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ALL TORIC LOCAL COMPLETE INTERSECTION SINGULARITIES ADMIT PROJECTIVE CREPANT RESOLUTIONS

DIMITRIOS I. DAIS, CHRISTIAN HAASE AND GÜNTER M. ZIEGLER

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Abstract. It is known that the underlying spaces of all abelian quotient singularities which are embeddable as complete intersections of hypersurfaces in an affine space can be overall resolved by means of projective torus-equivariant crepant birational morphisms in all dimensions. In the present paper we extend this result to the entire class of toric local complete intersection singularities. Our strikingly simple proof makes use of Nakajima's classification theorem and of some techniques from toric and discrete geometry.

1. Introduction.

1.1. Motivation. In the past two decades *crepant* birational morphisms were mainly used in algebraic geometry to reduce the canonical singularities of algebraic (not necessarily proper) d–folds, $d \geq 3$, to $\boldsymbol{Q}$-factorial terminal singularities, and to treat *minimal models* in high dimensions. From the late eighties onwards, crepant full desingularizations $\hat{Y} \to Y$ of projective varieties Y with trivial dualizing sheaf and mild singularities (like quotient or toroidal singularities) play also a crucial role in producing *Calabi-Yau manifolds*, which serve as internal target spaces for non-linear supersymmetric sigma models in the framework of physical string-theory. This explains the recent mathematical interest in both local and global versions of the existence problem of *smooth* birational models of such Y's.

Locally, the high-dimensional *McKay correspondence* (cf. [IR, R]) for the underlying spaces $\boldsymbol{C}^d/G$, $G \subset \mathrm{SL}(d, \boldsymbol{C})$, of Gorenstein quotient singularities was proven by Batyrev [B, Theorem 8.4]. It states that the following two quantities are equal: the ranks of the non-trivial (=even) cohomology groups $H^{2k}(\hat{X}, \boldsymbol{C})$ of the overlying spaces $\hat{X}$ of crepant, full desingularizations $\hat{X} \to X = \boldsymbol{C}^d/G$ on the one hand and the number of conjugacy classes of G having the weight (also called "age") k on the other. Moreover, a one-to-one correspondence of McKay-type is also true for torus-equivariant, crepant, full desingularizations $\hat{X} \to X = X_\sigma$ of the underlying spaces of Gorenstein toric singularities [BD, §4]. Again, the non-trivial (even) cohomology groups of the $\hat{X}$'s have the "expected" dimensions, which in this case are determined by the Ehrhart polynomials of the corresponding lattice polytopes. Thus in both situations the ranks of the cohomology groups of $\hat{X}$'s turn out to be independent of the particular choice of a crepant resolution. Also in both situations, a crepant resolution exists always

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if $d \leq 3$, but not in general: for $d \geq 4$ there are, for instance, lots of *terminal* Gorenstein singularities in both classes.

We believe that a purely algebraic, sufficient condition for the existence of projective, crepant, full resolutions in *all* dimensions is to require our singularities to be, in addition, *local complete intersections* (*l.c.i.'s*). In the toric category, where the existence question can be translated into a question concerning the existence of specific lattice triangulations of lattice polytopes, this conjecture was verified for abelian quotient singularities in [DHZ98] via a Theorem by Kei-ichi Watanabe [W]. (For non-abelian groups acting on $\boldsymbol{C}^d$, it remains open.) Furthermore, the authors of [DHZ98, cf. §8(iii)] asked for geometric analogues of the joins and dilations occuring in their Reduction Theorem also for toric non-quotient l.c.i.-singularities. As we shall see below, such a characterization (in a somewhat different context) is indeed possible by making use of another beautiful Classification Theorem, due to Haruhisa Nakajima [N], which generalizes Watanabe's results to the entire class of toric l.c.i.'s. Based on this classification we prove the following:

MAIN THEOREM 1.1. *The underlying spaces of all toric l.c.i.-singularities admit torus-equivariant, projective, crepant, full resolutions (i.e., smooth minimal models) in all dimensions.*

An extended version of this paper is electronically available as [DHaZ]. Families of Gorenstein non-l.c.i. toric singularities that have such special full resolutions seem to exist only rarely. For a discussion of this problem for certain families of abelian quotient non-l.c.i. singularities the reader is referred to [DHH, DH, DHZ99].

1.2. Notions from convex geometry.

1.2.1. In the following we refer to $\boldsymbol{Z}^d$ as *the lattice*; $(\boldsymbol{Z}^d)^\vee$ is the *dual lattice* of integral linear forms. The *convex hull* and the *affine hull* of a set $S \subset \boldsymbol{R}^d$ are denoted by $\mathrm{conv}(S)$ and $\mathrm{aff}(S)$, respectively, and the *dimension* of S is the dimension of $\mathrm{aff}(S)$. A *lattice polytope* is the convex hull of finitely many integral points; it is *elementary* (also called *empty* or *lattice point free*) if its vertices are the only lattice points it contains. A lattice simplex $\mathfrak{s}$ whose vertices form an affine lattice basis for $\mathrm{aff}(\mathfrak{s}) \cap \boldsymbol{Z}^d$ is called *unimodular* (or *basic*). Every unimodular simplex is elementary, but the converse is not true in dimensions $d \geq 3$. Two sets $S \subset \boldsymbol{R}^d$ and $S' \subset \boldsymbol{R}^{d'}$ are *lattice equivalent* if there is an affine map $\mathrm{aff}(S) \to \mathrm{aff}(S')$ that maps $\boldsymbol{Z}^d \cap \mathrm{aff}(S)$ bijectively onto $\boldsymbol{Z}^{d'} \cap \mathrm{aff}(S')$ and which maps S to S'; e.g., all d-dimensional unimodular simplices embedded in $\boldsymbol{R}^{d'}$ $(d' \geq d)$ are lattice equivalent to the *standard simplex* $\mathfrak{s}^{(d)}$ which is defined to be the convex hull of the origin $\mathbf{0}$ together with the standard unit vectors $\mathbf{e}_i$ $(1 \leq i \leq d)$ in $\boldsymbol{R}^d$.

For $S \subset \boldsymbol{R}^d$ let $\mathrm{pos}(S)$ denote the set of all real, non-negative linear combinations of elements of S. A set $\sigma \subset \boldsymbol{R}^d$ is a *cone* if it equals $\mathrm{pos}(S)$ for some S. A cone σ is *polyhedral* if its generating set S can be chosen to be finite, and *rational* if, in addition, one may have $S \subset \boldsymbol{Z}^d$. If $\sigma \cap (-\sigma) = \{\mathbf{0}\}$, we say that σ is *pointed.* A cone is *simplicial* if it is generated by an $\boldsymbol{R}$-linearly independent set. A simplicial cone is *unimodular* if it is lattice equivalent to the cone generated by the standard basis in $\mathrm{R}^{\dim \sigma}$. If $P \subset \boldsymbol{R}^{d-1}$ is a lattice polytope, then

$\sigma_P := \mathrm{pos}(P \times \{1\}) \subset \boldsymbol{R}^d$ is a pointed, rational cone which will be referred to as the cone *spanned* by P. Given a (rational/polyhedral) cone σ, the set $\sigma^\vee = \{\mathbf{x} \in (\boldsymbol{R}^d)^\vee : \langle \mathbf{x}, \sigma \rangle \geq 0\}$ is also a (rational/polyhedral) cone, the *dual cone* of σ. A set of the form $\{\mathbf{y} \in \sigma : \langle \mathbf{x}, \mathbf{y} \rangle = 0\}$ for fixed $\mathbf{x} \in \sigma^\vee$ is a *face* of σ.

1.2.2. A *polyhedron* is a finite intersection of closed halfspaces in $\boldsymbol{R}^d$. A *polyhedral complex* Σ is a finite collection of polyhedra such that the faces of any member belong to Σ and such that the intersection of any two members is a face of each of them. The *support* $|\Sigma|$ of such a Σ is the union of its members. A single polyhedron together with all of its faces forms a polyhedral complex that we denote by the same symbol as the single object. We say that Σ' is a *subdivision* of Σ if every polyhedron in Σ' is contained in a polyhedron of Σ and $|\Sigma'| = |\Sigma|$. A polyhedral complex that consists of rational pointed cones is a *fan*, whereas a polyhedral complex all of whose members are lattice polytopes will be called a *complex of lattice polytopes*. A fan is *simplicial* (resp. *unimodular*) if all of its members are. A complex of lattice polytopes $\mathcal{T}$ that subdivides $|\Sigma|$ into simplices is a (*lattice*) *triangulation*. A lattice triangulation is *unimodular* if its faces are.

An integral Σ-linear convex *support function* is defined to be a continuous function $\omega : |\Sigma| \to \boldsymbol{R}$, with $\omega(|\Sigma| \cap \boldsymbol{Z}^d) \subset \boldsymbol{Z}$, which is affine on each $\sigma \in \Sigma$, and convex on the entire $|\Sigma|$. If the domains of linearity of such an ω are exactly the maximal polyhedra of Σ, then ω is said to be *strictly convex*. For a polyhedral complex Σ that is equipped with an integral Σ-linear strictly convex support function ω, we write $\Sigma = \Sigma_\omega$, and call Σ *coherent*.

1.3. Notions from algebraic and toric geometry.

1.3.1. Let R denote a local Noetherian ring with maximal ideal $\mathfrak{m}$. R is *regular* if $\dim(R) = \dim(\mathfrak{m}/\mathfrak{m}^2)$. R is said to be a *complete intersection* (*c.i.*) if there exists a regular local ring R', such that $R \cong R'/\langle f_1, \ldots, f_q \rangle$ for a finite set $\{f_1, \ldots, f_q\} \subset R'$ whose cardinality equals $q = \dim(R') - \dim(R)$. R is called *Cohen-Macaulay* if $\mathrm{depth}(R) = \dim(R)$, where its depth is defined to be the maximum of the lengths of all regular sequences whose members belong to $\mathfrak{m}$. Such an R is *Gorenstein* if $\mathrm{Ext}_R^{\dim(R)}(R/\mathfrak{m}, R) \cong R/\mathfrak{m}$. The hierarchy of the above types of R's reads:

$$\text{regular} \Longrightarrow \text{c.i.} \Longrightarrow \text{Gorenstein} \Longrightarrow \text{Cohen-Macaulay}.$$

An arbitrary Noetherian ring R and its associated affine scheme $\mathrm{Spec}(R)$ are called regular, Cohen-Macaulay, or Gorenstein, respectively, if all the localizations $R_\mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ of R are of this type. In particular, if all $R_\mathfrak{m}$'s are c.i.'s, then one says that R is a *local complete intersection* (*l.c.i.*).

1.3.2. Throughout the paper we consider only complex varieties $(X, \mathcal{O}_X)$, i.e., integral separated schemes of finite type over $\boldsymbol{C}$, and work within the analytic category. The algebraic properties of 1.3.1 can be defined for the whole X via its affine coverings, and pointwise via the stalks $\mathcal{O}_{X,x}$ of the structure sheaf at $x \in X$. By $\mathrm{Sing}(X) = \{x \in X : \mathcal{O}_{X,x} \text{ non-regular}\}$ we denote the *singular* and by $\mathrm{Reg}(X) = X \setminus \mathrm{Sing}(X)$ the *regular locus* of X. A *partial desingularization* $f : \hat{X} \to X$ of X is a proper holomorphic morphism of complex varieties with $\hat{X}$ normal, such that there is a nowhere dense analytic set $S \subset X$, with $S \cap \mathrm{Sing}(X) \neq \emptyset$, whose inverse image $f^{-1}(S) \subset \hat{X}$ is nowhere dense and such that the restriction of f to $\hat{X} \setminus f^{-1}(S)$ is biholomorphic. $f : \hat{X} \to X$ is called a *full desingularization* of X (or *full resolution of singularities* of X) if $\mathrm{Sing}(X) \subseteq S$ and $\mathrm{Sing}(\hat{X}) = \emptyset$.

1.3.3. A Weil divisor K_X of a normal complex variety X is *canonical* if its sheaf $\mathcal{O}_X(K_X)$ of fractional ideals is isomorphic to the sheaf of the (regular in codimension 1) Zariski differentials or, equivalently, if $\mathcal{O}_{\mathrm{Reg}(X)}(K_X)$ is isomorphic to the sheaf $\Omega^{\dim X}_{\mathrm{Reg}(X)}$ of the highest regular differential forms on $\mathrm{Reg}(X)$. As it is known, a Cohen-Macaulay variety X is Gorenstein if and only if K_X is Cartier, i.e., if and only if $\mathcal{O}_X(K_X)$ is invertible. A birational morphism $f : X' \to X$ between normal Gorenstein complex varieties is called *non-discrepant* or simply *crepant*, if the difference $K_{X'} - f^*(K_X)$ (which is uniquely determined up to rational equivalence) vanishes. Furthermore, $f : X' \to X$ is *projective* if X' admits an f-ample Cartier divisor.

1.3.4. Let $\sigma \subset \boldsymbol{R}^d$ be a pointed rational cone and $\sigma^\vee$ its dual. Then the semigroup ring $\boldsymbol{C}[\sigma^\vee \cap (\boldsymbol{Z}^d)^\vee]$ defines an affine complex variety

$$X_\sigma := \mathrm{Spec}(\boldsymbol{C}[\sigma^\vee \cap (\boldsymbol{Z}^d)^\vee]) .$$

If $\sigma = \sigma_P$ is spanned by P, we also write X_P instead of X_{σ_P}. A general *toric variety* X_Σ associated with a fan Σ is the identification space $X_\Sigma := (\bigsqcup_{\sigma\in\Sigma} X_\sigma)/{\sim}$ over the equivalence relation $\sim$ defined by the property: $X_{\sigma_1} \ni u_1 \sim u_2 \in X_{\sigma_2}$ if and only if there is a face τ of both σ_1, σ_2, and $u_1 = u_2$ within X_τ. X_Σ is always normal and Cohen-Macaulay, and has at most rational singularities. Moreover, X_Σ admits a canonical group action which extends the multiplication of the algebraic torus $X_{\{\mathbf{0}\}} \cong (\boldsymbol{C}^*)^d$. This action partitions X_Σ into orbits that are in one-to-one correspondence to the faces of Σ. We denote by D_σ the closure of the orbit corresponding to σ. The notion of *equivariance* will always be used with respect to this action. We refer to [E, F, KKMS, O] for further reading.

2. From the dictionary. In this section, we review some entries of the dictionary that translates between convex and toric geometry.

2.1. Gorenstein property, ampleness and smoothness. The torus invariant prime divisors on X_Σ are the orbit closures D_ϱ, for all one-dimensional cones $\varrho \in \Sigma$. So the torus invariant Weil divisors are just formal $\mathbf{Z}$-linear combinations of the D_ϱ. A divisor $\sum \lambda_\varrho D_\varrho$ is Cartier if for every maximal cone $\sigma \in \Sigma$ there is an element $\ell_\sigma \in (\boldsymbol{Z}^d)^\vee$ such that $\lambda_\varrho = \langle \ell_\sigma, p(\varrho)\rangle$, for every one-dimensional subcone ϱ of σ, where $p(\varrho)$ is the primitive lattice vector that generates ϱ. The divisor K_Σ, all whose λ_ϱ's are -1, is canonical. From this we deduce:

PROPOSITION 2.1. *X_Σ is Gorenstein if and only if every cone in Σ is spanned by some lattice polytope.*

Any subdivision Σ' of a fan Σ induces a proper equivariant birational morphism $f^{\Sigma}_{\Sigma'} : X_{\Sigma'} \to X_\Sigma$. By the ampleness criterion for torus invariant divisors (see [KKMS, Theorem 13, p. 48]) one gets:

PROPOSITION 2.2. *The morphism $f^{\Sigma}_{\Sigma'}$ is projective if and only if Σ' is a coherent subdivision of Σ.*

On the other hand, since each $X_{\sigma'} \subseteq X_{\Sigma'}$ is smooth if and only if σ' is unimodular (cf. O, [Theorem 1.10]), we obtain the following:

PROPOSITION 2.3. *The morphism $f^{\Sigma}_{\Sigma'}$ is a full desingularization of X_Σ if and only if Σ' is unimodular.*

It is well-known that projective full desingularizations always exist for any X_Σ (see [KKMS, §I.2]). Nevertheless, asking about conditions under which such $f^{\Sigma}_{\Sigma'}$ are, in addition, crepant, for X_Σ Gorenstein, leads to an existence problem of special lattice triangulations which is subtle in general.

2.2. Crepant resolutions via triangulations. Let X_P be the underlying space of a Gorenstein toric singularity defined by means of the cone $\sigma = \sigma_P$. Let Σ' denote a simplicial subdivision of σ. Then K_{X_P} is trivial, $X_{\Sigma'}$ is $\boldsymbol{Q}$-Gorenstein, and the discrepancy with respect to the partial desingularization morphism $f^{\sigma}_{\Sigma'}$ equals

$$K_{X_{\Sigma'}} - f^{\sigma\,*}_{\Sigma'}(K_{X_P}) = \sum \mu_{\varrho'} D_{\varrho'} .$$

The summation is taken through all one-dimensional faces ϱ' of Σ' not belonging to σ, and $\mu_{\varrho'}$'s are rationals > -1. If $X_{\Sigma'}$ itself is Gorenstein, then these coefficients are non-negative integers and can be written as $\mu_{\varrho'} = \langle \ell_{\sigma'}, p(\varrho') \rangle - 1$ (with $\ell_{\sigma'}$ as defined in 2.1 for every maximal cone $\sigma' \in \Sigma'$, cf. [F, p. 61]). Hence, combining the vanishing of the discrepancy with Propositions 2.2 and 2.3, one gets the following:

PROPOSITION 2.4. *The proper birational morphism $f^{\sigma}_{\Sigma'}$ is crepant if and only if the fan $\Sigma' = \Sigma'(\mathcal{T})$ is induced by a lattice triangulation $\mathcal{T} = \mathcal{T}_P$ of P, where $\Sigma' := \{\sigma_{\mathbf{s}} :$ $\mathbf{s}$ simplex of $\mathcal{T}\}$. In this case, $f^{\sigma}_{\Sigma'} : X_{\Sigma'} \to X_P$ is projective if and only if $\mathcal{T}$ is coherent, and forms a full desingularization of X_P if and only if $\mathcal{T}$ is unimodular.*

2.3. Nakajima's classification. The question about the geometric description of the polytopes which span the cones defining affine toric l.c.i.-varieties was completely answered by Nakajima in [N]. His original definition reads as follows:

DEFINITION 2.5 (Nakajima Polytopes). Associate to a sequence $\mathfrak{g} = (\mathbf{g}_1, \ldots, \mathbf{g}_r)$ of nonzero forms in $(\mathbf{Z}^d)^\vee$ $(d > r)$, satisfying $g_{ij} = 0$ for $j > i$, the following sequence of polytopes:

$$P^{(1)} = \{(1, 0, \ldots, 0)\} \subset \boldsymbol{R}^d ,$$
$$P^{(i+1)} = \mathrm{conv}(P^{(i)} \cup \{(\mathbf{x}', \langle \mathbf{g}_i, \mathbf{x}\rangle, 0, \ldots, 0) \in \boldsymbol{R}^d : \mathbf{x} = (\mathbf{x}', 0, 0, \ldots, 0) \in P^{(i)}\}) .$$

Call $\mathfrak{g}$ *admissible* if $\langle \mathbf{g}_i, \mathbf{x}\rangle \geq 0$ for all $\mathbf{x} \in P^{(i)}$. For $\mathfrak{g}$ admissible we call $P_{\mathfrak{g}} := P^{(r+1)}$ the *Nakajima polytope* or *lci polytope* associated to $\mathfrak{g}$.

CLASSIFICATION THEOREM 2.6 (Nakajima [N]). *An affine toric variety X_σ is a local complete intersection if and only if σ is lattice equivalent to a cone spanned by some Nakajima polytope.*

For our purposes a recursive definition is more suitable.

LEMMA 2.7 (Inductive Characterization). *A lattice polytope $P \subset \boldsymbol{R}^d$ is a Nakajima polytope if and only if it is a point $P = \{\mathbf{p}\} \subset \boldsymbol{Z}^d$ or*

$$P = \{\mathbf{x} = (\mathbf{x}', x_d) \in F \times \boldsymbol{R} : 0 \le x_d \le \langle \mathbf{g}, \mathbf{x}' \rangle\}, \tag{2.1}$$

where the facet $F \subset \boldsymbol{R}^{d-1}$ is a Nakajima polytope, and $\mathbf{g} \in (\mathbf{Z}^{d-1})^\vee$ is a functional with non-negative values on F (cf. Figure 1).

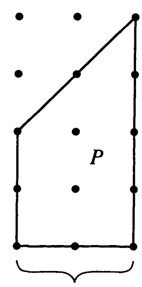

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FIGURE 1.

PROOF. Let $\mathfrak{g}$ be an admissible sequence. Then the corresponding Nakajima polytope $P_\mathfrak{g}$ has the following description by inequalities:

$$P_\mathfrak{g} = \{\mathbf{x} \in \boldsymbol{R}^d : x_1 = 1 \text{ and } 0 \le x_{i+1} \le \langle \mathbf{g}_i, \mathbf{x} \rangle \text{ for } 1 \le i \le d-1\}, \tag{2.2}$$

so that $P_\mathfrak{g}$ can be reconstructed from the facet $F = P_{\mathfrak{g}'}$ and $\mathbf{g} = \mathbf{g}_r$, where $\mathfrak{g}'$ is the truncated sequence $(\mathbf{g}_1, \ldots, \mathbf{g}_{r-1})$. Conversely, given the situation (2.1), F is some $P_{\mathfrak{g}'}$. Then we can append $\mathbf{g}$ to $\mathfrak{g}'$ and obtain an admissible sequence for P. □

3. Proof of the main theorem. By Proposition 2.4 and Theorem 2.6 we reduce the proof of Theorem 1.1 to the existence of coherent unimodular triangulations for all Nakajima polytopes of any dimension. We do not lose generality if we henceforth assume that the considered Nakajima polytope $P \subset \boldsymbol{R}^d$ is full-dimensional. (Otherwise, $\sigma_P = \tilde{\sigma}_P \oplus \{\mathbf{0}\}$ leads to the splitting $\mathrm{Sing}(X_P) = \mathrm{Sing}(X_{\tilde{\sigma}_P}) \times (\boldsymbol{C}^*)^{d-\dim P}$ which does not cause any difficulties for our desingularization problem.) We shall proceed by induction on the dimension of P. Zero- and one-dimensional polytopes always admit unique such triangulations. For the induction step we proceed as follows: According to Lemma 2.7, we may assume that for a given $P_\mathfrak{g}$, the facet F (as in (2.1)) is already endowed with a coherent unimodular triangulation $\mathcal{T}_F$. This triangulation induces a coherent subdivision

$$\mathcal{S}_P = \{(\mathfrak{s} \times \boldsymbol{R}) \cap P_\mathfrak{g} : \mathfrak{s} \in \mathcal{T}_F\}$$

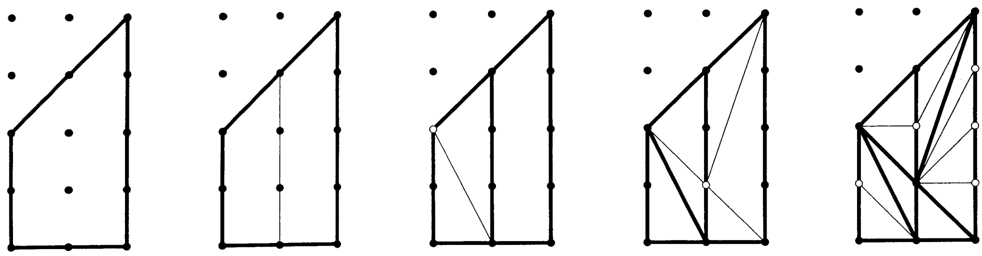

FIGURE 2.

of $P_{\mathfrak{g}}$ into "chimneys" over the simplices of $\mathcal{T}_F$. The second step is to refine $\mathcal{S}_P$ coherently by "pulling vertices" until we obtain a triangulation into elementary simplices. (Cf. Figure 2.) We shall prove that any such triangulation is automatically unimodular.

DEFINITION 3.1 (Pulling Vertices; cf. [L]). Consider a polytope $P \subset \boldsymbol{R}^d$ and let $\mathcal{S}$ be a subdivision of P. For any $v \in P$ we define a refinement $\mathrm{pull}_v(\mathcal{S})$ of $\mathcal{S}$ (called the *pulling* of v) by the following properties:

(i) $\mathrm{pull}_v(\mathcal{S})$ contains all $Q \in \mathcal{S}$ for which $v \notin Q$, and

(ii) if $v \in Q \in \mathcal{S}$, then $\mathrm{pull}_v(\mathcal{S})$ contains all the polytopes $\mathrm{conv}(F \cup v)$, where F is a facet of Q such that $v \notin F$.

If $\mathcal{S}$ is coherent, with $\mathcal{S} = \mathcal{S}_\omega$ as in §1.2, then $\mathrm{pull}_v(\mathcal{S})$ is obtained by "pulling v from below". This means that, defining $\omega'(v) = \omega(v) - \varepsilon$ for ε small enough, and $\omega' = \omega$ on the remaining lattice points, and extending ω' by the maximal convex function ω'' whose values at the lattice points are not greater than the given ones, we get $\mathrm{pull}_v(\mathcal{S}) = \mathcal{S}_{\omega''}$. Hence:

LEMMA 3.2 (Coherency of Pullings). *If $\mathcal{S}$ is coherent, then so is* $\mathrm{pull}_v(\mathcal{S})$.

We revert now to the initial subdivision $\mathcal{S} = \mathcal{S}_P$ of a Nakajima polytope P. Fix an arbitrary ordering $\{v_1, \dots, v_k\}$ of all lattice points of $P \cap \boldsymbol{Z}^d$, and define the new subdivision

$$\mathcal{T}_P := \mathrm{pull}_{v_k}(\mathrm{pull}_{v_{k-1}}(\cdots \mathrm{pull}_{v_1}(\mathcal{S}_P))) .$$

$\mathcal{T}_P$ is a lattice triangulation of P, and is coherent by Lemma 3.2. It remains to show that $\mathcal{T}_P$ is unimodular.

LEMMA 3.3 (Chimney Lemma). *Let $\pi : \boldsymbol{R}^d \to \boldsymbol{R}^{d-1}$ be the deletion of the last coordinate. Let $\mathfrak{s}$ be an elementary lattice simplex whose projection $\mathfrak{s}' = \pi(\mathfrak{s})$ is unimodular. Then $\mathfrak{s}$ itself is unimodular.*

PROOF. We can assume that $\mathfrak{s}$ is full-dimensional. After a unimodular transformation of $\boldsymbol{R}^{d-1}$ we can suppose that $\mathfrak{s}'$ is the standard $(d-1)$-simplex. After a translation in the last coordinate, $\mathfrak{s}$ contains the origin. Now we can shift the lines $\pi^{-1}(\mathbf{e}_i)$ independently so that we finally obtain a simplex with non negative last coordinate that contains $\mathfrak{s}^{(d-1)} \times \{\mathbf{0}\}$. The fact that $\mathfrak{s}$ is elementary implies that the additional vertex has the last coordinate 1. $\square$

PROOF OF THEOREM 1.1. By construction, all simplices of the above lattice triangulation $\mathcal{T}_P$ are elementary because $P \cap \boldsymbol{Z}^d$ coincides with the set of vertices of $\mathcal{T}_P$. Since their projections under π are unimodular simplices of $\mathcal{T}_F$, they have to be unimodular themselves by the Chimney Lemma. □

4. Applications.

4.1. Computing Betti numbers. In this paragraph we are interested in the cohomology groups of $X_{\Sigma'}$. They are trivial in odd dimension and their ranks in even dimensions can be calculated from the admissible sequence $\mathfrak{g}$ by combinatorial means, as we now explain.

Let $P \subset \boldsymbol{R}^d$ be a lattice polytope. Then the number of lattice points in kP $(k \in \boldsymbol{Z}_{\geq 0})$ is a polynomial from $\boldsymbol{Q}[k]$, the so-called Ehrhart polynomial:

$$\mathrm{Ehr}(P, k) := \mathrm{card}(kP \cap \boldsymbol{Z}^d) = \sum_{i=0}^{\dim P} a_i(P) k^i .$$

The corresponding Ehrhart series

$$\mathfrak{Ehr}(P, t) = \sum_{k \geq 0} \mathrm{Ehr}(P, k) t^k = \frac{\sum_{j=0}^{\dim P} \delta_j(P) t^j}{(1-t)^{\dim P + 1}}$$

gives rise to the δ-*vector* $(\delta_0(P), \ldots, \delta_{\dim P}(P))$ of P. The coordinates of this vector can be written as $\boldsymbol{Z}$-linear combinations of the coefficients $a_i(P)$ of $\mathrm{Ehr}(P, k)$ by the formula

$$\delta_j(P) = \sum_{i=0}^{\dim P} \left(\sum_{\nu=0}^{j} (-1)^{\nu} \binom{\dim P + 1}{\nu} (j - \nu)^i \right) a_i(P) . \tag{4.1}$$

These are actually the only numbers one needs for the computation of the desired ranks by the following:

THEOREM 4.1 (Cohomology Ranks [BD]). *Let $f_{\Sigma'}^{\sigma_P} : X_{\Sigma'} \to X_P$ be any equivariant crepant full resolution. Then*

$$\mathrm{rk}\, H^k(X_{\Sigma'}) = \begin{cases} \delta_j(P) & \text{if } k = 2j , \\ 0 & \text{if } k \text{ is odd} . \end{cases}$$

In particular, these ranks are independent of the choices of the unimodular triangulation $\mathcal{T}$ of P by means of which one constructs the fan $\Sigma' = \Sigma'(\mathcal{T})$.

In our specific case, where P is lattice equivalent to a $P_{\mathfrak{g}}$, one deduces for its dilations:

$$kP_{\mathfrak{g}} = \{\mathbf{x} \in \boldsymbol{R}^d : x_1 = k \text{ and } 0 \leq x_{i+1} \leq \langle \mathfrak{g}_i, \mathbf{x} \rangle \text{ for } 1 \leq i \leq d-1\} ,$$

by making use of the inequalities (2.2). Thus, the Ehrhart polynomial of $P_{\mathfrak{g}}$ equals

$$\mathrm{Ehr}(P_{\mathfrak{g}}, k) = \sum_{\nu_1=0}^{k g_{1,1}} \sum_{\nu_2=0}^{k g_{2,1} + \nu_1 g_{2,2}} \cdots \sum_{\nu_d=0}^{k g_{d,1} + \sum \nu_i g_{d-1,i}} 1 .$$

Writing it in an extended form we first determine its coefficients $a_i(P_{\mathfrak{g}})$ and then compute the cohomology ranks according to Theorem 4.1 by the formula (4.1) for the $\delta_j(P)$'s. For instance, the Ehrhart polynomial of the d-dimensional Nakajima polytope $P_{\mathfrak{g}}$ for $d \leq 3$ equals

$$\mathrm{Ehr}(P_{\mathfrak{g}}, k) = g_{1,1}k + 1\,, \quad \text{for } d = 1$$

$$\mathrm{Ehr}(P_{\mathfrak{g}}, k) = \left(\frac{1}{2}g_{2,2}g_{1,1}^2 + g_{2,1}g_{1,1}\right)k^2 + \left(g_{1,1} + \frac{1}{2}g_{2,2}g_{1,1} + g_{2,1}\right)k + 1\,, \quad \text{for } d = 2$$

and

$$\begin{aligned}\mathrm{Ehr}(P_{\mathfrak{g}}, k) = \Big(& g_{3,1}g_{2,1}g_{1,1} + \frac{1}{2}g_{3,2}g_{2,1}g_{1,1}^2 + \frac{1}{2}g_{3,3}g_{2,1}^2g_{1,1} + \frac{1}{6}g_{3,3}g_{2,2}^2g_{1,1}^3 \\ &+ \frac{1}{2}g_{3,3}g_{2,2}g_{2,1}g_{1,1}^2 + \frac{1}{2}g_{3,1}g_{2,2}g_{1,1}^2 + \frac{1}{3}g_{3,2}g_{2,2}g_{1,1}^3\Big)k^3 \\ &+ \Big(g_{2,1}g_{1,1} + \frac{1}{2}g_{3,3}g_{2,1}^2 + \frac{1}{2}g_{3,1}g_{2,2}g_{1,1} + \frac{1}{4}g_{3,3}g_{2,2}^2g_{1,1}^2 + g_{3,1}g_{2,1} \\ &+ \frac{1}{2}g_{2,2}g_{1,1}^2 + \frac{1}{2}g_{3,2}g_{1,1}^2 + \frac{1}{2}g_{3,2}g_{2,2}g_{1,1}^2 + \frac{1}{4}g_{3,3}g_{2,2}g_{1,1}^2 \\ &+ \frac{1}{2}g_{3,3}g_{2,1}g_{1,1} + g_{3,1}g_{1,1} + \frac{1}{2}g_{3,2}g_{2,1}g_{1,1} + \frac{1}{2}g_{3,3}g_{2,2}g_{2,1}g_{1,1}\Big)k^2 \\ &+ \Big(\frac{1}{2}g_{3,2}g_{1,1} + g_{2,1} + g_{1,1} + \frac{1}{2}g_{3,3}g_{2,1} + \frac{1}{2}g_{2,2}g_{1,1} + g_{3,1} + \frac{1}{12}g_{3,3}g_{2,2}^2g_{1,1} \\ &+ \frac{1}{6}g_{3,2}g_{2,2}g_{1,1} + \frac{1}{4}g_{3,3}g_{2,2}g_{1,1}\Big)k + 1\,, \quad \text{for } d = 3\,.\end{aligned}$$

4.2. Nakajima polytopes are Koszul. A graded $\boldsymbol{C}$-algebra $R = \bigoplus_{i \geq 0} R_i$ is a *Koszul algebra* if the R-module $\boldsymbol{C} \cong R/\mathfrak{m}$ (for $\mathfrak{m}$ a maximal homogeneous ideal) has a linear free resolution, i.e., if there exists an exact sequence

$$\cdots \longrightarrow R^{n_{i+1}} \xrightarrow{\varphi_{i+1}} R^{n_i} \xrightarrow{\varphi_i} \cdots \xrightarrow{\varphi_2} R^{n_1} \xrightarrow{\varphi_1} R^{n_0} \longrightarrow R/\mathfrak{m} \longrightarrow 0$$

of graded free R-modules all of whose matrices (determined by the φ_i's) have entries which are forms of degree 1. Every Koszul algebra is generated by its component of degree 1 and is defined by relations of degree 2.

The algebra $R = \boldsymbol{C}[\sigma_P \cap \boldsymbol{Z}^{d+1}]$ associated with a lattice polytope $P \subset \boldsymbol{R}^d$ has a natural grading $R_i = \boldsymbol{C}[\sigma_P \cap (\boldsymbol{Z}^d \times \{i\})]$. Call P *Koszul* if R is Koszul. Bruns, Gubeladze and Trung gave a sufficient condition for the Koszulness of P. In order to formulate it, we need the notion of a "non-face" of a lattice triangulation $\mathcal{T}$ of P. A subset $F \subset P \cap \boldsymbol{Z}^d$ is a *face* (of $\mathcal{T}$) if $\mathrm{conv}(F)$ is; otherwise F is said to be a *non-face*.

PROPOSITION 4.2 (Koszulness and Triangulations [BGT, 2.1.3]). *If the lattice polytope P has a coherent unimodular triangulation whose minimal non-faces* (*with respect to inclusion*) *are* 1*-dimensional* (*i.e., consist of* 2 *points*), *then P is Koszul.*

This property is satisfied for the triangulations constructed in Section 3.

COROLLARY 4.3. *Nakajima polytopes are Koszul.*

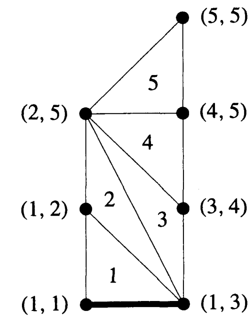

FIGURE 3.

PROOF. Once more we proceed by induction. Let $P = P_{\mathfrak{g}} \subset \boldsymbol{R}^d$ be a Nakajima polytope and $F = P \cap (\boldsymbol{R}^{d-1} \times \{0\})$ the Nakajima facet, both triangulated (by $\mathcal{T}_P$ and $\mathcal{T}_F$, respectively) as in Section 3. Let $\pi : \boldsymbol{R}^d \to \boldsymbol{R}^{d-1}$ denote the projection. By construction π maps faces of $\mathcal{T}_P$ to faces of $\mathcal{T}_F$. Choose a non-face $\mathfrak{n} \subseteq P \cap \boldsymbol{Z}^d$ of $\mathcal{T}_P$. We have to consider two cases:

(i) The projection $\pi(\mathfrak{n})$ is a face of $\mathcal{T}_F$.

(ii) It is not.

In the first case we stay within the chimney over $\pi(\mathfrak{n}) \in \mathcal{T}_F$. Thus, we may assume that $\pi(P) = \pi(\mathfrak{n})$. For all interior points x of $\pi(P)$ the linear ordering of the maximal simplices of $\mathcal{T}_P$ by their intersections with the line $\pi^{-1}(x)$ is the same, say, $\mathfrak{s}_1, \dots, \mathfrak{s}_r$. Assign to each lattice point $p \in P \cap \boldsymbol{Z}^d$ two numbers (m, M): $M(p) = \max\{i : p \in \mathfrak{s}_i\}$ and $m(p) = \min\{i : p \in \mathfrak{s}_i\}$, as illustrated in Figure 3 for a 2-dimensional Nakajima polytope.

Two vertices p, p' belong to the same maximal simplex $\mathfrak{s}_i$ of $\mathcal{T}_P$ if and only if $m(p)$, $m(p') \leq i \leq M(p), M(p')$. Such an index i can be found if and only if $m(p) \leq M(p')$ and $m(p') \leq M(p)$. Furthermore, if $p \in \mathfrak{s}_i \cap \mathfrak{s}_j$, then also $p \in \mathfrak{s}_k$ for all k between i and j.

Let $n_\uparrow \in \mathfrak{n}$ be a vertex with maximal m and $n_\downarrow \in \mathfrak{n}$ a vertex with minimal M. Since $\mathfrak{n}$ is a non-face, we have $m(n_\uparrow) > M(n_\downarrow)$. Hence $n_\uparrow$ and $n_\downarrow$ do not belong to a common maximal simplex, and $\{n_\downarrow, n_\uparrow\}$ is therefore a 1-dimensional non-face of $\mathcal{T}_P$.

In the second case, by induction $\pi(\mathfrak{n})$ contains a 1-dimensional non-face of $\mathcal{T}_F$. Any two vertices in $\mathfrak{n}$ with this projection form a 1-dimensional non-face of $\mathcal{T}_P$. □

5. Examples. In this section we apply our results to two classes of examples which are "extreme" in the sense that they achieve the lowest respectively highest possible numbers of faces of a $(d-1)$-dimensional Nakajima polytope. Moreover, we give the concrete binomial equations for the underlying spaces X_P of the corresponding l.c.i.-singularities, and present closed formulae for the cohomology ranks of the overlying spaces X_Σ of any crepant full desingularization $X_\Sigma \to X_P$.

5.1. Dilations of the standard simplex. Our first class of examples is the family of dilated standard simplices $k\mathfrak{s}^{(d-1)}$. These polytopes have the Nakajima description $P_{\mathfrak{g}}$ by the admissible sequence $\mathfrak{g} = (k\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{d-1})$ and, being simplices, they achieve for all i the minimal number of i-faces any $(d-1)$-polytope can have. This $P_{\mathfrak{g}}$ spans the cone $\sigma = \{\mathbf{x} \in \boldsymbol{R}^d : 0 \leq x_d \leq \ldots \leq x_2 \leq kx_1\}$, and its dual cone equals

$$\sigma^\vee = \operatorname{pos}(\mathbf{e}_d, \mathbf{e}_{d-1} - \mathbf{e}_d, \ldots, \mathbf{e}_2 - \mathbf{e}_3, k\mathbf{e}_1 - \mathbf{e}_2).$$

The semigroup $\sigma^\vee \cap \boldsymbol{Z}^d$ has the Hilbert basis (minimal generating system) $\mathcal{H} = \{\mathbf{f}_1, \ldots, \mathbf{f}_d, \mathbf{g}\}$, with $\mathbf{f}_1 = \mathbf{e}_d$, $\mathbf{f}_i = \mathbf{e}_i - \mathbf{e}_{i+1}$ for $i = 2, \ldots, d-1$, $\mathbf{f}_d = k\mathbf{e}_1 - \mathbf{e}_2$ and $\mathbf{g} = \mathbf{e}_1$. Introducing variables u_i and t to correspond to the evaluation of the torus characters at $\mathbf{f}_i$ and $\mathbf{g}$, respectively, we see that there is only one additive dependency $(k\mathbf{g}) - (\sum \mathbf{f}_i) = 0$, which means that

$$X_\sigma = \{(t, u_1, \ldots, u_d) \in \boldsymbol{C}^{d+1} : t^k - \prod u_i = 0\}$$

is a $(d; k)$-hypersurface (cf. [DHZ98, 5.11]).

After computing the coefficients of the Erhart polynomial of $k\mathfrak{s}^{(d-1)}$, we obtain by Theorem 4.1 the following formula for any equivariant crepant full desingularization $f_\Sigma^\sigma : X_\Sigma \to X_\sigma$ of X_σ:

$$\operatorname{rk} H^{2j}(X_\Sigma) = \sum_{i=0}^{j} (-1)^i \binom{d}{i} \binom{k(j-i)+d-1}{d-1}.$$

Besides the triangulations of $k\mathfrak{s}^{(d-1)}$ constructed by the inductive procedure of Section 3, there is also another coherent unimodular triangulation which is directly induced by a hyperplane arrangement. The families of hyperplanes

$$H_{i,j}(k') = \{\mathbf{x} \in \boldsymbol{R}^{d-1} : x_i - x_j = k'\} \ (i, j, k' \in \boldsymbol{Z}) \quad \text{and}$$
$$H_i(k') = \{\mathbf{x} \in \boldsymbol{R}^{d-1} : x_i = k'\} \ (i, k' \in \boldsymbol{Z})$$

form an arrangement $\mathfrak{H}$ that triangulates the entire ambient space $\boldsymbol{R}^d$ coherently and unimodularly. Since the facets of $k\mathfrak{s}^{(d-1)}$ span hyperplanes of $\mathfrak{H}$, it is enough to consider the restriction of this $\mathfrak{H}$-triangulation to $k\mathfrak{s}^{(d-1)}$. The advantage of this new triangulation is the uniform nature of its vertex stars. (A vertex star consists of the union of all simplices which contain the reference vertex.) The cones spanned by the simplices of each triangulation-vertex which is not a vertex of $k\mathfrak{s}^{(d-1)}$ determine a fan defining an exceptional prime divisor of $f_\Sigma^\sigma : X_\Sigma \to X_\sigma$. In particular, the compactly supported exceptional prime divisors of f_Σ^σ correspond to the vertices lying in the interior of $k\mathfrak{s}^{(d-1)}$ and the star of each of them is nothing but the $\mathfrak{H}$-triangulation of a polytope which is lattice equivalent to the lattice zonotope

$$W^{(d-1)} = \operatorname{conv}([-1, 0]^{d-1} \cup [0, 1]^{d-1}).$$

This suffices to show that each compactly supported exceptional prime divisor of f_Σ^σ comes from the crepant full $\mathfrak{H}$-desingularization of a projective toric Fano variety which admits an embedding of degree $\binom{2(d-1)}{d-1}$ into $\boldsymbol{P}_{\boldsymbol{C}}^{d(d-1)}$.

5.2. Products of intervals. Our second class of examples is the family of hyper-intervals $[\mathbf{0}, \mathbf{a}] = [0, a_1] \times \cdots \times [0, a_{d-1}]$ for integers $a_i > 0$. These polytopes have the Nakajima description $P_{\mathfrak{g}}$ by the admissible sequence $\mathfrak{g} = (a_1\mathbf{e}_1, \ldots, a_{d-1}\mathbf{e}_1)$, and for all i achieve the maximal number of i-faces any $(d-1)$-dimensional Nakajima polytope can have. The cone spanned by this P_g is $\sigma = \{\mathbf{x} \in \boldsymbol{R}^d : 0 \leq x_i \leq a_i x_d\}$,

$$\sigma^\vee = \operatorname{pos}(\mathbf{e}_1, \ldots, \mathbf{e}_{d-1}, a_1\mathbf{e}_d - \mathbf{e}_1, \ldots, a_{d-1}\mathbf{e}_d - \mathbf{e}_{d-1}),$$

and $\sigma^\vee \cap \mathbf{Z}^d$ is generated by $\mathcal{H} = \{\mathbf{f}_1, \ldots, \mathbf{f}_{d-1}, \mathbf{g}_1, \ldots, \mathbf{g}_{d-1}, \mathbf{h}\}$ with $\mathbf{f}_i = \mathbf{e}_i$ and $\mathbf{g}_i = a_i\mathbf{e}_d - \mathbf{e}_i$ for $i = 1, \ldots, d-1$, and $\mathbf{h} = \mathbf{e}_d$. Introduce one variable u_i for each $\mathbf{f}_i$, v_i for $\mathbf{g}_i$, and t for $\mathbf{h}$. The $d-1$ equations $(\mathbf{f}_i + \mathbf{g}_i) - (a_i\mathbf{h}) = 0$ form a basis for the lattice of integral linear dependences of $\mathcal{H}$, giving rise to the equations $u_i v_i - t^{a_i} = 0$.

Moreover, using the multiplicative behavior of the Ehrhart polynomial for products of polytopes, one obtains by Theorem 4.1 and (4.1) the following formula for any equivariant crepant full desingularization $f_\Sigma^\sigma : X_\Sigma \to X_\sigma$ of X_σ:

$$\operatorname{rk} H^{2j}(X_\Sigma) = \sum_{i=0}^{d-1} \sum_{\nu=0}^{j} (-1)^\nu \binom{d}{\nu} (j-\nu)^i \operatorname{sym}_i(\mathbf{a}),$$

where $\operatorname{sym}_i(\mathbf{x}) = \sum_{1 \leq \mu_1 < \ldots < \mu_i \leq d-1} x_{\mu_1} \cdots x_{\mu_i}$ denotes the elementary symmetric polynomial of degree i.

Note that this example also admits the $\mathfrak{H}$-triangulation discussed in 5.1.

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DIMITRIOS I. DAIS
SECTION OF ALGEBRA & GEOMETRY
MATHEMATICS DEPARTMENT
UNIVERSITY OF IOANNINA
GR-45110 IOANNINA
GREECE

E-mail address: DDais@CC.UoI.Gr

CHRISTIAN HAASE AND GÜNTER M. ZIEGLER
FACHBEREICH MATHEMATIK, MA 6-2
TU-BERLIN
STRASSE DES 17. JUNI 136
D-10623 BERLIN
GERMANY

E-mail address: [Haase/Ziegler]@Math.TU-Berlin.De