_2$ and (16) now follows from (14) with $Q = P$. Conversely, if $P \in \Delta_2$, then there exists $q > a$ such that $P(y)/y^q$ is decreasing (see [KK]), and hence, by (16)

$$CP(t) \geq t^a \int_0^t \frac{P(y)}{y^{a+1}} dy \geq C_1 P(t) .$$

For $m = 0, 1, 2, \dots$, define

$$A_m = t^a \int_0^t \frac{P(y)}{y^{a+1}} \frac{(\log(t/y))^m}{m!} dy .$$

Then, by (16),

$$\begin{aligned} A_m &= t^a \int_0^t \frac{P(y)}{y^{a+1}} \left(\int_y^t \frac{(\log(t/s))^{m-1}}{(m-1)!} \frac{ds}{s} \right) dy \\ &= t^a \int_0^t \frac{(\log(t/s))^{m-1}}{(m-1)! s^{a+1}} \left(s^a \int_y^s \frac{P(y)}{y^{a+1}} dy \right) ds \\ &\leq CA_{m-1} . \end{aligned}$$

Therefore, $A_m \leq C^m A_0 = C^{m+1} P(t)$. Choose $1 < C < M$ and $\varepsilon < 1/M$. Then

$$\sum_{m=0}^{\infty} \varepsilon^m A_m \leq \sum_{m=0}^{\infty} \frac{A_m}{M^m} \leq C P(t) \sum_{m=0}^{\infty} \left(\frac{C}{M}\right)^m = C_2 P(t) .$$

Also

$$\sum_{m=0}^{\infty} \varepsilon^m A_m = \sum_{m=0}^{\infty} t^a \int_0^t \frac{P(y)}{y^{a+1}} \frac{(\varepsilon \log(t/y))^m}{m!} dy = t^a \int_0^t \frac{P(y)}{y^{a+1}} \left(\frac{t}{y}\right)^{\varepsilon} dy ,$$

and hence

$$t^{a+\varepsilon} \int_0^t \frac{P(y)}{y^{a+\varepsilon+1}} dy \leq C_2 P(t) .$$

But, since $P \in \Delta_2$, this implies (15) with $Q = P$, and so S_a satisfies the (P) modular inequality. □

We now consider the operator $\tilde{S}_b$ with $b > 0$.

PROPOSITION 4.3. *Assume that $b > 0$ and $\chi_{(0,1)} \in L$. Then, if $\tilde{S}_b$ satisfies a (P) modular inequality for L, $P \in \Delta_2$. Moreover, in this case, the Δ_2 constant for P is less than or equal to $C((2+b)/b)^b$, where C is the constant arising from the (P) modular inequality.*

PROOF. It is enough to see that the set $E = (0, 1)$ and $d = 2$ satisfies the condition of Proposition 3.1. But $\tilde{S}_b \chi_{(0,1)}(x) = b(x^{-1/b} - 1)\chi_{(0,1)}(x)$. It then follows that $|\{x; |\tilde{S}_b \chi_{(0,1)}(x)| > 2\}| = (b/(b+2))^b \neq 0$, and hence

$$P(y) \leq C \left(\frac{b+2}{b}\right)^b P\left(\frac{y}{2}\right), \quad y > 0 .$$ □

We shall also need the following lemma.

LEMMA 4.4. *Let $M \geq 0$. If f is a decreasing function on $[M, \infty)$ and $0 < p \leq q \leq \infty$, then, for every $x \geq 2M$,*

$$\left(\int_x^{\infty} (t^{1/p} f(t))^q \frac{dt}{t}\right)^{1/q} \leq C \left(\int_M^{\infty} f^p(t) dt\right)^{1/p} ,$$

where the constant depends only on p and q.

PROOF. The result follows from a straightforward modification of the case $M = 0$ given in [St. p. 273]. □

THEOREM 4.5. *Let $0 < b < \infty$ and assume that $\tilde{S}_b$ is defined on decreasing functions.*

(i) *If $\tilde{S}_b$ satisfies a (P, Q) modular inequality, then there exists a constant C such that, for every $t > 0$,*

$$t^b \int_t^{\infty} \frac{P(y)}{y^{b+1}} dy \leq C Q(t) . \tag{18}$$

(ii) *If there exists an* $\varepsilon > 0$ *such that*

$$t^{b-\varepsilon}\int_t^\infty \frac{P(y)}{y^{b-\varepsilon+1}}dy \le CQ(t)\,, \tag{19}$$

then $\tilde{S}_b$ *satisfies a* (P, Q) *modular inequality.*

(iii) $\tilde{S}_b$ *satisfies a* (P) *modular inequality if and only if there exists a constant* C *such that, for every* $t > 0$, (18) *holds for* $Q = P$.

PROOF. (i) It is enough to check the hypothesis on the functions $f = t\chi_{(0,r)}$.

(ii) Let us consider first the case $b > 1$. Choose $\varepsilon > 0$ such that $b - \varepsilon > 1$. Then, it follows from the weighted (conjugate) Hardy inequalities ([Mu]) that $\tilde{S}_b : L^{b-\varepsilon} \to L^{b-\varepsilon}$ is bounded and therefore, for some constant C, the function $A(t) = Ct^{b-\varepsilon}$ is an admissible function for $\tilde{S}_b$ and every subset of $L_0(\boldsymbol{R}^+)$.

Consequently, the function $A(t) = C_1 t^{b-\varepsilon}$ is an admissible function for $\tilde{S}_b$ and $\tilde{\tilde{L}}$ and, since $|\operatorname{supp} \tilde{S}_b f| \le |\operatorname{supp} f|$, we can apply Theorem 3.5(ii). Hence, if for some ε',

$$P(t) + C_1(1+\varepsilon')^{b-\varepsilon}t^{b-\varepsilon}\int_t^\infty \frac{dP(z)}{z^{b-\varepsilon}} \le CQ(t)\,,$$

then we see that $\tilde{S}_b$ satisfies a (P, Q) modular inequality.

Since we may assume that the integral on the left side of (19) is bounded, it follows that $P(y)/y^{b-\varepsilon} \to 0$ as $y \to \infty$. Integration by parts argument then shows that (19) implies the above inequality.

Let now $0 < b \le 1$. Then, we do not know if the $A(t) = C_1 t^{b-\varepsilon}$ is an admissible function for $\tilde{S}_b$ and $\tilde{\tilde{L}}$, but the inequality (11) still holds. To see this, we have to apply Lemma 4.4 as follows. Let f be a decreasing function and set $g = f\chi_{\{|f|\le y\}}$ with $y > 0$. Choose $\varepsilon > 0$ such that $\alpha = b - \varepsilon > 0$. Applying Lemma 4.4 with $p = \alpha$ and $q = 1$, it then follows that, if $x \ge 2\lambda_f(y)$,

$$\begin{aligned}\tilde{S}_b g(x) &= x^{-1/b}\int_x^\infty g(s)s^{1/b-1}ds = x^{-1/b}\int_x^\infty g(s)s^{1/b-1}s^{1/\alpha}s^{-1/\alpha}ds\\ &\le x^{-1/\alpha}\int_x^\infty g(s)s^{1/\alpha-1}ds \le Cx^{-1/\alpha}\|g\|_\alpha\,.\end{aligned}$$

Therefore, for every $z > 0$,

$$\begin{aligned}|\{x > 0; |\tilde{S}_b g(x)| > z\}| &\le 2\lambda_f(y) + |\{x \ge 2\lambda_f(y); |\tilde{S}_b g(x)| > z\}|\\ &\le 2\lambda_f(y) + |\{x \ge 2\lambda_f(y); Cx^{-1/\alpha}\|g\|_\alpha > z\}|\\ &\le 2\lambda_f(y) + \left(\frac{C\|g\|_\alpha}{z}\right)^\alpha,\end{aligned}$$

and hence, for every $\varepsilon > 0$,

$$\begin{aligned}\lambda_{\tilde{S}_b f}(y) &\leq \left|\left\{x > 0; |\tilde{S}_b(f-g)(x)| \geq \frac{\varepsilon y}{(1+\varepsilon)}\right\}\right| + \left|\left\{x; |\tilde{S}_b g(x)| \geq \frac{y}{(1+\varepsilon)}\right\}\right| \\ &\leq |\{x; |\tilde{S}_b(f-g)(x)| > 0\}| + |\{x; (1+\varepsilon)|\tilde{S}_b g(x)| > y\}| \\ &\leq C(|\{x; |f(x)| > y\}| + |\{x \leq 2\lambda_f(y); (1+\varepsilon)|\tilde{S}_b g(x)| > y\}| \\ &\quad + |\{x > 2\lambda_f(y); (1+\varepsilon)|\tilde{S}_b g(x)| > y\}|) \\ &\leq C\lambda_f(y) + \int_{\{|f(x)|\leq y\}} \left(\frac{(1+\varepsilon)|f(x)|}{y}\right)^{\alpha} dx\,,\end{aligned}$$

which is the inequality (11). The proof now proceeds as for the case $b > 1$.

(iii) If $\tilde{S}_b$ satisfies a (P) modular inequality, then by (i), (18) holds with $Q = P$.

Conversely, if (18) holds with $Q = P$, then it follows that $P(y)/y^b$ tends to zero when y tends to infinity, and an integration by parts shows that (18) is equivalent to

$$\int_t^\infty \frac{dP(y)}{y^b} \leq Ct^{-b}\int_0^t dP(y)\,.$$

This implies that dP satisfies a B_b condition (see [AM]), and hence it is known (see for example Lemma 3 of [CW]) that there exists an $\varepsilon > 0$ such that $dP \in B_{p-\varepsilon}$. Again an integration by parts shows that

$$t^{b-\varepsilon}\int_t^\infty \frac{P(y)}{y^{b-\varepsilon+1}}dy \leq CP(t)\,,$$

and the result follows from (ii). □

If $b = \infty$, we have the following result for the conjugate Hardy operator.

THEOREM 4.6. *Assume that, for every $r > 0$, $\chi_{(0,r)} \in L$. Then*

(i)

$$\int_0^\infty P\left(\int_t^\infty \frac{f(s)}{s}ds\right)dt \leq C\int_0^\infty P(f(t))dt\,, \qquad f \in L \tag{20}$$

if and only if $P \in \Delta_2$.

(ii) *If either P or $Q \in \Delta_2$, then $\tilde{S}$ satisfies a (P, Q) modular inequalities if and only if $P \leq CQ$.*

PROOF. (i) If the inequality (20) holds, we have that $P \in \Delta_2$ by Proposition 3.1, since obviously

$$|\{x; |\tilde{S}\chi_{(0,1)}(x)| > 2\}| = e^{-2} \neq 0\,.$$

Conversely, if $P \in \Delta_2$, then (see [KK]) there exists $p > 0$ such that $P(t)/t^p$ is equivalent to a decreasing function and hence

$$\int_t^\infty \frac{P(y)}{y^{p+2}}dy \leq C\frac{P(t)}{t^{p+1}}\,.$$

An integration by parts shows that $t^{p+1}\int_t^\infty (1/y^{p+1})dP(y) \leq CP(t)$ and, since we already know that $A(t) = t^{b+1}$ is admissible for $\tilde{S}$, we get (i) from Theorem 3.5 (ii).

(ii) Suppose P or Q satisfies Δ_2. Then, by (i) a (P) or (Q) modular inequality is satisfied. Since $P \le CQ$, we get the (P, Q) modular inequality in either case.

Conversely, if we apply the (P, Q) modular inequality to the functions $f(x) = t\chi_{(0,1)}(x)$, we get

$$\int_0^1 P\left(t \log \frac{1}{x}\right) dx \le CQ(t),$$

and with $z = t\log(1/x)$, we obtain

$$\frac{1}{e}P(t) \le \frac{1}{t}\int_t^\infty P(z)e^{-z/t}dz \le \int_0^1 P\left(t\log\frac{1}{x}\right)dx \le CQ(t). \qquad \square$$

Theorem 2.1(i) and Theorem 4.6 now yields a characterization of a (P, Q) modular inequality for the Hilbert transform.

COROLLARY 4.7. *Suppose either P or Q satisfies the Δ_2 condition. Then, the (P, Q) modular inequality for the Hilbert transform*

$$\int_{\boldsymbol{R}} P(|Hf(x)|)dx \le C\int_{\boldsymbol{R}} Q(|f(x)|)dx \tag{21}$$

is satisfied for $f \in L_0(dx)$ if and only if $P \le CQ$ and

$$t\int_0^t \frac{P(s)}{s^2}ds \le CQ(t). \tag{22}$$

PROOF. Clearly (21) is equivalent to

$$\int_0^\infty P((Hf)^*(x))dx \le C\int_0^\infty Q(f^*(x))dx.$$

But, since (see [S])

$$(Hf)^*(x) \le C_1\left[\frac{1}{x}\int_0^x f^*(t)dt + \int_x^\infty \frac{f^*(t)}{t}dt\right] \le C_2(Hf^*)^*(x),$$

it follows that (21) is satisfied if and only if

$$\int_0^\infty P\left[\frac{1}{x}\int_0^x f^*(t)dt\right]dx \le C\int_0^\infty Q(f^*(x))dx,$$

and

$$\int_0^\infty P\left[\int_x^\infty \frac{f^*(t)}{t}dt\right]dx \le C\int_0^\infty Q(f^*(x))dx$$

is satisfied. Then, by Theorem 2.1(i) and Theorem 4.5 (ii), this holds if and only if $P \le CQ$ and (22) holds. $\square$

Finally, we give a short proof of an interpolation theorem proved by Miyamoto in [M] in the case where P is continuous, $P \in \Delta_2$ and $P(x) = 0$ if and only if $x = 0$. As we shall see, these conditions can be removed.

THEOREM 4.8. *Let T be a quasilinear operator such that T is of weak type (a, a) and (b, b), where $0 < a < b < \infty$. Then, T satisfies a (P, Q) modular inequality for every measurable function f with*

$$Q(t) = \max\left(t^a \int_0^t \frac{P(s)}{s^{a+1}} ds,\ t^b \int_t^\infty \frac{P(s)}{s^{b+1}} ds\right).$$

PROOF. It follows from the definition of Q that

$$\lim_{t\to 0} \frac{P(t)}{t^a} = \lim_{t\to\infty} \frac{P(t)}{t^b} = 0.$$

Now, fix $y > 0$ and write $f = f_1 + f_2$, where $f_1(x) = f(x)$ if $|f(x)| > y$ and zero otherwise. Then, by assumption

$$\begin{aligned}\lambda^{\nu}_{Tf}(y) &\le \lambda^{\nu}_{Tf_1}(y/(2K)) + \lambda^{\nu}_{Tf_2}(y/(2K)) \\ &\le C\left[\int_{\{|f(x)|>y\}} \left(\frac{|f(x)|}{y}\right)^a d\mu(x) + \int_{\{|f(x)|<y\}} \left(\frac{|f(x)|}{y}\right)^b d\mu(x)\right],\end{aligned}$$

and therefore

$$\begin{aligned}\int_{\mathcal{N}} P(|Tf(x)|) d\nu(x) &= \int_0^\infty \lambda^{\nu}_{Tf}(y) dP(y) \\ &\le C \int_0^\infty \left[\int_{\{|f(x)|>y\}} \left(\frac{|f(x)|}{y}\right)^a d\mu(x) + \int_{\{|f(x)|<y\}} \left(\frac{|f(x)|}{y}\right)^b d\mu(x)\right] dP(y) \\ &= C\left[\int_{\mathcal{M}} |f(x)|^a \int_0^{|f(x)|} \frac{dP(y)}{y^a} d\mu(x) + \int_{\mathcal{M}} |f(x)|^b \int_{|f(x)|}^\infty \frac{dP(y)}{y^b} d\mu(x)\right] \\ &= C[I_1 + I_2].\end{aligned}$$

Since $P(t) \le bQ(t)$, using an integration by parts, we obtain that

$$I_1 = \int_{\mathcal{M}} |f(x)|^a \left[\frac{P(|f(x)|)}{|f(x)|^a} + a\int_0^{|f(x)|} \frac{P(y)}{y^{a+1}} dy\right] d\mu(x) \le C \int_{\mathcal{M}} Q(|f(x)|) d\mu(x).$$

The estimate for I_2 follows similarly. □

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