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DECOMPOSITION OF KILLING VECTOR FIELDS ON TANGENT SPHERE BUNDLES

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Abstract. Given an orientable Riemannian manifold, we consider the bundle of oriented orthonormal frames and the tangent sphere bundle over it, which admit natural Riemannian metrics defined by the Riemannian connection. We show that there is a natural homomorphism between the Lie algebras of fiber preserving Killing vector fields on these bundles. In particular, for any orientable Riemannian manifold of dimension two, we show that the homomorphism is extended to an isomorphism between these Lie algebras.

1. Introduction. It is well-known that the tangent bundle and the bundle of orthonormal frames over a Riemannian manifold admit natural Riemannian metrics defined by the Riemannian connection. In fact, let (M, g) be a connected, orientable Riemannian manifold of dimension $n \geq 2$, and $SO(M)$ the bundle of oriented orthonormal frames over M. Then, for any fixed positive number λ, a Riemannian metric G on $SO(M)$ is defined by

$$G(Z, W) = {}^t\theta(Z) \cdot \theta(W) + \frac{\lambda^2}{2}\mathrm{trace}({}^t\omega(Z) \cdot \omega(W)) \quad \text{for} \quad Z, W \in T_uSO(M), \quad u \in SO(M), \tag{1.1}$$

where θ and ω denote the canonical form and the Riemannian connection form on $SO(M)$, respectively.

In this paper we shall prove that there is a natural homomorphism between the Lie algebra of fiber preserving Killing vector fields on the tangent sphere bundle over M and that of fiber preserving Killing vector fields on $(SO(M), G)$. In their paper [8], Takagi and Yawata studied the Lie algebra of Killing vector fields on $(SO(M), G)$ with $\lambda = \sqrt{2}$ and proved that there exist natural lifts $\Psi_{SO(M)}(X) \in \mathfrak{i}(SO(M), G)$ for each $X \in \mathfrak{i}(M, g)$ and $\Phi_{SO(M)}(\phi) \in \mathfrak{i}(SO(M), G)$ for each $\phi \in \mathfrak{D}^2(M)_0$, where $\mathfrak{i}(M, g)$ and $\mathfrak{i}(SO(M), G)$ denote respectively the set of Killing vector fields on (M, g) and $(SO(M), G)$, and $\mathfrak{D}^2(M)_0$ the set of parallel two-forms on (M, g). Refining their results, we shall prove that the mappings $\Psi_{SO(M)} : \mathfrak{i}(M, g) \to \mathfrak{i}(SO(M), G)$ and $\Phi_{SO(M)} : \mathfrak{D}^2(M)_0 \to \mathfrak{i}(SO(M), G)$ are simultaneously factored through in terms of natural lifts to the tangent sphere bundle over M.

To be precise, let TM be the tangent bundle of M, and g^S the Sasaki metric on TM. For a given positive number λ, we consider the tangent sphere bundle $T^\lambda M$ over M. The total space of $T^\lambda M$ is defined to be $\{X \in TM; g(X, X) = \lambda^2\}$, and gives rise to a hypersurface of (TM, g^S). We denote the induced metric on $T^\lambda M$ also by g^S. We show a certain

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relation between the Riemannian metrics g^S and G in Section 2. In Konno [4], we studied the fiber preserving Killing vector fields on $(T^\lambda M, g^S)$ and prove that there exist natural lifts $\Psi_{T^\lambda M}(X) \in \mathfrak{i}(T^\lambda M, g^S)$ for each $X \in \mathfrak{i}(M, g)$ and $\Phi_{T^\lambda M}(\phi) \in \mathfrak{i}(T^\lambda M, g^S)$ for each $\phi \in \mathfrak{D}^2(M)_0$. Then, regarding $SO(M)$ as the total space of a principal fiber bundle over the base manifold $T^\lambda M$ (cf. Nagy [6]), we prove that $\Psi_{SO(M)}$ and $\Phi_{SO(M)}$ are simultaneously factored through $\Psi_{T^\lambda M}$ and $\Phi_{T^\lambda M}$, respectively. Namely, we have the following.

THEOREM 1.1. *Let (M, g) be a connected, orientable Riemannian manifold and λ a positive number. Then there exists a unique homomorphism Ψ of the Lie algebra of fiber preserving Killing vector fields on $(T^\lambda M, g^S)$ into the Lie algebra of fiber preserving Killing vector fields on $(SO(M), G)$ such that $\Psi_{SO(M)} = \Psi \circ \Psi_{T^\lambda M}$ and $\Phi_{SO(M)} = \Psi \circ \Phi_{T^\lambda M}$.*

In Section 3, we define the vector field $\Psi(Z)$ on $(SO(M), G)$ for any Killing vector field Z on $(T^\lambda M, g^S)$ by using the Riemannian connection form on $SO(M)$, and prove in Section 4 that Ψ is a homomorphism of the Lie algebra of fiber preserving Killing vector fields on $(T^\lambda M, g^S)$.

When $\dim M = 2$, we can refine Theorem 1.1 as follows: The tangent sphere bundle $(T^\lambda M, g^S)$ is isometric to $(SO(M), G)$, and there exists an isomorphism $\Psi : \mathfrak{i}(T^\lambda M, g^S) \to \mathfrak{i}(SO(M), G)$ such that $\Psi_{SO(M)} = \Psi \circ \Psi_{T^\lambda M}$ and $\Phi_{SO(M)} = \Psi \circ \Phi_{T^\lambda M}$. Moreover, we can determine the structure of the Lie algebra of Killing vector fields on $(SO(M), G)$, without assuming the completeness of the Riemannian manifold. Namely, we obtain the following.

THEOREM 1.2. *Let (M, g) be a connected, orientable two-dimensional Riemannian manifold and λ a positive number. If $(T^\lambda M, g^S)$ admits a Killing vector field which does not preserve the fibers, then (M, g) is a space of constant curvature $1/\lambda^2$. For the structure of the Lie algebra of Killing vector fields on $(T^\lambda M, g^S)$, we have the following*:

(i) *If (M, g) is not a space of constant curvature $1/\lambda^2$, then*

$$\mathfrak{i}(T^\lambda M, g^S)/\Psi_{T^\lambda M}(\mathfrak{i}(M, g)) \cong \Phi_{T^\lambda M}(\mathfrak{D}^2(M)_0)\,.$$

In this case, the center of $\mathfrak{i}(T^\lambda M, g^S)$ is $\Phi_{T^\lambda M}(\mathfrak{D}^2(M)_0)$.

(ii) *If (M, g) is a space of constant curvature $1/\lambda^2$, then*

$$\mathfrak{i}(T^\lambda M, g^S)/\Psi_{T^\lambda M}(\mathfrak{i}(M, g)) \cong \mathrm{span}\{\Phi_{T^\lambda M}(\phi), S, [\Phi_{T^\lambda M}(\phi), S]; \phi \in \mathfrak{D}^2(M)_0\}\,,$$

where S denotes the geodesic spray on $(T^\lambda M, g^S)$. In this case, the center of $\mathfrak{i}(T^\lambda M, g^S)$ is trivial.

This result is proved in Section 5. It has been known by Tanno [9] that, conversely, if (M, g) is a space of constant curvature $1/\lambda^2$, then the tangent sphere bundle $(T^\lambda M, g^S)$ always admits a Killing vector field which is not of fiber preserving.

When (M, g) is the unit two-sphere in the Euclidean three-space with the standard metric, it follows from Theorem 1.2 that the tangent sphere bundle $(T^1 M, g^S)$ is isometric to the three-dimensional real projective space of constant sectional curvature 1/4, which was proved, for instance, by Klingenberg and Sasaki in [2].

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2. The Riemannian metric on $SO(M)$. In this section, we fix our notation used throughout this paper and prove a certain relation between the Sasaki metric g^S on $T^\lambda M$ and the metric G on $SO(M)$ defined in the Introduction.

Let ∇ denote the Riemannian connection of (M, g), and $\pi : TM \to M$ be the bundle projection of the tangent bundle TM of M. Recall that the connection map $K : TTM \to TM$ corresponding to ∇ is defined to be

$$K(Z) = \lim_{t \to 0} \frac{\tau_0^t(X(t)) - X}{t} \quad \text{for } Z \in T_X TM\,, \quad X \in TM\,,$$

where $X(t)$, $-\varepsilon < t < \varepsilon$, is a differentiable curve on TM satisfying $X(0) = X$, $\dot{X}(0) = Z$, and $\tau_0^t(X(t))$ denotes the parallel displacement of $X(t)$ from $\pi(X(t))$ to $\pi(X)$ along the geodesic arc joining $\pi(X(t))$ and $\pi(X)$ in a normal neighborhood of $\pi(X)$. We define distributions H and V on TM by

$$H_X = \mathrm{Ker}(K|_{T_X TM})\,, \qquad V_X = \mathrm{Ker}(\pi_*|_{T_X TM})\,,$$

where X is in TM. The space H_X is called the horizontal subspace of $T_X TM$ and V_X the vertical subspace of $T_X TM$. The tangent space $T_X TM$ of TM is decomposed into a direct sum $T_X TM = V_X \oplus H_X$. Then the Sasaki metric g^S on TM is defined by

$$g^S(Z, W) = g(\pi_*(Z), \pi_*(W)) + g(K(Z), K(W)) \quad \text{for } Z, W \in T_X TM\,, \quad X \in TM\,.$$

The space H_X is orthogonal to V_X with respect to the Sasaki metric.

Let $\mathfrak{D}^2(M)$ denote the Lie algebra of two-forms on M, and $\mathfrak{D}^2(M)_0$ be the Lie subalgebra of parallel two-forms in $\mathfrak{D}^2(M)$ with respect to ∇. We shall identify $\mathfrak{D}^2(M)$ with the set of all skew-symmetric tensor fields of type $(1, 1)$ on M in the usual manner. For each $\phi \in \mathfrak{D}^2(M)$, there exists a unique vector field ϕ^L on $T^\lambda M$ such that

$$(\pi|_{T^\lambda M})_*(\phi^L{}_Y) = 0\,, \quad (K|_{T_Y T^\lambda M})(\phi^L{}_Y) = \phi(Y) \quad \text{for any } Y \in T^\lambda M\,.$$

Given a Killing vector field X on (M, g), since the tensor field ∇X is regarded as an element of $\mathfrak{D}^2(M)$, we then define the vector field X^L on $T^\lambda M$ by

$$X^L = X^H + (\nabla X)^L\,, \tag{2.1}$$

where X^H denotes the horizontal lift of X. It follows from Corollary in [4] that X^L and ϕ^L are fiber preserving Killing vector fields on $(T^\lambda M, g^S)$. We recall that $\Psi_{T^\lambda M}$ is the mapping of $\mathfrak{i}(M, g)$ into $\mathfrak{i}(T^\lambda M, g^S)$ defined by $\Psi_{T^\lambda M}(X) = X^L$ for $X \in \mathfrak{i}(M, g)$, and that $\Phi_{T^\lambda M}$ is the mapping of $\mathfrak{D}^2(M)_0$ into $\mathfrak{i}(T^\lambda M, g^S)$ defined by $\Phi_{T^\lambda M}(\phi) = \phi^L$ for $\phi \in \mathfrak{D}^2(M)_0$.

We consider $SO(M)$ as a principal fiber bundle over the base manifold M with structure group $SO(n)$, the special orthogonal group of $n \times n$-matrices, and denote it simply by P. Let $\pi_P : P \to M$ denote its bundle projection, and ω_P be the Riemannian connection form on

P. Let $(\cdot,\cdot)$ denote the canonical inner product on the n-dimensional real vector space $\boldsymbol{R}^n$. We regard each $u \in P$ as an isometry of $(\boldsymbol{R}^n, (\cdot,\cdot))$ onto $(T_{\pi_P(u)}M, g|_{\pi_P(u)})$ as follows: For $u = (X_1, \dots, X_n) \in P$,

$$u(e_i) = X_i \quad \text{for } e_i = {}^t(0, \dots, \overset{(i)}{1}, \dots, 0) \in \boldsymbol{R}^n, \quad 1 \le i \le n.$$

Let $\mathfrak{o}(n)$ be the Lie algebra of $SO(n)$. For $\phi \in \mathfrak{D}^2(M)$, we define an $\mathfrak{o}(n)$-valued function $\phi^\sharp$ on P and a vector field ϕ^{L_P} on P respectively by

$$(2.2) \quad \phi^\sharp(u) = u^{-1} \circ \phi_{\pi_P(u)} \circ u \quad \text{for } u \in P \quad \text{and} \quad \omega_P(\phi^{L_P}) = \phi^\sharp, \ (\pi_P)_*(\phi^{L_P}) = 0.$$

Given a Killing vector field X on (M, g), the vector field X^{L_P} on P is defined by

$$(2.3) \qquad X^{L_P} = X^{H_P} + (\nabla X)^{L_P},$$

where X^{H_P} denotes the horizontal lift of X. For any $X \in \mathfrak{i}(M, g)$ and $\phi \in \mathfrak{D}^2(M)_0$, X^{L_P} and ϕ^{L_P} give rise to fiber preserving Killing vector fields on (P, G), which can be seen in the same manner as in [8]. We define the mapping Ψ_P of $\mathfrak{i}(M, g)$ into $\mathfrak{i}(P, G)$ by $\Psi_P(X) = X^{L_P}$ for $X \in \mathfrak{i}(M, g)$, and also the mapping Φ_P of $\mathfrak{D}^2(M)_0$ into $\mathfrak{i}(P, G)$ by $\Phi_P(\phi) = \phi^{L_P}$ for $\phi \in \mathfrak{D}^2(M)_0$.

Let us identify $SO(n-1)$ with a subgroup of $SO(n)$ given by

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}; a \in SO(n-1) \right\}.$$

The set of oriented orthonormal frames over M, or $SO(M)$, can be regarded as the total space of a principal fiber bundle over the base manifold $T^\lambda M$ with structure group $SO(n-1)$. In fact, the bundle projection $\pi_Q : SO(M) \to T^\lambda M$ is defined by

$$\pi_Q(u) = \lambda \cdot X_n \quad \text{for } u = (X_1, \dots, X_n) \in SO(M),$$

and the structure group $SO(n-1)$ acts on $SO(M)$ on the right as follows:

$$ua = \left(\sum_{k_1} a^{k_1}{}_1 X_{k_1}, \dots, \sum_{k_{n-1}} a^{k_{n-1}}{}_{n-1} X_{k_{n-1}}, X_n \right) \quad \text{for } a = (a^i{}_j) \in SO(n-1).$$

Each a in $SO(n)$ defines a diffeomorphism $R_a : u \in SO(M) \mapsto ua \in SO(M)$. We denote this principal fiber bundle simply by Q.

We define an inner product $\langle\cdot,\cdot\rangle$ on the vector space $\mathfrak{o}(n)$ by $\langle A, C\rangle = \mathrm{trace}({}^t A \cdot C)$ for $A, C \in \mathfrak{o}(n)$. Let $\mathfrak{o}(n-1)^\perp$ denote the orthogonal complement of $\mathfrak{o}(n-1)$ in $\mathfrak{o}(n)$, and $p : \mathfrak{o}(n) \to \mathfrak{o}(n-1)$ be the orthogonal projection. Define $\omega_Q = p \circ \omega_P$. We remark that *$\omega_Q$ is a connection form on Q*. Indeed, by a direct computation, we can see that $\omega_Q(A^*) = A$ for $A \in \mathfrak{o}(n-1)$ and $R_a{}^*\omega_Q = \mathrm{ad}(a^{-1})\omega_Q$ for $a \in SO(n-1)$, where A^* denotes the fundamental vector field corresponding to $A \in \mathfrak{o}(n)$.

We now define the horizontal and the vertical subspaces of the tangent spaces of P and Q. Let N denote either the bundle P or Q. Distributions H_N and V_N on $SO(M)$ are defined by

$$(H_N)_u = \mathrm{Ker}(\omega_N|_{T_u SO(M)}), \quad (V_N)_u = \mathrm{Ker}((\pi_N)_*|_{T_u SO(M)}),$$

where u is in $SO(M)$. The space $(H_N)_u$ is called the horizontal subspace of T_uN and $(V_N)_u$ the vertical subspace of T_uN. At each point u in $SO(M)$, the tangent space $T_uSO(M)$ is decomposed into a direct sum $T_uSO(M) = (H_N)_u \oplus (V_N)_u$. Given a vector field Z on $T^\lambda M$, there exists a unique vector field Z^{H_N} on $SO(M)$ such that

$$(\pi_N)_*(Z^{H_N}) = Z\,, \quad \omega_N(Z^{H_N}) = 0\,,$$

which is called the horizontal lift of Z to N.

Let N be a Riemannian manifold with metric h. Let $\mathfrak{F}(N)$ denote the ring of C^∞-functions on N, $\mathfrak{X}(N)$ the $\mathfrak{F}(N)$-module of vector fields on N, and $\mathfrak{i}(N, h)$ the Lie algebra of Killing vector fields on (N, h), respectively. Suppose N has a structure of a fiber bundle. Then a vector field X on N is called a fiber preserving vector field if any element of the local one-parameter group of local transformations of X maps each fiber of N to another fiber. Suppose further that N is one of the fiber bundles $T^\lambda M$, P, and Q. For a vector field W on N, we call W horizontal (resp. vertical) if the tangent vector W_p is in the horizontal (resp. vertical) subspace of T_pN for each point p of N. A vector field Z on N is of fiber preserving if and only if the commutator product $[W, Z]$ is vertical for any vertical vector field W on N.

A useful relation between g^S and G is given by the following.

THEOREM 2.1. (i) *For a given* $\lambda > 0$, *we have*

$$G(Z, W) = g^S((\pi_Q)_*Z, (\pi_Q)_*W) + \frac{\lambda^2}{2}\langle\omega_Q(Z), \omega_Q(W)\rangle \quad \textit{for } Z, W \in T(SO(M))\,.$$

(ii) *Let* ∇^S *and* D *denote the Riemannian connections of* $(T^\lambda M, g^S)$ *and* $(SO(M), G)$, *respectively. Then we have*

$$G(D_{X^{H_Q}}Y^{H_Q}, Z^{H_Q}) = g^S(\nabla^S{}_XY, Z) \quad \textit{for } X, Y, Z \in \mathfrak{X}(T^\lambda M)\,.$$

To prove Theorem 2.1 we need the following lemma.

LEMMA 2.2. *Let Z and W be vector fields on $T^\lambda M$ and A be in $\mathfrak{o}(n-1)$. Then we have* $G(Z^{H_Q}, W^{H_Q}) = g^S(Z, W)$.

PROOF. Since each tangent space of $T^\lambda M$ is decomposed into the direct sum of the horizontal subspace and the vertical subspace, it suffices to verify the identity for the following three cases for each u in $SO(M)$.

Case 1. $Z_{\pi_Q(u)}$ and $W_{\pi_Q(u)}$ are both in $H_{\pi_Q(u)}$. The identity holds in this case, because the projections $\pi|_{T^\lambda M}$ and π_P are Riemannian submersions, and $(X^H)^{H_Q} = X^{H_P}$ holds for any X in $\mathfrak{X}(M)$.

Case 2. $Z_{\pi_Q(u)}$ is in $H_{\pi_Q(u)}$, but $W_{\pi_Q(u)}$ is in $V_{\pi_Q(u)}$. Since there exists a vector field X on M such that $Z_{\pi_Q(u)} = X^H{}_{\pi_Q(u)}$, we have

$$G(Z^{H_Q}, W^{H_Q})_u = G(X^{H_P}, W^{H_Q})_u = 0 = g^S(Z, W)_{\pi_Q(u)}\,.$$

Case 3. $Z_{\pi_Q(u)}$ and $W_{\pi_Q(u)}$ are both in $V_{\pi_Q(u)}$. Then there exists A in $\mathfrak{o}(n-1)^\perp$ such that $Z^{H_Q}{}_u = A^*{}_u$. Setting

$$A = \left(\begin{array}{ccc|c} & & & \xi_1 \\ & 0 & & \vdots \\ & & & \xi_{n-1} \\ \hline -\xi_1 & \cdots & -\xi_{n-1} & 0 \end{array}\right),$$

we have $G(Z^{H_Q}, Z^{H_Q})_u = \lambda^2 \sum_{k=1}^{n-1}(\xi_k)^2$. Furthermore, putting $\exp tA = (a^i{}_j(t))$, $-\varepsilon < t < \varepsilon$, and $u = (X_1, \dots, X_n)$, we have

$$g^S(Z, Z) = \left\| \frac{d}{dt}\{(\pi_Q \circ R_{\exp tA})(u)\}_{t=0} \right\|^2 = \lambda^2 \sum_{k=1}^{n} \{\dot{a}^k{}_n(0)\}^2 = \lambda^2 \sum_{k=1}^{n-1} (\xi_k)^2 ,$$

and hence $G(Z^{H_Q}, Z^{H_Q})_u = g^S(Z, Z)_{\pi_Q(u)}$. □

We are now in a position to prove Theorem 2.1. Since the tangent space at $u \in SO(M)$ is orthogonally decomposed into

$$T_u SO(M) = \{X^{H_Q}{}_u;\ X \in T_{\pi_Q(u)} T^\lambda M\} \oplus \{A^*{}_u;\ A \in \mathfrak{o}(n-1)\}, \tag{2.4}$$

the statement (i) of Theorem 2.1 follows from Lemma 2.2. From Lemma 2.2 and the above decomposition, we know that the projection π_Q is the Riemannian submersion. Hence, by the O'Neill's formula [7], the statement (ii) holds. We proved Theorem 2.1.

REMARK 2.3. Let Z and W be in $\mathfrak{X}(SO(M))$. By (1.1) and (i) of Theorem 2.1, we have

$$((\pi_Q)^* g^S)(Z, W) = (\theta(Z), \theta(W)) + \frac{\lambda^2}{2}\langle \omega_P(Z), \omega_P(W)\rangle - \frac{\lambda^2}{2}\langle \omega_Q(Z), \omega_Q(W)\rangle .$$

Putting $\lambda = 1$ in the formula above, we obtain

$$(\pi_Q)^* g^S = \sum_{i=1}^{n} (\theta_i)^2 + \sum_{i=1}^{n} (\omega_{in})^2 ,$$

where one-forms θ_i and ω_{in}, $i = 1, \dots, n$, on $SO(M)$ are defined respectively by $\theta_i(\cdot) = (\theta(\cdot), e_i)$ and $\omega_{in}(\cdot) = (\omega(\cdot)e_n, e_i)$. This formula is proved by Musso and Tricerri [5, Proposition 6.1].

3. The lifts of Killing vector fields on tangent sphere bundles. Given a Killing vector field Z on $T^\lambda M$, we shall define the lift Z^{L_Q} of Z to $SO(M)$, and find necessary and sufficient condition for Z^{L_Q} also to be a Killing vector field for G.

We first define $A_{ij} \in \mathfrak{o}(n)$, $i, j = 1, \dots, n$, by $A_{ij} = 0$ if $i = j$,

$$A_{ij} = \begin{array}{c} \begin{array}{ccccc} & (i) & & (j) & \end{array} \\ \left(\begin{array}{ccccc} & \vdots & & \vdots & \\ \cdots & 0 & \cdots & -1 & \cdots \\ & \vdots & & \vdots & \\ \cdots & 1 & \cdots & 0 & \cdots \\ & \vdots & & \vdots & \end{array} \right) \end{array} \begin{array}{c} \\ (i) \\ \\ (j) \\ \end{array} \qquad \text{if} \quad i < j\,,$$

and $A_{ij} = -A_{ji}$ if $i > j$.

For A_{ij}, we recall here, without proof, the following well-known facts which will be frequently used in the following argument.

LEMMA 3.1. *Put $A_i = A_{ni}$ for $i = 1, \dots, n-1$. When $n \geq 3$, the set $\{A_1, \dots, A_{n-1}\}$ is a basis of $\mathfrak{o}(n-1)^{\perp}$ and $\{A_{ij};\ 1 \leq i < j \leq n-1\}$ is a basis of $\mathfrak{o}(n-1)$. Moreover, we have $[A_i, A_j] = A_{ij}$, $[A_{ij}, A_k] = \delta_{ik}A_j - \delta_{jk}A_i$, and $[A_{ij}, A_{kl}] = \delta_{jl}A_{ik} - \delta_{jk}A_{jl} - \delta_{il}A_{jk} + \delta_{ik}A_{jl}$ for $i, j, k, l = 1, \dots, n-1$, where δ_{ij} denotes the Kronecker delta.*

Given a Killing vector field Z on $T^{\lambda}M$, we define an $\mathfrak{o}(n-1)$-valued function $F(Z)$ on $SO(M)$ by

$$((F(Z)(u)) \cdot e_j, e_i) = \frac{1}{\lambda^2} G(D_{A_j{}^*} Z^{H_Q}, A_i{}^*)_u \quad \text{for } u \in SO(M)\,. \tag{3.1}$$

To see that $F(Z)$ is $\mathfrak{o}(n-1)$-valued, we first note $A_i{}^*{}_u$ is in $(H_Q)_u$, and there exist X_i in $\mathfrak{X}(T^{\lambda}M)$ with $i = 1, \dots, n-1$ such that $A_i{}^*{}_u = (X_i{}^{H_Q})_u$. It then follows from these and (ii) in Theorem 2.1 that

$$((F(Z)(u)) \cdot e_j, e_i) = \frac{1}{\lambda^2} G(D_{A_j{}^*} Z^{H_Q}, A_i{}^*)_u = \frac{1}{\lambda^2} g^S(\nabla^S{}_{X_j} Z, X_i)_{\pi_Q(u)}\,,$$

which shows that $F(Z)$ is $\mathfrak{o}(n-1)$-valued. We then define the vector field Z^{L_Q} on $SO(M)$ by

$$Z^{L_Q}{}_u = Z^{H_Q}{}_u + ((F(Z)(u)))^*{}_u \quad \text{at } u \in SO(M)\,, \tag{3.2}$$

and get the mapping $\Psi : \mathfrak{i}(T^{\lambda}M, g^S) \to \mathfrak{X}(SO(M))$ by $\Psi(Z) = Z^{L_Q}$. We call Z^{L_Q} the lift of a Killing vector field Z on $T^{\lambda}M$.

LEMMA 3.2. *If Z is a Killing vector field on $(T^{\lambda}M, g^S)$, then $G(Z^{L_Q}, A_{ij}{}^*) = G(D_{A_i{}^*} Z^{H_Q}, A_j{}^*)$.*

PROOF. At each point $u \in SO(M)$, we set $F = (F(Z))(u)$ and $F^i{}_j = (Fe_j, e_i)$. Then we have

$$G(Z^{L_Q}, A_{ij}{}^*)_u = \frac{\lambda^2}{2} \mathrm{trace}({}^tF \cdot A_{ij}) = \lambda^2 F^j{}_i = G(D_{A_i{}^*} Z^{H_Q}, A_j{}^*)_u\,,$$

proving Lemma 3.2. $\square$

PROPOSITION 3.3. *Let Z be a Killing vector field on $(T^\lambda M, g^S)$. Then Z^{L_Q} is a Killing vector field on $(SO(M), G)$ if and only if Z^{L_Q} satisfies the following equation*:

$$L_{Z^{L_Q}} G(X^{H_P}, A^*) = 0 \quad \text{for any } X \in \mathfrak{X}(M) \text{ and } A \in \mathfrak{o}(n-1)\,.$$

To prove Proposition 3.3, we need several lemmas.

LEMMA 3.4. *Let Ω denote the curvature form of ∇. For any $A, C \in \mathfrak{o}(n)$ and $\xi, \eta, \zeta \in \boldsymbol{R}^n$, we have the following*:

$$G([B(\xi), B(\eta)], A^*) = -\lambda^2 \langle \Omega(B(\xi), B(\eta)), A\rangle\,, \quad G([B(\xi), B(\eta)], B(\zeta)) = 0\,,$$

$$[A^*, B(\xi)] = B(A\xi)\,, \quad [A^*, C^*] = [A, C]^*\,,$$

$$G(D_{B(\xi)}B(\eta), A^*) = -\frac{\lambda^2}{2}\langle \Omega(B(\xi), B(\eta)), A\rangle\,, \quad G(D_{B(\xi)}B(\eta), B(\zeta)) = 0\,,$$

$$G(D_{B(\xi)}A^*, C^*) = 0\,, \quad G(D_{B(\xi)}A^*, B(\eta)) = \frac{\lambda^2}{2}\langle \Omega(B(\xi), B(\eta)), A\rangle\,,$$

$$G(D_{A^*}B(\xi), B(\eta)) = \frac{\lambda^2}{2}\langle \Omega(B(\xi), B(\eta)), A\rangle + (A\xi, \eta)\,,$$

$$G(D_{A^*}B(\xi), C^*) = 0\,, \quad D_{A^*}C^* = \frac{1}{2}[A, C]^*\,,$$

where $B(\xi)$ denotes the standard horizontal vector field corresponding to $\xi \in \boldsymbol{R}^n$.

PROOF. We prove only the first identity, because the others can be seen in a similar way as in the proof of lemma 1 in [8]. By the structure equation of E. Cartan, we have

$$G([B(\xi), B(\eta)], A^*) = \frac{\lambda^2}{2}\langle \omega_P([B(\xi), B(\eta)]), \omega_P(A^*)\rangle = -\lambda^2\langle \Omega(B(\xi), B(\eta)), A\rangle\,,$$

which shows the first identity. □

From this lemma, it is easy to see that the tensor DA^* on $SO(M)$ is skew-symmetric with respect to G, hence A^* is a Killing vector field on $SO(M)$.

To prove Proposition 3.3, we now find a condition which is equivalent to $L_{Z^{L_Q}} G = 0$.

LEMMA 3.5. *If Z is a Killing vector field on $(T^\lambda M, g^S)$, then $L_{Z^{L_Q}} G(X^{H_Q}, Y^{H_Q}) = 0$ holds for any X, Y in $\mathfrak{X}(T^\lambda M)$.*

PROOF. Since π_Q is the Riemannian submersion, the above identity holds. □

LEMMA 3.6. *If Z is a Killing vector field on $(T^\lambda M, g^S)$, then $L_{Z^{L_Q}} G(A^*, C^*) = 0$ holds for any A, C in $\mathfrak{o}(n-1)$.*

PROOF. It suffices to show that $L_{Z^{L_Q}} G(A_{ij}{}^*, A_{kl}{}^*) = 0$ for $1 \le i, j, k, l \le n-1$. Since $A_{ij}{}^*$, $A_{kl}{}^*$ are Killing vector fields on $(SO(M), G)$, we have by Lemmas 3.1 and 3.2 that

$$\begin{aligned} L_{Z^{L_Q}} G(A_{ij}{}^*, A_{kl}{}^*) =& A_{ij}{}^* G(Z^{L_Q}, A_{kl}{}^*) + A_{kl}{}^* G(A_{ij}{}^*, Z^{L_Q}) \\ =& \delta_{ik}\{G(D_{A_j{}^*} Z^{H_Q}, A_l{}^*) + G(D_{A_l{}^*} Z^{H_Q}, A_j{}^*)\} \\ & - \delta_{jl}\{G(D_{A_k{}^*} Z^{H_Q}, A_i{}^*) + G(D_{A_i{}^*} Z^{H_Q}, A_k{}^*)\}\,. \end{aligned}$$

The formula above vanishes, because Z is a Killing vector field on $(T^\lambda M, g^S)$. $\square$

LEMMA 3.7. *If Z is a Killing vector field on $(T^\lambda M, g^S)$, then $L_{Z^{LQ}} G(A^*, C^*) = 0$ holds for any A in $\mathfrak{o}(n-1)$ and C in $\mathfrak{o}(n-1)^\perp$.*

PROOF. There exist functions a^{kl} with $k, l = 1, \dots, n-1$ on $SO(M)$ such that

$$Z^{LQ} = Z^{HQ} + \sum_{k<l} a^{kl} A_{kl}{}^*, \tag{3.3}$$

which implies that

$$G([Z^{LQ}, A^*], C^*) = G([Z^{HQ} + \sum_{k<l} a^{kl} A_{kl}{}^*, A^*], C^*) = 0,$$

where $G([A_{kl}{}^*, A^*], C^*) = 0$ and $G(A_{kl}{}^*, C^*) = 0$ hold, since $[A_{kl}, A]$ and A_{kl} are in $\mathfrak{o}(n-1)$. By these formulas, we see that $L_{Z^{LQ}} G(A^*, C^*) = -G(A^*, [Z^{LQ}, C^*])$. Since C^* is a Killing vector field, we further have

$$L_{Z^{LQ}} G(A^*, C^*) = C^* G(A^*, Z^{LQ}) - G([C^*, A^*], Z^{LQ}). \tag{3.4}$$

When $A = A_{ij}$, $i \neq j$, and $C = A_i$, it is verified that $C^* G(A^*, Z^{LQ}) = G([C^*, A^*], Z^{LQ})$ in the following way. From Lemma 3.2 and the assumption that Z is a Killing vector field on $T^\lambda M$, we have

$$\begin{aligned} A_i{}^* G(Z^{LQ}, A_{ij}{}^*) &= A_i{}^* G(D_{A_i{}^*} Z^{HQ}, A_j{}^*) = -A_i{}^* G(D_{A_j{}^*} Z^{HQ}, A_i{}^*) \\ &= -A_i{}^* A_j{}^* G(Z^{HQ}, A_i{}^*) + A_i{}^* G(Z^{HQ}, D_{A_j{}^*} A_i{}^*), \end{aligned}$$

where $D_{A_j{}^*} A_i{}^*$ is vertical on Q by Lemmas 3.4 and 3.1. Hence the second term of the right hand side of the above formula equals zero. On the other hand, for the first term, we compute that

$$\begin{aligned} &- A_i{}^* A_j{}^* G(Z^{HQ}, A_i{}^*) \\ &\quad = A_{ji}{}^* G(Z^{HQ}, A_i{}^*) - A_j{}^* G(D_{A_i{}^*} Z^{HQ}, A_i{}^*) - A_j{}^* G(Z^{HQ}, D_{A_i{}^*} A_i{}^*). \end{aligned}$$

Since Z is a Killing vector field on $T^\lambda M$, we see $G(D_{A_i{}^*} Z^{HQ}, A_i{}^*) = 0$ by (ii) of Theorem 2.1. The formula $D_{A_i{}^*} A_i{}^* = 0$ holds trivally by Lemmas 3.1 and 3.4. Since $A_{ji}{}^*$ is a Killing vector field, we have

$$A_{ji}{}^* G(Z^{HQ}, A_i{}^*) = G([A_{ji}{}^*, Z^{HQ}], A_i{}^*) + G(Z^{HQ}, [A_{ji}{}^*, A_i{}^*]) = G([A_i{}^*, A_{ij}{}^*], Z^{LQ}),$$

where we use (3.2) and the fact that $[A_{ij}{}^*, A_i{}^*] = A_j{}^*$ is horizontal on Q. Hence we have $A_i{}^* G(A_{ij}{}^*, Z^{LQ}) = G([A_i{}^*, A_{ij}{}^*], Z^{LQ})$, and $L_{Z^{LQ}} G(A_{ij}{}^*, A_i{}^*) = 0$ by (3.4).

When $A = A_{ij}$ and $C = A_k$ with $k \neq i, j$, we see from Lemma 3.1 that

$$[A_k{}^*, A_{ij}{}^*] = 0. \tag{3.5}$$

Since $A_{ki}{}^*$ is a Killing vector field, we have by (3.5) that

$$A_{ki}{}^* G(Z^{HQ}, A_j{}^*) = G([A_{ki}{}^*, Z^{HQ}], A_j{}^*) + G(Z^{HQ}, [A_{ki}{}^*, A_j{}^*]) = 0. \tag{3.6}$$

Applying (3.5) to (3.4) and using Lemma 3.2, we see

$$L_{Z^{LQ}} G(A_{ij}{}^*, A_k{}^*) = A_k{}^* G(D_{A_i{}^*} Z^{HQ}, A_j{}^*),$$

and, by (3.6), we further have that

$$A_k{}^*G(D_{A_i{}^*}Z^{H_Q}, A_j{}^*) = A_i{}^*G(D_{A_k{}^*}Z^{H_Q}, A_j{}^*) .$$

Therefore $L_{Z^{L_Q}}G(A_{ij}{}^*, A_k{}^*)$ is symmetric with respect to i, k, and is skew-symmetric with respect to i, j. Hence we have that $L_{Z^{L_Q}}G(A_{ij}{}^*, A_k{}^*) = 0$. □

We are now in a position to complete the proof of Proposition 3.3. At each point u in $SO(M)$, the tangent space $T_uSO(M)$ is decomposed ([6]) into a direct sum:

$$T_uSO(M) = (H_P)_u \oplus \{A^*{}_u; A \in \mathfrak{o}(n-1)^{\perp}\} \oplus \{C^*{}_u; C \in \mathfrak{o}(n-1)\} . \tag{3.7}$$

Lemmas 3.5, 3.6 and 3.7 together with this decomposition imply that Z^{L_Q} is a Killing vector field on $SO(M)$ if and only if Z^{L_Q} satisfies the equation of Proposition 3.3. We thus proved Proposition 3.3.

4. The proof of Theorem 1.1. In this section, we prove Theorem 1.1. Let Z be a fiber preserving Killing vector field on $T^\lambda M$. We first show that the lift Z^{L_Q} is also a Killing vector field on $(SO(M), G)$.

LEMMA 4.1. *Let Z be a Killing vector field on $T^\lambda M$. Then we have*

$$L_{Z^{L_Q}}G(X^{H_P}, A_{ij}{}^*) = G([A_i{}^*, Z^{H_Q}], D_{X^{H_P}}A_j{}^*) - G([A_j{}^*, Z^{H_Q}], D_{X^{H_P}}A_i{}^*)$$

for any X in $\mathfrak{X}(M)$ and A_{ij} with $1 \le i, j \le n-1$.

PROOF. Recall that Z^{L_Q} is represented as (3.3). We first prove the following identities:

$$A_i{}^*G([Z^{H_Q}, X^{H_P}], A_j{}^*) = A_i{}^*A_j{}^*G(X^{H_P}, Z^{H_Q}) , \tag{4.1}$$

$$G([A_i{}^*, [Z^{H_Q}, X^{H_P}]], A_j{}^*) = 2G([A_i{}^*, Z^{H_Q}], D_{X^{H_P}}A_j^*) - X^{H_P}G(Z^{L_Q}, A_{ij}{}^*) , \tag{4.2}$$

$$G\left(\left[\sum_{k<l} a^{kl}A_{kl}{}^*, X^{H_P}\right], A_{ij}^*\right) = -X^{H_P}G(Z^{L_Q}, A_{ij}{}^*) , \tag{4.3}$$

$$\begin{aligned} G([Z^{H_Q}, X^{H_P}], A_{ij}{}^*) = {} & A_i{}^*A_j{}^*G(X^{H_P}, Z^{H_Q}) - 2G([A_i{}^*, Z^{H_Q}], D_{X^{H_P}}A_j{}^*) \\ & + X^{H_P}G(Z^{L_Q}, A_{ij}{}^*) . \end{aligned} \tag{4.4}$$

Since $Z \in \mathfrak{i}(T^\lambda M, g^S)$ and $A_j^* \in \mathfrak{i}(SO(M), G)$, we have

$$\begin{aligned} A_i{}^*G([Z^{H_Q}, X^{H_P}], A_j{}^*) &= A_i{}^*Z^{H_Q}G(X^{H_P}, A_j{}^*) - A_i{}^*G(X^{H_P}, [Z^{H_Q}, A_j{}^*]) \\ &= A_i{}^*A_j{}^*G(X^{H_P}, Z^{H_Q}) . \end{aligned}$$

This shows (4.1).

(4.2) is proved as follows: Using $[A_i{}^*, X^{H_P}] = 0$ together with Jacobi's identity, we have

$$\begin{aligned}G([A_i{}^*, [Z^{H_Q}, X^{H_P}]], A_j{}^*) &= G([[A_i{}^*, Z^{H_Q}], X^{H_P}], A_j{}^*)\\ &= G(D_{[A_i{}^*, Z^{H_Q}]}X^{H_P}, A_j{}^*) - G(D_{X^{H_P}}[A_i{}^*, Z^{H_Q}], A_j{}^*)\\ &= -G(X^{H_P}, D_{[A_i{}^*, Z^{H_Q}]}A_j{}^*) - X^{H_P}G([A_i{}^*, Z^{H_Q}], A_j{}^*)\\ &\quad + G([A_i{}^*, Z^{H_Q}], D_{X^{H_P}}A_j{}^*)\,.\end{aligned}$$

Since $A_j{}^*$ and $A_i{}^*$ are in $\mathfrak{i}(SO(M), G)$, we have

$$\begin{aligned}-G(X^{H_P}, D_{[A_i{}^*, Z^{H_Q}]}A_j{}^*) &= G(D_{X^{H_P}}A_j{}^*, [A_i{}^*, Z^{H_Q}])\,,\\ -X^{H_P}G([A_i{}^*, Z^{H_Q}], A_j{}^*) &= -X^{H_P}A_i{}^*G(Z^{H_Q}, A_j{}^*) + X^{H_P}G(Z^{H_Q}, [A_i{}^*, A_j{}^*])\\ &= -X^{H_P}G(Z^{L_Q}, A_{ij}{}^*)\,.\end{aligned}$$

Hence (4.2) follows.

It follows from (3.3) that

$$G\left(\left[\sum_{k<l} a^{kl}A_{kl}{}^*, X^{H_P}\right], A_{ij}{}^*\right) = -X^{H_P}G(Z^{L_Q}, A_{ij}{}^*)\,,$$

which proves (4.3).

Since $A_i{}^*$ is a Killing vector field, we have by (4.1) and (4.2) that

$$\begin{aligned}G([Z^{H_Q}, X^{H_P}], A_{ij}{}^*) &= A_i{}^*G([Z^{H_Q}, X^{H_P}], A_j{}^*) - G([A_i{}^*, [Z^{H_Q}, X^{H_P}]], A_j{}^*)\\ &= A_i{}^*A_j{}^*G(X^{H_P}, Z^{H_Q}) - 2G([A_i{}^*, Z^{H_Q}], D_{X^{H_P}}A_j{}^*)\\ &\quad + X^{H_P}G(Z^{L_Q}, A_{ij}{}^*)\,.\end{aligned}$$

This proves (4.4).

Using these identities (4.3) and (4.4), we prove Lemma 4.1. By (3.3), we obtain

$$L_{Z^{L_Q}}G(X^{H_P}, A_{ij}{}^*) = -G([Z^{H_Q}, X^{H_P}], A_{ij}{}^*) - G\left(\left[\sum_{k<l} a^{kl}A_{kl}{}^*, X^{H_P}\right], A_{ij}{}^*\right)\,.$$

From (4.3) and (4.4), we see that the above formula equals

$$-A_i{}^*A_j{}^*G(X^{H_P}, Z^{H_Q}) + 2G([A_i{}^*, Z^{H_Q}], D_{X^{H_P}}A_j{}^*)\,.$$

We then have

$$\begin{aligned}&L_{Z^{L_Q}}G(X^{H_P}, A_{ij}{}^*)\\ &\quad = -\frac{1}{2}A_{ij}{}^*G(X^{H_P}, Z^{H_Q}) + G([A_i{}^*, Z^{H_Q}], D_{X^{H_P}}A_j{}^*) - G([A_j{}^*, Z^{H_Q}], D_{X^{H_P}}A_i{}^*)\end{aligned}$$

Hence we obtain Lemma 4.1. □

Using Lemma 4.1, we next show that $L_{Z^{L_Q}}G(X^{H_P}, A_{ij}{}^*) = 0$, which is a condition for Z^{L_Q} to be in $\mathfrak{i}(SO(M), G)$ by Proposition 3.3. From Lemma 3.4, each $D_{X^{H_P}}A_i{}^*$ is horizontal on P, so that, from Lemma 4.1, it suffices to show that $[A_i{}^*, Z^{H_Q}]$, $i = 1, \ldots, n$, is vertical on P. Let U (Resp. W) be a horizontal (resp. vertical) vector field on $T^\lambda M$. From the assumption that Z preserves the fibers on $T^\lambda M$, we have by Theorem 2.1 that

$$G(D_{Z^{H_Q}}W^{H_Q}, U^{H_Q}) - G(D_{W^{H_Q}}Z^{H_Q}, U^{H_Q}) = 0\,. \tag{4.5}$$

Then, from (4.5), it is verified that

$$G([A_i{}^*, Z^{H_Q}], B(e_j)) = 0 \quad (\text{or } G([A_i{}^*, Z^{L_Q}], B(e_j)) = 0)\,. \tag{4.6}$$

It follows from (4.6) that $[A_i{}^*, Z^{H_Q}]$ is vertical on P. Therefore $L_{Z^{L_Q}}G(X^{H_P}, A_{ij}{}^*) = 0$ holds, and Z^{L_Q} is a Killing vector field on $(SO(M), G)$ by Proposition 3.3.

Next, we show a lemma which completes the proof of Theorem 1.1.

LEMMA 4.2. *$(X^L)^{L_Q} = X^{L_P}$ and $(\phi^L)^{L_Q} = \phi^{L_P}$ for any X in $\mathfrak{i}(M, g)$ and ϕ in $\mathfrak{D}^2(M)$.*

PROOF. Given a vector field W on M, there exists a unique vector field W^V on TM, called the vertical lift of W, such that

$$\pi_*(W^V{}_Y) = 0\,, \quad K(W^V{}_Y) = W_{\pi(Y)} \quad \text{for any } Y \in TM\,.$$

For any Y in TM, the vector $W^V{}_Y$ at Y depends only on the connection ∇ and the given vector $W_{\pi(Y)}$. Let V_Y be the vertical space of T_YTM. We define $I_Y := K|_Y$, which is an isomorphism from V_Y to the tangent space $T_{\pi(Y)}M$. Let $u = (Y_1, \dots, Y_n)$ be an arbitrary point in $SO(M)$. Set $\exp tA_i = (a_{(i)}{}^k{}_l(t))$. Then we obtain

$$\begin{aligned}(\pi_Q)_*(A_i{}^*{}_u) &= \frac{d}{dt}\{(\pi_Q)\circ(R_{\exp tA_i})(u)\}_{t=0} = \frac{d}{dt}\left\{\lambda\sum_{k=1}^{n-1}a_{(i)}{}^k{}_n(t)Y_k\right\}_{t=0}\\ &= I_{Y_n}{}^{-1}\left(\lambda\sum_{k=1}^{n-1}\dot a_{(i)}{}^k{}_n(0)Y_k\right) = \lambda Y_i{}^V{}_{Y_n} = (\lambda u(e_i))^V{}_{Y_n}\,,\end{aligned}$$

which implies that

$$A_i{}^*{}_u = \{(\lambda u(e_i))^V{}_{Y_n}\}^{H_Q}{}_u\,. \tag{4.7}$$

We shall use this in the following argument.

To prove the first formula in Lemma 4.2, it suffices to show that

$$(\pi_P)_*((X^L)^{L_Q}) = (\pi_P)_*(X^{L_P})\,, \quad \omega_P((X^L)^{L_Q}) = \omega_P(X^{L_P})\,. \tag{4.8}$$

Note that, putting $F = F(X^L)$, we get

$$(X^L)^{L_Q} = (X^L)^{H_Q} + F^* = (X^H + (\nabla X)^L)^{H_Q} + F^* = X^{H_P} + ((\nabla X)^L)^{H_Q} + F^*\,,$$

which gives rise to the decomposition of (3.7) for $(X^L)^{L_Q}$. Then the first identity of (4.8) follows from (2.3) and the decomposition above.

For the second identity of (4.8), it suffices to prove the following identities for each u in $SO(M)$ and l, i, j with $1 \le l, i, j \le n-1$,

$$(\omega_P(((\nabla X)^L)^{H_Q}{}_u)\cdot e_n, e_l) = (((\nabla X)^\sharp(u))\cdot e_n, e_l)\,, \tag{4.9}$$

$$(\omega_P(F^*)\cdot e_i, e_j) = (((\nabla X)^\sharp(u))\cdot e_i, e_j)\,, \tag{4.10}$$

where $(\nabla X)^\sharp$ is defined by (2.2).

Indeed, setting

$$((\nabla X)^L)^{H_Q} = \sum_{k=1}^{n-1} \xi^k A_k{}^*, \quad \xi^k \in \mathfrak{F}(SO(M)), \tag{4.11}$$

we see that

$$((\nabla X)^L)_{\pi_Q(u)} = \sum_{k=1}^{n-1} \xi^k(u) \cdot (\pi_Q)_* \left(\frac{d}{dt}\{(R_{\exp tA_k})(u)\}_{t=0} \right) = \lambda \sum_{k=1}^{n-1} \xi^k(u) \cdot I_{X_n}{}^{-1}(X_k),$$

and hence

$$\xi^l(u) = \frac{1}{\lambda} g^S(((\nabla X)^L)_{\pi_Q(u)}, I_{X_n}{}^{-1}(X_l)) = \frac{1}{\lambda} g((\nabla X)(\lambda X_n), X_l) = (((\nabla X)^\sharp(u))e_n, e_l).$$

Therefore it follows that

$$\begin{aligned}(\omega_P(((\nabla X)^L)^{H_Q}{}_u)e_n, e_l) &\\ &= \left(\omega_P \left(\sum_{k=1}^{n-1} \xi^k(u) \cdot A_k{}^*{}_u \right) e_n, e_l \right) = \xi^l(u) = (((\nabla X)^\sharp(u))e_n, e_l),\end{aligned}$$

which proves (4.9).

Next, we show (4.10). Using (3.1), (4.7), (ii) of Theorem 2.1, and (2.1) in order, we obtain

$$\begin{aligned}(\omega_P(F^*) \cdot e_i, e_j) &= \frac{1}{\lambda^2} G(D_{A_i{}^*}(X^L)^{H_Q}, A_j{}^*)_u \\ &= \frac{1}{\lambda^2} g^S(\nabla^S{}_{\lambda u(e_i)^V} X^H, \lambda u(e_j)^V)_{X_n} \\ &\quad + \frac{1}{\lambda^2} g^S(\nabla^S{}_{\lambda u(e_i)^V} (\nabla X)^L, \lambda u(e_j)^V)_{X_n}.\end{aligned}$$

Note here that the first term in the right hand side above vanishes. In fact, by (4.7) and Theorem 2.1, we see

$$g^S(\nabla^S{}_{\lambda u(e_i)^V} X^H, \lambda u(e_j)^V)_{X_n} = G(D_{A_i{}^*}(X^H)^{H_Q}, A_j{}^*)_u = -G(X^{H_P}, D_{A_i{}^*}A_j{}^*)_u = 0,$$

since X^{H_P} is horizontal and $D_{A_i{}^*}A_j{}^*$ are vertical on P. On the other hand, we see

$$\begin{aligned}\frac{1}{\lambda^2} g^S(\nabla^S{}_{\lambda u(e_i)^V} (\nabla X)^L, \lambda u(e_j)^V)_{X_n} &= g^S(\nabla^S{}_{X_i{}^V} (\nabla X)^L, X_j{}^V)_{X_n} \\ &= g(\nabla_{X_i} X, X_j)_{\pi(X_n)} = (((\nabla X)^\sharp(u))e_i, e_j).\end{aligned}$$

In consequence, we obtain (4.10), which completes the proof of the first formula of Lemma 4.2. The proof of the second formula proceeds in the same way as that of the first one. □

Now we prove Theorem 1.1. From the fact proved in the beginning of this section, it is known that the mapping Ψ defined in Section 3 is regarded as a mapping of the Lie algebra of fiber preserving Killing vector fields on $T^\lambda M$ into $\mathfrak{i}(SO(M), G)$. Let Z be a fiber preserving Killing vector field on $T^\lambda M$. It is easy to see that the image $\Psi(Z) = Z^{L_Q}$ preserves the fibers on P.

In fact, using (3.3), we have the following for $1 \leq i, j \leq n-1$.

$$G([Z^{L_Q}, A_{ij}{}^*], B(e_k)) = G([Z^{H_Q}, A_{ij}{}^*], B(e_k)) + G\left(\left[\sum_{k<1} a^{kl} A_{kl}{}^*, A_{ij}{}^*\right], B(e_k)\right) = 0\,.$$

This formula and (4.6) imply

$$G([Z^{L_Q}, A_{ij}{}^*], B(e_k)) = 0 \quad \text{for } 1 \leq i, j, k \leq n\,. \tag{4.12}$$

Hence Z^{L_Q} preserves the fibers on P.

We remark that the mapping Ψ is a homomorphism, which is proved in the following way. Note that each $T^\lambda M$ is an integral manifold of the distribution $\{TT^\lambda M; \lambda > 0\}$. For a given chart (U, f) of M, a chart $(\pi^{-1}(U), \tilde{f})$ of the tangent bundle TM is defined by:

$$\tilde{f}\left(\sum_{i=1}^n y^i \left(\frac{\partial}{\partial x^i}\right)_p\right) = (x^1(p), \dots, x^n(p), y^1, \dots, y^n)\,, \quad {}^t(y^1, \dots, y^n) \in \boldsymbol{R}^n\,,$$

where $f(p) = (x^1(p), \dots, x^n(p))$ for $p \in U$. Using these charts (cf. [4], Section 2), we easily see that

$$[X^L, Y^L] = [X, Y]^L\,, \quad [\phi^L, \psi^L] = -[\phi, \psi]^L\,, \quad [X^L, \phi^L] = -[\nabla X, \phi]^L \tag{4.13}$$

for any $X, Y \in \mathfrak{i}(M, g)$ and $\phi, \psi \in \mathfrak{D}^2(M)_0$. On the other hand, in the same manner as in [8], it is verified that

$$\begin{aligned} &[X^{L_P}, Y^{L_P}] = [X, Y]^{L_P}\,, \quad [\phi^{L_P}, \psi^{L_P}] = -[\phi, \psi]^{L_P}\,, \\ &[X^{L_P}, \phi^{L_P}] = -[\nabla X, \phi]^{L_P} \end{aligned} \tag{4.14}$$

for $X, Y \in \mathfrak{i}(M, g)$ and $\phi, \psi \in \mathfrak{D}^2(M)_0$. Since there exist uniquely $X \in \mathfrak{i}(M, g)$ and $\phi \in \mathfrak{D}^2(M)_0$ such that $Z = X^L + \phi^L$ [4], it follows from formulas (4.13), (4.14), and Lemma 4.2 that Ψ is a homomorphism. Since Ψ satisfies $\Psi_{SO(M)} = \Psi \circ \Psi_{T^\lambda M}$ and $\Phi_{SO(M)} = \Psi \circ \Phi_{T^\lambda M}$, the uniqueness of such homomorphism follows from that of the decompositions of the fiber preserving Killing vector fields on $(T^\lambda M, g^S)$ and $(SO(M), G)$. This completes the proof of Theorem 1.1.

5. The case of dimension two. In this section we assume that (M, g) is two-dimensional. Since the connection form of the bundle Q then vanishes, Theorem 2.1 says that $G = (\pi_Q)^* g^S$ and the mapping $\pi_Q : (SO(M), G) \to (T^\lambda M, g^S)$ is an isometry. From Proposition 3.3, we can define the one-to-one homomorphism $\Psi : \mathfrak{i}(T^\lambda M, g^S) \to \mathfrak{i}(SO(M), G)$ by $\Psi(Z) = Z^{L_Q}$ for $Z \in \mathfrak{i}(T^\lambda M, g^S)$.

To prove the first part of Theorem 1.2, we suppose that there exists a Killing vector field Z on $T^\lambda M$ which does not preserve fibers. Set $J := (\pi_Q)_*(A_1{}^*)$, which is a vertical Killing vector field on $T^\lambda M$ satisfying $\|J\| = \lambda$. For each positive integer l, let us define Killing vector field W_l on $(T^\lambda M, g^S)$ and open set U_l of $T^\lambda M$ as follows:

$$W_1 = [J, Z]\,, \quad W_{l+1} = [J, W_l]\,, \quad U_l = \{Y \in T^\lambda M; (W_l)_Y \neq 0\}\,.$$

Then, we have the following lemma.

LEMMA 5.1. (i) *W_l is a horizontal Killing vector field on $(T^\lambda M, g^S)$, which satisfies $g^S(W_l, W_{l+1}) = 0$ and $g^S(W_{l+1}, W_{l+1}) = -g^S(W_l, W_{l+2})$ for $l \geq 1$.*

(ii) *$U_l = T^\lambda M$, and $\|W_l\|$ is a constant function on $T^\lambda M$ for $l \geq 1$.*

(iii) *$\|W_l\|^2 = \lambda^2 \langle \Omega(W_l, W_{l-1}), A_1 \rangle$ for $l \geq 2$.*

PROOF. (i) Put $W_0 = Z$. Since Killing vector fields constitute the Lie algebra, it is proved by induction that W_l is a Killing vector field on $(T^\lambda M, g^S)$. It follows from

$$g^S(J, W_l) = g^S(J, [J, W_{l-1}]) = -\frac{1}{2} W_{l-1} g^S(J, J) = 0$$

that W_l is horizontal on $T^\lambda M$. Hence we have

$$g^S(W_l, W_{l+1}) = g^S(W_l, [J, W_l]) = -W_l g^S(W_l, J) + g^S([W_l, W_l], J) = 0\,.$$

Since J is a Killing vector field on $(T^\lambda M, g^S)$, we have

$$g^S(W_{l+1}, W_{l+1}) = J g^S(W_l, W_{l+1}) - g^S(W_l, [J, W_{l+1}]) = -g^S(W_l, W_{l+2})\,.$$

(ii) Using the second formula of (i), it is proved by induction that $U_m \supset U_{m+1}$ for $m \geq 1$ and $U_m \subset U_{m+1}$ for $m \geq 2$. It follows that $U_m = U_2$ for $m \geq 2$.

We next show that U_2 is not empty. To do this, we suppose that U_2 is an empty set, and derive a contradiction. If U_2 is an empty set, that is $[J, W_1] = 0$ on $T^\lambda M$, the Killing vector field W_1 preserves the fibers on $T^\lambda M$. Hence, by Corollary in [4], there exist X in $\mathfrak{i}(M, g)$ and ϕ in $\mathfrak{D}^2(M)_0$ such that

$$W_1 = X^L + \phi^L = X^H + (\nabla X + \phi)^L\,.$$

Since W_1 is horizontal by (i), we have $\nabla X + \phi = 0$. It follows that $\nabla\nabla X = -\nabla\phi = 0$, and hence $R(Y, Y')X = 0$ on M for any $Y, Y' \in \mathfrak{X}(M)$, that is $(\pi(U_1), g)$ is flat. But this contradicts the fact that a Killing vector field $Z|_{U_1}$, which does not preserve fibers, exists on U_1. Because, if $(\pi(U_1), g)$ is flat, then the distribution H_P is integrable, and $((\pi|_{T^\lambda M})^{-1}(\pi(U_1)), g^S)$ is also flat, which can be easily seen from the formula for the curvature tensor of $(T^\lambda M, g^S)$ (cf. Blair [1] and Section 3 of [4]). Hence there exists an open set $U_1{}'$ of U_1 such that $((\pi|_{T^\lambda M})^{-1}(\pi(U_1{}')), g^S)$ is isometric to an open set of $\boldsymbol{R}^3/\Gamma$, where Γ is the free group generated by $2\pi\lambda e_3 \in \boldsymbol{R}^3$, which contains a whole fiber. But, on such an open set, there exists no Killing vector field which does not preserve fibers. On account of these facts, we conclude that U_2 is not empty.

Since W_l and J are Killing vector fields, it follows that

$$(5.1)\qquad \begin{cases} W_l(\|W_l\|^2) = 2g^S([W_l, W_l], W_l) = 0\,, \\ W_{l+1}(\|W_l\|^2) = -2W_l g^S(W_{l+1}, W_l) + 2g^S(W_{l+1}, [W_l, W_l]) = 0\,, \\ J(\|W_l\|^2) = 2g^S([J, W_l], W_l) = 2g^S(W_{l+1}, W_l) = 0\,. \end{cases}$$

So, $\|W_m\|$ is a constant function on each connected component of U_m for $m \geq 2$. Then, because of the continuity of the vector field W_m, we see that $U_m = T^\lambda M$. Hence we conclude that $U_l = T^\lambda M$ for any $l \geq 1$. This proves the assertion (ii).

(iii) Since W_l and J are in $\mathfrak{i}(T^\lambda M, g^S)$, we have by Lemma 3.4 that

$$\begin{aligned} g^S(W_{l+1}, W_{l+1}) &= g^S([J, W_l], W_{l+1}) = g^S(\nabla^S{}_J W_l, W_{l+1}) - g^S(\nabla^S{}_{W_l} J, W_{l+1}) \\ &= -g^S(\nabla^S{}_{W_{l+1}} W_l, J) + g^S(\nabla^S{}_{W_{l+1}} J, W_l) = \lambda^2 \langle \Omega(W_{l+1}, W_l), A_1 \rangle . \end{aligned}$$

This completes the proof of Lemma 5.1. □

It follows from (i) of Lemma 5.1 that $\|W_{l+1}\| / \|W_l\|$ is independent of the number l. Hence, from (ii) and (iii) of Lemma 5.1, we know that the Gaussian curvature of (M, g) is equal to the constant $c = \|W_{l+1}\| \cdot (\lambda^2 \|W_l\|)^{-1}$ on M. We show that c can be computed in a different way.

LEMMA 5.2. *For each $l \geq 1$, we have*

$$\nabla^S{}_{W_l} W_l = 0 \quad \textit{and} \quad g^S(R(W_l, W_{l+1})W_{l+1}, W_l) = (c\lambda \|W_l\| \cdot \|W_{l+1}\|/2)^2 .$$

PROOF. Since it follows from (5.1) and (i) of Lemma 5.1 that

$$g^S(\nabla^S{}_{W_l} W_l, W_l) = 0 , \quad g^S(\nabla^S{}_{W_l} W_l, W_{l+1}) = 0 , \quad g^S(\nabla^S{}_{W_l} W_l, J) = 0 , \tag{5.2}$$

we get $\nabla^S{}_{W_l} W_l = 0$, which implies that $(\nabla^S{}_{W_{l+1}} \nabla^S W_l) W_{l+1} = \nabla^S{}_{W_{l+1}} \nabla^S W_{l+1} W_l$. Since any Killing vector field W on $(T^\lambda M, g^S)$ satisfies the following differential equation

$$(\nabla^S{}_Y \nabla^S W)(Y') + R(W, Y)Y' = 0 , \quad Y, Y' \in \mathfrak{X}(T^\lambda M) ,$$

we have

$$g^S(R(W_l, W_{l+1})W_{l+1}, W_l) = -g^S(\nabla^S{}_{W_{l+1}} \nabla^S{}_{W_{l+1}} W_l, W_l) = \|\nabla^S{}_{W_{l+1}} W_l\|^2 ,$$

where (5.1) is used. From the second identity of (5.2) together with the fact that W_l is a Killing vector field, we know that $\nabla^S{}_{W_{l+1}} W_l$ are vertical on $T^\lambda M$, and hence it follows from Lemma 3.4 that

$$\|\nabla^S{}_{W_{l+1}} W_l\| = \left| \frac{1}{\lambda} g^S(\nabla^S{}_{W_{l+1}} W_l, J) \right| = \left| \frac{\lambda}{2} \langle \Omega(W_l, W_{l+1}), A_1 \rangle \right| = \frac{c\lambda}{2} \|W_l\| \cdot \|W_{l+1}\| . \quad \square$$

On the other hand, by a formula of the curvature tensor of $(T^\lambda M, g^S)$ (cf. Blair [1] and Section 3 of [4]), we have the following: For an arbitrary point Y in $T^\lambda M$, put $(\pi|_{T^\lambda M})(Y) = Y^\flat$ and $(\pi|_{T^\lambda M})_*((W_l)_Y) = X_l$ for $l \geq 1$. Then it holds that

$$\begin{aligned} & g^S(R(W_l, W_{l+1})W_{l+1}, W_l)_Y \\ &\quad = g(R(X_l, X_{l+1})X_{l+1}, X_l) + \frac{1}{4} g(R(Y^\flat, R(X_{l+1}, X_{l+1})Y^\flat)X_l, X_l) \\ &\qquad + \frac{1}{4} g(R(Y^\flat, R(X_l, X_{l+1})Y^\flat)X_{l+1}, X_l) + \frac{1}{2} g(R(Y^\flat, R(X_l, X_{l+1})Y^\flat)X_{l+1}, X_l) \\ &\quad = \|W_l\|^2 \cdot \|W_{l+1}\|^2 \left(c - \frac{3}{4} c^2 \lambda^2 \right) . \end{aligned}$$

From Lemma 5.2 and the formula above, we get $c = 0$ or $c = 1/\lambda^2$. However, in the proof of Lemma 5.1, we see that if $c = 0$, then there exists no Killing vector field Z which does not preserve fibers. Hence (M, g) is a space of constant curvature $1/\lambda^2$, which proves the first part of Theorem 1.2.

Now we decompose the Killing vector field Z and prove the second part of Theorem 1.2. There exists a unique vector field S on $T^\lambda M$, called the geodesic spray on $T^\lambda M$, such that

$$(\pi|_{T^\lambda M})_*(S_Y) = Y\,, \quad (K|_{TT^\lambda M})(S_Y) = 0 \quad \text{for any } Y \in T^\lambda M\,.$$

Since the mapping π_Q is an isometry, Theorem E in [9] says that $\lambda \cdot B(e_2)$, which is the lift of S, is a Killing vector field on $SO(M)$. Indeed, we can see $(\pi|_{T^\lambda M})_*(\{(\pi_Q)_*(\lambda B(e_2))\}_Y) = Y$ for each Y in $T^\lambda M$.

It then follows that both

$$B_1 := \frac{1}{\lambda}[J, S] = (\pi_Q)_*(B(e_1)) \quad \text{and} \quad B_2 := \frac{1}{\lambda} S = (\pi_Q)_*(B(e_2))$$

are in $\mathfrak{i}(T^\lambda M, g^S)$. Since W_l is horizontal, there exist functions $b^1{}_l$ and $b^2{}_l$ on $T^\lambda M$ such that $W_l = b^1{}_l B_1 + b^2{}_l B_2$. We show that both $b^1{}_l$ and $b^2{}_l$ are constant on $T^\lambda M$. In fact, for $m = 1, 2$, we have

$$0 = g^S(\nabla^S{}_J W_l, B_m) + g^S(J, \nabla^S{}_{B_m} W_l) = \delta_{1m}(Jb^1{}_l) + \delta_{2m}(Jb^2{}_l)\,,$$

from which we get

$$Jb^m{}_l = 0\,. \tag{5.3}$$

For any vector fields Y and Y' on $T^\lambda M$, we have

$$\begin{aligned} 0 &= g^S(\nabla^S{}_Y W_l, Y') + g^S(Y, \nabla^S{}_{Y'} W_l) \\ &= (Yb^1{}_l)g^S(B_1, Y') + (Yb^2{}_l)g^S(B_2, Y') + (Y'b^1{}_l)g^S(Y, B_1) + (Y'b^2{}_l)g^S(Y, B_2)\,. \end{aligned}$$

Setting $Y = Y' = B_1$ (resp. $Y = Y' = B_2$) in the formulas above, we get

$$B_1 b^1{}_l = 0 \quad (\text{resp. } B_2 b^2{}_l = 0)\,. \tag{5.4}$$

Moreover, we have

$$\begin{aligned} 0 &= g^S(\nabla^S{}_Y W_{l+1}, Y') + g^S(Y, \nabla^S{}_{Y'} W_{l+1}) \\ &= g^S(\nabla^S{}_Y(b^2{}_l B_1 - b^1{}_l B_2), Y') + g^S(Y, \nabla^S{}_{Y'}(b^2{}_l B_1 - b^1{}_l B_2)) \\ &= (Yb^2{}_l)g^S(B_1, Y') - (Yb^1{}_l)g^S(B_2, Y') + (Y'b^2{}_l)g^S(B_1, Y) - (Y'b^1{}_l)g^S(B_2, Y)\,. \end{aligned}$$

Setting $Y = Y' = B_1$ (resp. $Y = Y' = B_2$) in the formulas above, we get

$$B_1 b^2{}_l = 0 \quad (\text{resp. } B_2 b^1{}_l = 0)\,. \tag{5.5}$$

These formulas (5.3), (5.4) and (5.5) imply that both $b^1{}_l$ and $b^2{}_l$ are constant on $T^\lambda M$, and hence $W_l = (\pi_Q)_*(B(b^1{}_l e_1 + b^2{}_l e_2))$.

Setting $Z' = Z - W_4$, we have

$$[J, Z'] = W_1 - (\pi_Q)_*([A_1{}^*, [A_1{}^*, [A_1{}^*, [A_1{}^*, B(b^1{}_1 e_1 + b^2{}_1 e_2)]]]]) = 0\,,$$

which implies that Z' is a fiber preserving Killing vector field on $T^{\lambda}M$. It follows that there exist X in $\mathfrak{i}(M, g)$ and ψ in $\mathfrak{D}^2(M)_0$ such that $Z' = X^L + \psi^L$. Hence we decompose Z as

$$Z = W_4 + Z' = \alpha \cdot S + \beta \cdot [J, S] + X^L + \psi^L ,$$

where $\alpha = \lambda^{-2} g^S([J, [J, Z]], S)$ and $\beta = \lambda^{-2} g^S([J, [J, Z]], [J, S])$.

The following formulas for the bracket products are proved in the same manner as in [8].

LEMMA 5.3. (i) *Let (M, g) be a connected, orientable two-dimensional Riemannian manifold and λ a positive number. Then for any $X, Y \in \mathfrak{i}(M, g)$ and $\phi, \psi \in \mathfrak{D}^2(M)_0$ it holds that*

$$[X^L, Y^L] = [X, Y]^L , \quad [\phi^L, \psi^L] = 0 , \quad [X^L, \phi^L] = 0 .$$

Furthermore, if (M, g) is a space of constant curvature $1/\lambda^2$, then for $m = 1, 2$, it holds that

$$[B_1, B_2] = -\frac{1}{\lambda^2} J , \quad [X^L, B_m] = 0 , \quad [J, B_m] = \delta_{1m} B_1 - \delta_{2m} B_2 .$$

Accordingly, these facts and Corollary in [4] lead us to the second part of Theorem 1.2.

REFERENCES

[1] D. E. BLAIR, When is the tangent sphere bundle locally symmetric?, Geometry and Topology, World Sci. Publishing, Singapore (1989), 15–30.

[2] W. KLINGENBERG AND S. SASAKI, On the tangent sphere bundle of a 2-sphere, Tôhoku Math. J. 27 (1975), 49–56.

[3] T. KONNO AND S. TANNO, Geodesics and Killing vector fields on the tangent sphere bundle, Nagoya Math. J. 151 (1998), 91–97.

[4] T. KONNO, Killing vector fields on tangent sphere bundles, Kodai Math. J. 21 (1998), 61–72.

[5] E. MUSSO AND F. TRICERRI, Riemannian metrics on tangent bundles, Ann. Mat. Pura Appl. 150 (1988), 1–19.

[6] P. T. NAGY, Geodesics on the tangent sphere bundle of a Riemannian manifold, Geom. Dedicata 7 (1978), 233–243.

[7] B. O'NEILL, The fundamental equations of a submersion, Michigan Math. J. 13 (1966), 459–469.

[8] H. TAKAGI AND M. YAWATA, Infinitesimal isometries of frame bundles with natural Riemannian metric, Tôhoku Math. J. 43 (1991), 103–115; II, 46 (1994), 341–355.

[9] S. TANNO, Killing vectors and geodesic flow vectors on tangent bundles, J. Reine Angew. Math. 282 (1976), 162–171.

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