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ASYMPTOTICALLY SATURATED TORIC ALGEBRAS

FLORIN AMBRO

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Abstract. We establish finite generation of certain invariant graded algebras defined on toric log Fano fibrations. These are the toric version of FGA algebras, introduced by Shokurov in connection with the existence of flips.

Introduction. Asymptotically saturated algebras were introduced by Shokurov in his proof of the existence of 4-fold flips [7]. His approach is to reduce the existence of flips, by induction on the dimension, to the finite generation of graded algebras which are asymptotically saturated with respect to log Fano fibrations. Shokurov established the finite generation of these algebras in dimensions one and two, and conjectured this to hold in any dimension. In this paper we verify the toric case of this conjecture.

In a recent development, Hacon and M^{c}Kernan [3] reduced the existence of flips to the existence of smaller dimensional minimal models. They simplified Shokurov's reduction argument by studying extensions of pluricanonical forms, and so they end up with asymptotically saturated algebras of a particular kind, whose finite generation follows from the existence of smaller dimensional minimal models. Though asymptotically saturated algebras may be useful in other contexts, their finite generation on log Fano fibrations is no longer interesting for the existence of flips.

THEOREM 0.1. *Let $\pi : X \to S$ be a proper surjective toric morphism with connected fibers, and B an invariant $\boldsymbol{Q}$-divisor on X such that (X, B) has Kawamata log terminal singularities and $-(K + B)$ is π-nef.*

(1) *Let $\mathcal{L} \subseteq \bigoplus_{i=0}^{\infty} \pi_* \mathcal{O}_X(iD)$ be an invariant graded $\mathcal{O}_S$-subalgebra which is asymptotically saturated with respect to $(X/S, B)$, where D is an invariant divisor on X. Then $\mathcal{L}$ is finitely generated.*

(2) *Up to isomorphism, there are only finitely many rational maps $X \dashrightarrow \operatorname{Proj}(\bar{\mathcal{L}})$, where $\mathcal{L}$ is an $\mathcal{O}_S$-algebra as in (1) and $\bar{\mathcal{L}}$ is its integral closure in $\boldsymbol{C}(X)$.*

The toric case of asymptotic saturation, the key property ensuring finite generation in (1), can be explicitly written down as a diophantine system (see Lemma 4.2). To see this in a special case, let S be a point and let $X = T_N \operatorname{emb}(\Delta)$ be a torus embedding for some lattice N. Let M be the lattice dual to N, and consider a compact convex set of maximal dimension

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$\square$ in $M_{\boldsymbol{R}}$. This defines a toric graded $\boldsymbol{C}$-algebra

$$\mathcal{R}(\square) = \bigoplus_{i=0}^{\infty} \left(\bigoplus_{m \in M \cap i\square} \boldsymbol{C} \cdot \chi^m \right),$$

which is finitely generated if and only if $\square$ is a rational polytope. One can easily construct a Weil divisor D on X such that $\mathcal{R}(\square)$ is a subalgebra of $\bigoplus_{i=0}^{\infty} H^0(X, iD)$. On the other hand, the log discrepancies of the log pair (X, B) with respect to toric valuations can be encoded in a positive function $\psi \colon N_{\boldsymbol{R}} \to \boldsymbol{R}$, and ψ determines a rule to enlarge any convex set in $M_{\boldsymbol{R}}$ to an open convex neighborhood. Asymptotic saturation of $\mathcal{R}(\square)$ with respect to (X, B) means that the lattice points of the neighborhood of $j\square$ are already contained in $j\square$, for every sufficiently divisible positive integer j (see Definition 2.5). This diophantine property restricts the way that $\square$ can be approximated with rational points from the outside.

The key technical tool behind Theorem 0.1 is a known result in the geometry of numbers, the Flatness Theorem [4]: for any convex body $\square \subset M \otimes_{\boldsymbol{Z}} \boldsymbol{R}$, there exists $e \in N \setminus 0$ such that $\max_{m \in \square} \langle m, e \rangle - \min_{m \in \square} \langle m, e \rangle$ is bounded from above by a constant depending only on the dimension of M and the number of lattice points contained in $\square$.

The outline of this paper is as follows. In Section 4 we explicitly describe toric asymptotic saturation and reduce Theorem 0.1 to the special case when the algebra is normal and associated with a convex set, the equivalent of $\mathcal{R}(\square)$ above. The rest of the paper is devoted to this special case. In Section 1 we collect some elementary results on convex sets and their support functions, and on diophantine approximation. In Section 2 we characterize asymptotic saturation in geometric terms (Theorem 2.6) and obtain a boundedness result (Theorem 2.7). These are used in Section 3 to prove Theorem 0.1, by induction on the dimension.

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1. Preliminary. We collect in this section elementary results on convex sets, toric geometry and diophantine approximation. We refer the reader to Oda [5] for basic notions and terminology on toric varieties and convex sets.

Throughout this section, N is a lattice, with dual lattice M. We have a duality pairing $\langle \cdot, \cdot \rangle \colon M_{\boldsymbol{R}} \times N_{\boldsymbol{R}} \to \boldsymbol{R}$, defined over $\boldsymbol{Z}$.

1.1. CONVEX SETS AND SUPPORT FUNCTIONS. Fix a *convex rational polyhedral cone* $\sigma \subseteq N_{\boldsymbol{R}}$, that is, σ is spanned by finitely many elements of N. We denote by $\mathcal{S}(\sigma)$ the set of all functions $h \colon \sigma \to \boldsymbol{R}$ satisfying the following properties:

1) *Positive homogeneity*: $h(te) = t \cdot h(e)$ for $t \geq 0, e \in \sigma$.
2) *Upper convexity*: $h(e_1 + e_2) \geq h(e_1) + h(e_2)$ for $e_1, e_2 \in \sigma$.

THEOREM 1.1. *The following properties hold*:

(i) *Every function h in $\mathcal{S}(\sigma)$ is continuous.*

(ii) *Let $(h_i)_{i \geq 1}$ be a sequence of functions in $\mathcal{S}(\sigma)$ which converges pointwise, and set $h(e) = \lim_{i \to \infty} h_i(e)$ for $e \in \sigma$. Then $h \in \mathcal{S}(\sigma)$, and the sequence $(h_i)_i$ converges uniformly to h on compact subsets of σ.*

PROOF. This is a special case of [6, Theorems 10.1 and 10.8].

For a function $h\colon \sigma \to \boldsymbol{R}$, define

$$\square_h = \{m \in M_{\boldsymbol{R}}\,;\ \langle m, e\rangle \geq h(e) \text{ for all } e \in \sigma\}\,.$$

A *convex polytope* $K \subset M_{\boldsymbol{R}}$ is the convex hull of a finite set in $M_{\boldsymbol{R}}$. A *rational convex polytope* is the convex hull of a finite set in $M_{\boldsymbol{Q}}$. A *rational convex polyhedral set* is the intersection of finitely many rational affine half spaces in $M_{\boldsymbol{R}}$. We denote by $\mathcal{C}(\sigma^\vee)$ the set of all nonempty closed convex sets $\square \subseteq M_{\boldsymbol{R}}$ satisfying the following two properties:

1) $\square + \sigma^\vee = \square$.

2) $\square \subseteq K + \sigma^\vee$ for some convex polytope $K \subset M_{\boldsymbol{R}}$.

The *support function* $h_\square\colon \sigma \to \boldsymbol{R}$ of $\square \in \mathcal{C}(\sigma^\vee)$ is defined by

$$h_\square(e) = \inf_{m\in\square} \langle m, e\rangle\,.$$

THEOREM 1.2. *The maps $\square \mapsto h_\square$ and $h \mapsto \square_h$ are inverse to each other, inducing a bijection $\mathcal{C}(\sigma^\vee) \simeq \mathcal{S}(\sigma)$. Under this correspondence, the Minkowski sum $\square + \square'$ and a nonnegative scalar multiple $t\square$ correspond to $h_\square + h_{\square'}$ and $th_\square$, respectively.*

We omit the proof of this theorem, as it is similar to that of [5, Theorem A.18]. When $\sigma = N_{\boldsymbol{R}}$, this is the usual correspondence between compact convex sets and support functions. Note that $K + \sigma^\vee \in \mathcal{C}(\sigma^\vee)$ for every compact convex set $K \subset M_{\boldsymbol{R}}$, but not all elements of $\mathcal{C}(\sigma^\vee)$ are of this form. Such an example is $\sigma^\vee = \{(x, y) \in \boldsymbol{R}^2\,;\ x, y \geq 0\}$ and $\square = \{(x, y) \in \sigma^\vee\,;\ xy \geq 1\}$. Nevertheless, we have

LEMMA 1.3. *The following properties are equivalent for $\square \in \mathcal{C}(\sigma^\vee)$:*

(i) *$\square$ is a rational convex polyhedral set.*

(ii) *$\square = K + \sigma^\vee$ for some rational convex polytope K such that no vertex of K belongs to the Minkowski sum of $\sigma^\vee$ and the convex hull of the other vertices of K.*

Furthermore, K is uniquely determined by $\square$ if $\dim(\sigma) = \dim(N)$.

EXAMPLE 1.4. If $\sigma = N_{\boldsymbol{R}}$, then $K = \square$. If σ is the positive cone in $\boldsymbol{R}^d$ and $\square$ is a Newton polytope, then K is the convex hull of the compact faces of $\square$.

1.2. PROPER TORIC MORPHISMS WITH AFFINE BASE. Toric morphisms with affine base, which are proper, surjective and with connected fibers, are in one to one correspondence with fans having convex support.

Indeed, let Δ be a fan in a lattice N such that its support $|\Delta| = \bigcup_{\tau\in\Delta} \tau$ is a convex rational polyhedral cone. Let $\bar{N} = N/(N \cap |\Delta| \cap (-|\Delta|))$ and let $\bar{\sigma} \subset \bar{N}_{\boldsymbol{R}}$ be the image of $|\Delta|$ under the natural projection. Then $T_N \operatorname{emb}(\Delta) \to T_{\bar{N}}(\bar{\sigma})$ is a toric morphism with affine base, which is proper, surjective, with connected fibers.

Conversely, let $\pi\colon X \to S$ be a proper toric morphism of toric varieties, with S affine. Thus, $X = T_N \operatorname{emb}(\Delta)$, $S = T_{N'}(\sigma')$ and π corresponds to a lattice homomorphism $\varphi_{\boldsymbol{Z}}\colon N \to N'$ such that Δ is a finite fan in N, σ' is a strongly convex rational polyhedral cone in N' and

$|\Delta| = \varphi^{-1}(\sigma')$, respectively. In particular, $|\Delta|$ is a convex rational polyhedral cone. Then $\varphi_{\boldsymbol{Z}}$ factors through $\bar{N} = N/(N \cap |\Delta| \cap (-|\Delta|))$, and we have a commutative diagram

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "\pi"'] & & T_{\bar{N}}(\bar{\sigma}) \arrow[dl, "j"] \\ & S & \end{tikzcd}$$

where j is finite on its image.

Let now $D = \sum_{e \in \Delta(1)} d_e V(e)$ be an invariant $\boldsymbol{Q}$-divisor on X which is $\boldsymbol{Q}$-Cartier. This means that there exists a function $h\colon |\Delta| \to \boldsymbol{R}$ such that h is Δ-linear and $h(e) = -d_e$ for every $e \in \Delta(1)$. In particular, h is positively homogeneous. The $\boldsymbol{Q}$-divisor D is π-nef if h is upper convex; it is π-ample if for every maximal cone $\sigma \in \Delta$ there exists $m_\sigma \in \square_h$ such that $\sigma = \{e \in |\Delta|;\ h(e) = \langle m_\sigma, e\rangle\}$.

1.3. THE AMPLE FAN OF A CONVEX RATIONAL POLYHEDRAL SET. To each convex rational polyhedral set $\square \subseteq M_{\boldsymbol{R}}$ we associate a fan $\Delta_\square$ in a quotient lattice of N as follows. Assume first that $\dim(\square) = \dim(M)$. Let K be a rational polytope associated with $\square$ by Lemma 1.3, with vertices $v_1, \ldots, v_l$. The support function of $\square$ is $h(e) = \min_{j=1}^{l} \langle v_j, e\rangle$, and the cones $\{e \in |\Delta|;\ \langle v_j, e\rangle = h(e)\}$, for $1 \le j \le l$, form the maximal dimensional cones of a fan $\Delta_\square$ in N. The support $|\Delta_\square|$ is the unique convex cone $\sigma \subseteq N_{\boldsymbol{R}}$ such that $\square \in \mathcal{C}(\sigma^\vee)$.

If $\dim(\square) < \dim(M)$, choose a point $m_0 \in M_{\boldsymbol{Q}} \cap \square$ and define $N' = N/(N \cap (\square - m_0)^\perp)$. If M' is the dual lattice of N', then $M'_{\boldsymbol{R}}$ can be identified with the smallest vector subspace of $M_{\boldsymbol{R}}$ which contains $\square - m_0$. We have $\dim(\square - m_0) = \dim(M')$. The fan $\Delta_{\square - m_0}$ in N' defined above is independent of the choice of m_0, and we denote it again by $\Delta_\square$. Its support is a convex set.

DEFINITION 1.5. $\Delta_\square$ is called the *ample fan* of the rational convex polyhedral set $\square \subset M_{\boldsymbol{R}}$.

Assume now that $\pi : X \to S$ is a proper toric morphism with affine base S. We may write $X = T_N\,\mathrm{emb}(\Delta)$ and $S = T_{\bar{N}}(\bar{\sigma})$, and π corresponds to a lattice homomorphism $\varphi_{\boldsymbol{Z}}\colon N \to \bar{N}$ such that $|\Delta| = \varphi^{-1}(\bar{\sigma})$.

For a rational polyhedral convex set $\square \in \mathcal{C}(|\Delta|^\vee)$, the T_N-invariant $\mathcal{O}_S$-algebra

$$\mathcal{R}(\square) = \bigoplus_{i=0}^{\infty} \left(\bigoplus_{m \in M \cap i\square} \boldsymbol{C} \cdot \chi^m \right)$$

is normal and finitely generated. The induced toric rational map

$$\begin{tikzcd} X \arrow[rr, dashed, "\Phi"] \arrow[dr] & & \mathrm{Proj}(\mathcal{R}(\square)) \arrow[dl] \\ & S & \end{tikzcd}$$

is defined over S, and $\mathrm{Proj}(\mathcal{R}(\square))$ is the torus embedding of the ample fan $\Delta_\square$. If $\dim(\square) = \dim(M)$, then $\Delta_\square$ is a fan in N with $|\Delta_\square| = |\Delta|$, and hence Φ is birational in this case. The invariant $\boldsymbol{Q}$-divisor $\sum_{e\in\Delta_\square(1)} -h(e)V(e)$ is ample relative to S. If $\dim(\square) < \dim(M)$, then $\Delta_\square$ is a fan in N', whose support is the image of $|\Delta|$ under the natural projection.

1.4. DIOPHANTINE APPROXIMATION. Let $m \in M_{\boldsymbol{R}}$, $e \in N_{\boldsymbol{R}}$, I a positive integer and $\|\cdot\|$ a norm on $M_{\boldsymbol{R}}$.

THEOREM 1.6 (cf. [1]). *For every* $\varepsilon > 0$, *there exists a positive multiple* k *of* I *and there exists* $\bar{m} \in M$ *such that* $\langle \bar{m} - km, e\rangle \in (-\varepsilon, 0]$ *and* $\|\bar{m} - km\| < \varepsilon$. *Furthermore,* $\langle \bar{m} - km, e\rangle \neq 0$ *if* $e \notin \{e' \in N\,;\ \langle m, e'\rangle \in \boldsymbol{Q}\} \otimes_{\boldsymbol{Z}} \boldsymbol{R}$.

PROOF. We may find a decomposition $M = M' \oplus M''$, with dual decomposition $N = N' \oplus N''$, such that $m = m' + m''$, $m' \in M'_{\boldsymbol{Q}}$, $m'' \in M''_{\boldsymbol{R}}$ and $\{e'' \in N''\,;\ \langle m'', e''\rangle \in \boldsymbol{Q}\} = \{0\}$. Decompose $e = e' + e''$ with $e' \in N'_{\boldsymbol{R}}$ and $e'' \in N''_{\boldsymbol{R}}$. Note that $e \notin \{e' \in N\,;\ \langle m, e'\rangle \in \boldsymbol{Q}\} \otimes_{\boldsymbol{Z}} \boldsymbol{R}$ if and only if $e'' \neq 0$. Let k_1 be a positive integer such that $I | k_1$ and $k_1 m' \in M'$. Since $\{e'' \in N''\,;\ \langle k_1 m'', e''\rangle \in \boldsymbol{Q}\} = \{0\}$, we infer by [2, Chapter III, Theorem IV] that the subgroup generated by the class of $k_1 m''$ is dense in the torus $M''_{\boldsymbol{R}}/M''$. Equivalently, the set $\bigcup_{j\geq 1}(M'' + jk_1 m'')$ is dense in $M''_{\boldsymbol{R}}$. In particular, the following system has a solution for some $j \geq 1$:

$$\begin{cases} m''_j \in M''\,, \\ \langle m''_j + jk_1 m'', e''\rangle = \langle m''_j + jk_1 m''\,, e\rangle \in [0, \varepsilon)\,, \\ \|m''_j + jk_1 m''\| < \varepsilon\,. \end{cases}$$

If $e'' \neq 0$, we may also take $\langle m''_j + jk_1 m'', e''\rangle \neq 0$. Then $k = jk_1$ and $\bar{m} = km' - m''_j$ satisfy the desired properties.

LEMMA 1.7. *If* $\langle m, e\rangle \in \boldsymbol{Q}$, *the following properties are equivalent*:

(i) $m \in \{m' \in M\,;\ \langle m', e\rangle \in \boldsymbol{Q}\} \otimes_{\boldsymbol{Z}} \boldsymbol{R}$.
(ii) $e \in \{e' \in N\,;\ \langle m, e'\rangle \in \boldsymbol{Q}\} \otimes_{\boldsymbol{Z}} \boldsymbol{R}$.

PROOF. Assume that (i) holds. We may find a decomposition $N = N' \oplus N''$, with dual decomposition $M = M' \oplus M''$, such that $e = e' + e''$, $e' \in N'_{\boldsymbol{Q}}$, $e'' \in N''_{\boldsymbol{R}}$ and

$$\{m'' \in M''\,;\ \langle m'', e''\rangle \in \boldsymbol{Q}\} = \{0\}\,.$$

The assumption means that $m \in M'_{\boldsymbol{R}}$. We may find a decomposition $M' = M'_1 \oplus M'_2$, with dual decomposition $N' = N'_1 \oplus N'_2$, such that $m = m'_1 + m'_2$, $m'_1 \in M'_{1,\boldsymbol{Q}}$, $m'_2 \in M'_{2,\boldsymbol{R}}$ and

$$\{e'_2 \in N'_2\,;\ \langle m'_2, e'_2\rangle \in \boldsymbol{Q}\} = \{0\}\,.$$

We have $\langle m'_2, e'\rangle = \langle m, e'\rangle - \langle m'_1, e'\rangle = \langle m, e\rangle - \langle m'_1, e'\rangle \in \boldsymbol{Q}$. Therefore $e' \in N'_{1,\boldsymbol{Q}}$. Let $e'' = \sum_i r_i e''_i$, where $r_i \in \boldsymbol{R}$ and $\{e''_i\}_i$ is a basis of N''. Then $e', e''_i \in N_{\boldsymbol{Q}}$, $\langle m, e'\rangle \in \boldsymbol{Q}$ and $\langle m, e''_i\rangle = 0$. Therefore $e = e' + e'' \in \{e' \in N\,;\ \langle m, e'\rangle \in \boldsymbol{Q}\} \otimes_{\boldsymbol{Z}} \boldsymbol{R}$, and so (i) holds. The statement is symmetric in m and e, and hence the converse holds as well.

2. Asymptotic saturation. Throughout this section, we fix a lattice N and a convex rational polyhedral cone $\sigma \subseteq N_{\boldsymbol{R}}$.

DEFINITION 2.1. A *log discrepancy function* is a function $\psi : \sigma \to \boldsymbol{R}$ satisfying the following properties:

(i) ψ is positively homogeneous.
(ii) $\psi(e) > 0$ for $e \neq 0$.
(iii) ψ is continuous.

EXAMPLE 2.2. Let Δ be a fan in N with $|\Delta| = \sigma$. Let $B = \sum_{e \in \Delta(1)} b_e V(e)$ be an invariant $\boldsymbol{R}$-divisor on $X = T_N \operatorname{emb}(\Delta)$ such that $K + B$ is $\boldsymbol{R}$-Cartier and the log pair (X, B) has Kawamata log terminal singularities. Equivalently, there exists a function $\psi : \sigma \to \boldsymbol{R}$ such that $\psi(e) = 1 - b_e > 0$ for every $e \in \Delta(1)$, and ψ is Δ-linear. Then ψ is a *log discrepancy function*.

The terminology comes from the following property: let $e \in N^{\text{prim}} \cap \sigma$ be a primitive lattice point, corresponding to a toric valuation v_e of X. Then $\psi(e)$ is the log discrepancy of the log pair (X, B) at the valuation v_e.

LEMMA 2.3. *Let $\psi : \sigma \to \boldsymbol{R}$ be a log discrepancy function. Then $\{e \in \sigma\, ;\ \psi(e) \leq 1\}$ is a compact set.*

PROOF. Choose a norm $\|\cdot\|$ on $N_{\boldsymbol{R}}$. Since ψ is a log discrepancy function, the infimum $c_0 = \inf\{\psi(e)\, ;\ e \in \sigma, \|e\| = 1\}$ is a well defined positive real number. We have $\psi(e) \geq c_0\|e\|$, for $e \in \sigma$. In particular,

$$\{e \in \sigma\, ;\ \psi(e) \leq 1\} \subseteq \{e \in \sigma\, ;\ \|e\| \leq c_0^{-1}\}\,.$$

The left hand side is a closed set, since ψ is continuous, and the right hand side is a bounded set. Therefore the claim holds.

DEFINITION 2.4. For an arbitrary function $h : \sigma \to \boldsymbol{R}$, define

$$\overset{\circ}{\Box}_h = \{m \in M_{\boldsymbol{R}}\, ;\ \langle m, e\rangle > h(e) \ \text{ for all } e \in \sigma \setminus 0\}\,.$$

DEFINITION 2.5. Let $\Box \in \mathcal{C}(\sigma^\vee)$ and $\psi : \sigma \to \boldsymbol{R}$ a log discrepancy function. We say that

(i) $\Box$ is *ψ-saturated* if $M \cap \overset{\circ}{\Box}_{h_\Box - \psi} \subset \Box$, where $h_\Box \in \mathcal{S}(\sigma)$ is the support function of $\Box$. Note that $\Box = \Box_{h_\Box}$.
(ii) $\Box$ is *asymptotically ψ-saturated* if there exists a positive integer I such that $j\Box$ is ψ-saturated, for every $I \mid j$.

Note that saturation (asymptotic saturation) is invariant under lattice (rational) translations of the convex set.

THEOREM 2.6 (Characterization of asymptotic saturation). *Let $\Box \in \mathcal{C}(\sigma^\vee)$ be a rational polyhedral set and $\psi : \sigma \to \boldsymbol{R}$ a log discrepancy function. Let $N'' = \{e \in N\, ;\ \Box \ni$*

$m \mapsto \langle m, e\rangle \in \boldsymbol{R}$ constant}, with dual lattice M'', and define $\psi'' \colon N''_{\boldsymbol{R}} \to \boldsymbol{R}$ by

$$\psi''(e) = \psi(e)\,.$$

The ample fan $\Delta_{\square}$ is a fan in $N' = N/N''$ with support $\sigma' = \pi(\sigma)$, where $\pi \colon N_{\boldsymbol{R}} \to N'_{\boldsymbol{R}}$ is the natural projection. Define $\psi' \colon \sigma' \to \boldsymbol{R}$ by

$$\psi'(e') = \inf_{e \in \pi^{-1}(e')} \psi(e)\,.$$

Then $\square$ is asymptotically ψ-saturated if and only if the following hold:

(1) $M'' \cap \overset{\circ}{\square}_{-\psi''} = \{0\}$.

(2) *$\psi'(e') \leq 1$ for all $e' \in \Delta_{\square}(1)$. Recall that $\Delta_{\square}(1)$ denotes the set of primitive lattice points of the one-dimensional cones of $\Delta_{\square}$.*

PROOF. After a rational translation, we may assume that $0 \in \square$. In particular, $N'' = N \cap \square^{\perp}$ and $h(e) = h'(\pi(e))$, where $h \in \mathcal{S}(\sigma)$ and $h' \in \mathcal{S}(\sigma')$ are the support functions of $\square \subset M_{\boldsymbol{R}}$ and $\square \subset M'_{\boldsymbol{R}} = M' \otimes_{\boldsymbol{Z}} \boldsymbol{R}$, respectively.

Assume that (1) and (2) hold. Fix a positive integer j such that $I|j$ and $jh(N) \subseteq \boldsymbol{Z}$ and assume that $m \in M$ satisfies

$$\langle m, e\rangle > (jh - \psi)(e) \quad \text{for all } \ e \in \sigma \setminus 0\,.$$

Choose a decomposition $M = M' \oplus M''$, and decompose $m = m' + m''$. Since $h|_{N''} = 0$, we obtain $m'' \in M'' \cap Q''$, and hence $m'' = 0$ by (1). In particular, we have

$$\langle m, e'\rangle > jh'(e') - \psi'(e') \quad \text{for all } \ e' \in \sigma' \setminus 0\,.$$

For every $e' \in \Delta_{\square}(1)$, we have $\langle m, e'\rangle \in \boldsymbol{Z}$, and hence (2) gives $\langle m, e'\rangle \geq jh'(e')$. Since h' is $\Delta_{\square}$-linear, we obtain $\langle m, e'\rangle \geq jh'(e')$ for all $e' \in |\Delta_{\square}|$, hence $m \in \square_{jh'}$. Therefore jh is ψ-saturated.

For the converse, assume that jh is ψ-saturated for every $I|j$. We first check (1). Fix $m'' \in M'' \cap Q''$. Let $\|\cdot\|$ be a norm on $M_{\boldsymbol{R}}$, which is compatible with the decomposition $M = M' \oplus M''$. Since ψ is continuous, there exists $\varepsilon > 0$ such that

$$\langle m'', e\rangle + \psi(e) > 0 \quad \text{for all } \ e \in S(\sigma),\, \|e'\| < \varepsilon\,.$$

The rational convex polyhedral set $\square$ has the same dimension as $M'_{\boldsymbol{R}}$. Therefore there exists $m' \in M' \cap \operatorname{relint}(k\square)$, for some positive integer k. We have

$$\langle m', e'\rangle > kh'(e') \quad \text{for all } \ e' \in \sigma' \setminus 0\,.$$

The continuity of ψ implies that the following number is well defined:

$$t = -\inf\left\{\frac{\langle m'', e\rangle + \psi(e)}{\langle m', e'\rangle - kh'(e')}\,;\ e \in S(\sigma),\, \|e'\| \geq \varepsilon\right\}.$$

Let j be a positive multiple of I such that $j > t$. The identity

$$\langle jm' + m'', e\rangle - (jkh - \psi)(e) = j(\langle m', e'\rangle - kh'(e')) + \langle m'', e\rangle + \psi(e)$$

implies that $\langle jm' + m'', e\rangle > (jkh - \psi)(e)$ for every $e \in S(\sigma)$. Since jkh is ψ-saturated, we infer that $jm' + m'' \in \Box_{jkh}$. In particular, $jm' + m'' \in M'$, and hence $m'' = 0$. This proves (1).

For (2), fix $e' \in \Delta_\Box(1)$ and assume by contradiction that $\psi'(e') > 1$. We may find a basis $e_1, \ldots, e_d$ of N with $e_1 = e'$. Let $\|\cdot\|$ be the absolute value norm on $N_{\boldsymbol{R}}$ with respect to this basis and denote

$$S(\sigma) = \{e \in \sigma\,;\ \|e\| = 1\}\,.$$

The face $\{m \in \Box;\ \langle m, e_1\rangle = h(e_1)\}$ of $\Box$ is a positive dimensional convex polyhedral set, and hence there exists a 1-dimensional rational compact convex set $\Box_1$ with

$$\Box_1 \subset \operatorname{relint}(\{m \in \Box;\ \langle m, e_1\rangle = h(e_1)\})\,.$$

It is easy to see that there exists a positive real number t_1 such that $M \cap t\Box_1 \neq \emptyset$ for $t > t_1$. Consider the following set

$$C = \{e \in S(\sigma)\,;\ \psi(e) \leq \langle e_1^*, e\rangle\}\,.$$

Since ψ is continuous, the (possibly empty) set C is closed. Furthermore, $e_1 \notin C$ and $\Box_1$ is included in the relative interior of the face of $\Box$ corresponding to e_1, and hence $\langle m, e\rangle - h(e) > 0$ for $e \in C$ and $m \in \Box_1$. We infer that the following number is well defined

$$t_2 = \sup_{m \in \Box_1, e \in C} \frac{(e_1^* - \psi)(e)}{\langle m, e\rangle - h(e)}\,.$$

Let j be a positive multiple of I such that $j > \max(t_1, t_2)$. Since $j > t_1$, there exists $m_j \in M$ such that $m_j + e_1^* \in j\Box_1$. We have

$$\langle m_j, e\rangle - (jh - \psi)(e) = j(\langle (m_j + e_1^*)/j, e\rangle - h(e)) - (e_1^* - \psi)(e)\,.$$

Since $j > t_2$, we obtain $m_j \in \overset{\circ}{\Box}_{jh-\psi}$. Since $j\Box$ is ψ-saturated, we obtain $m_j \in \Box_{jh}$. This is a contradiction, since $\langle m_j, e_1\rangle = jh(e_1) - 1 < jh(e_1)$. This proves (2).

THEOREM 2.7. *Let $\psi\colon N_{\boldsymbol{R}} \to \boldsymbol{R}$ be a log discrepancy function such that $-\psi$ is upper convex and $M \cap \overset{\circ}{\Box}_{-\psi} = \{0\}$. Then there exists $e \in N \setminus 0$ such that $\psi(e) + \psi(-e) \leq C$, where C is a positive constant depending only on* $\dim(N)$.

PROOF. Let $\Box = \Box_{-\psi/2}$. Since ψ is positive, we have $0 \in \Box \subset \overset{\circ}{\Box}_{-\psi}$. Then $\Box$ is a compact convex set, of dimension $\dim(N) = d$, with support function $-\psi/2$, such that $M \cap \Box = \{0\}$. By [4, Theorem 4.1], there exists $e \in N \setminus 0$ such that

$$\max_{m \in \Box} \langle m, e\rangle - \min_{m \in \Box} \langle m, e\rangle \leq c_0 d^2 \lceil \sqrt[d]{1 + \#(M \cap \Box)}\rceil\,,$$

where c_0 is a positive universal constant and $\#(M \cap \Box)$ is the number of lattice points of $\Box$. In our case, this means that $\psi(e) + \psi(-e) \leq C = 2c_0 d^2 \lceil \sqrt[d]{2}\rceil$.

THEOREM 2.8 (Toric Asymptotic CCS). *Let $\psi\colon \sigma \to \boldsymbol{R}$ be a log discrepancy function. Let $\mathcal{M}(\psi)$ be the set of rational polyhedral sets $\Box \in \mathcal{C}(\sigma^\vee)$ such that $\Box$ is asymptotically ψ-saturated and $h_\Box - \psi$ is upper convex. Then the set of ample fans $\{\Delta_\Box\}_{\Box \in \mathcal{M}(\psi)}$ is finite.*

PROOF. Let $\square \in \mathcal{M}(\psi)$, with support function h. After a rational translation, we may assume $0 \in \square$. Let $N'' = N \cap \square^\perp$ and $d = \dim(N'')$. If $d = 0$, that is, $\dim(\square) = \dim(M)$, then the ample fan $\Delta_\square$ is a fan in N with $|\Delta_\square| = \sigma$, and by Theorem 2.6 we have

$$\Delta_\square(1) \subseteq N^{\text{prim}} \cap \{e \in N_{\boldsymbol{R}}\,;\ \psi(e) \le 1\}\,.$$

The right hand side is a finite set, and hence the number of fans $\Delta_\square$ is finite.

Assume now $d > 0$. We claim that N'' belongs to a finite set of sublattices of N. Since $h|_{N''} = 0$, $-\psi|_{N''_{\boldsymbol{R}}} = (h-\psi)|_{N''_{\boldsymbol{R}}}$ is an upper convex function. By assumption, $M'' \cap \mathring{\square}_{-\psi|_{N''_{\boldsymbol{R}}}} = \{0\}$. By Theorem 2.7, there exists $e_1 \in N'' \setminus 0$ such that $\psi(e_1) + \psi(-e_1) \le C$. We may assume that e_1 is a primitive element of N. Consider the lattice $N' = N/(\boldsymbol{Z} \cdot e_1)$ and let $\pi_{\boldsymbol{Z}} \colon N \to N'$ be the induced projection map. There exists $h' \colon \sigma' \to \boldsymbol{R}$ such that $h = h' \circ \pi$. Define $\psi'(e') = \inf_{e \in \pi^{-1}(e')} \psi(e)$. Then ψ' is a log discrepancy function on $N'_{\boldsymbol{R}}$ and $\square = \square_{h'} \in \mathcal{M}(\psi')$, by Lemma 2.9. We repeat this argument d times, until we obtain a basis $e_1, \dots, e_d$ of N'' with the following properties:

(i) $\psi(e_1) + \psi(-e_1) \le C$.

(ii) $\inf(\psi|_{e_k + \sum_{i=1}^{k-1} \boldsymbol{R} e_i}) + \inf(\psi|_{-e_k + \sum_{i=1}^{k-1} \boldsymbol{R} e_i}) \le C$ for $2 \le k \le d$.

By Lemma 2.3, e_1 belongs to a finite set. By Lemmas 2.9 and 2.3, e_k belongs to a finite set modulo $\sum_{i=1}^{k-1} \boldsymbol{Z} e_i$, for every k. Therefore N'', the subspace of N generated by $e_1, \dots, e_d$, belongs to a finite set of sublattices of N.

For N'' as above, let $N' = N/N''$. There exists $h' \colon \sigma' \to \boldsymbol{R}$ such that $h = h' \circ \pi$. Define the log discrepancy function $\psi' \colon \sigma' \to \boldsymbol{R}$ by $\psi'(e') = \inf_{e \in \pi^{-1}(e')} \psi(e)$. By Theorem 2.6 again, we have $\Delta_\square(1) \subseteq N'^{\text{prim}} \cap \{e' \in N'_{\boldsymbol{R}}\,;\ \psi'(e') \le 1\}$. Therefore there are finitely many ample fans $\Delta_\square$.

We have obtained the finiteness of ample fans when d is fixed. Since $1 \le d \le \dim(\sigma)$, there are only finitely many ample fans.

LEMMA 2.9 (Restriction of saturation). *Let $\psi \colon \sigma \to \boldsymbol{R}$ be a log discrepancy function, let $\square \in \mathcal{C}(\sigma^\vee)$ and let $\pi_{\boldsymbol{Z}} \colon N \to N'$ be a quotient lattice. We identity the dual lattice M' with $M \cap \operatorname{Ker}(\pi)^\perp \subset M$. The image $\sigma' = \pi(\sigma)$ is a rational convex polyhedral cone in $N'_{\boldsymbol{R}}$.*

Assume that $m_0 \in M \cap \square$. The convex set $\square' = (\square - m_0) \cap M'_{\boldsymbol{R}}$ belongs to $\mathcal{C}(\sigma'^\vee)$ and its support function $h' \in \mathcal{S}(\sigma')$ is computed as follows

$$h'(e') = \sup\{h(e) - \langle m_0, e\rangle\,;\ e \in \sigma \cap \pi^{-1}(e')\}\,.$$

Define a positively homogeneous function $\psi' \colon \sigma' \to \boldsymbol{R}$ by

$$\psi'(e') = h'(e') - \sup\{h(e) - \langle m_0, e\rangle - \psi(e)\,;\ e \in \sigma \cap \pi^{-1}(e')\}\,.$$

The following properties hold:

(i) *If $\square$ is ψ-saturated, then $\square'$ is ψ'-saturated.*

(ii) *For a positive integer k, define $\psi'_k \colon \sigma' \to \boldsymbol{R}$ by*

$$\psi'_k(e') = kh'(e') - \sup\{kh(e) - \langle m_0, e\rangle - \psi(e)\,;\ e \in \sigma \cap \pi^{-1}(e')\}\,.$$

If $\square$ is asymptotically ψ-saturated, then $\square'$ is asymptotically ψ'_k-saturated.

(iii) *If $h-\psi$ is upper convex, then $h'-\psi'$ is upper convex and ψ' is a log discrepancy function.*

(iv) $\psi'(e') \geq \inf_{e\in\sigma\cap\pi^{-1}(e')} \psi(e)$.

PROOF. We may assume $m_0 = 0$ after a translation of $\Box$.

(i) The inclusion $\mathring{\Box}_{h'-\psi'} \subseteq \mathring{\Box}$ is easy to see. Since $\Box$ is ψ-saturated, $M' \cap \mathring{\Box}_{h-\psi} \subset \Box$. Therefore $M' \cap \mathring{\Box}_{h'-\psi'} \subseteq M' \cap \Box = \Box'$.

(ii) Note first the identity

$$(\psi'_j - \psi'_k)(e') = (j-k) \sup_{\pi(e)=e'} h(e) + \sup_{\pi(e)=e'} (kh-\psi)(e) - \sup_{\pi(e)=e'} (jh-\psi)(e)\,,$$

which implies $\psi'_k \leq \psi'_j$ for $k \leq j$. By assumption, there exists a positive integer I such that jh is ψ-saturated for every $I\,|\,j$. Fix $k \geq 1$ and let j be a common multiple of I and k. By (i), jh' is ψ'_j-saturated. Since $\psi'_j \geq \psi'_k$, we infer that jh' is also ψ'_k-saturated.

(iii) The upper convexity of $h'-\psi'$ follows from the upper convexity of $h-\psi$ and the formula

$$(h'-\psi')(e') = \sup_{\pi(e)=e'} (h-\psi)(e)\,.$$

In particular, ψ' is a continuous function, being the diference of the continuous functions h' and $h'-\psi'$. Furthermore, ψ' is positively homogenous by its definition. Let $0 \neq e' \in \sigma'$. The restriction $\psi|_{\pi^{-1}(e')}$ is strictly positive, continuous and at least 1 outside some bounded subset, by Lemma 2.3. Therefore $\inf_{\pi(e)=e'} \psi(e) > 0$. We conclude from (iv) that $\psi'(e') > 0$.

(iv) This is a direct consequence of the definitions of h' and ψ'.

3. Rational polyhedral criterion.

THEOREM 3.1. *Let $\sigma \subseteq N_{\boldsymbol{R}}$ be a rational convex polyhedral cone, $\Box \in \mathcal{C}(\sigma^\vee)$ and $\psi : \sigma \to \boldsymbol{R}$ a log discrepancy function such that $\Box$ is asymptotically ψ-saturated.*

Then for every $e_1 \in \sigma \setminus 0$, there exist $m \in M_{\boldsymbol{Q}} \cap \Box$ and a rational convex polyhedral cone $\sigma_1 \subset \sigma$, with the following properties:

(i) $e_1 \in \mathrm{relint}(\sigma_1)$.

(ii) $h_\Box(e) = \langle m, e\rangle$ *for* $e \in \sigma_1$.

PROOF. Choose norms $\|\cdot\|$ on $N_{\boldsymbol{R}}$ and $M_{\boldsymbol{R}}$, defined as the maximum of the absolute values of the components with respect to some basis of N and its dual basis in M, respectively. Let $S(\sigma) = \{e \in \sigma\ ;\ \|e\| = 1\}$ and define the positive real number $\varepsilon(\psi)$ by the formula

$$-\varepsilon(\psi)^{-1} = \min\{\langle m, e\rangle/\psi(e)\ ;\ \|m\| = 1, e \in S(\sigma)\}\,.$$

The restriction of ψ to $S(\sigma)$ is a positive, continuous function, and hence $\varepsilon(\psi)$ is well defined. In particular,

$$\langle m, e\rangle + \psi(e) > 0 \quad \text{for } 0 \neq e \in \sigma,\ \ \|m\| < \varepsilon(\psi)\,.$$

Denote by $h \in \mathcal{S}(\sigma)$ the support function of $\Box$. We will prove the theorem in two steps.

(1) There exists $m \in M_{\boldsymbol{Q}} \cap \Box_h$ such that $\langle m, e_1\rangle = h(e_1)$.

Indeed, let τ be the unique face of σ which contains e_1 in its relative interior. We may find orthogonal decompositions

$$N = N' \oplus N'' , \quad M = M' \oplus M'' ,$$

where $N' = N \cap (\tau - \tau)$, $M' = M \cap \tau^{\perp}$ and M', N' and M'', N'' are dual lattices, respectively. If $N'' \neq 0$, let σ'' be the image of σ under the projection map $N_{\boldsymbol{R}} \to N''_{\boldsymbol{R}}$. Since $\tau \supseteq \sigma \cap (-\sigma)$, we infer that σ'' is a strongly rational convex polyhedral cone.

Since h is the support function of the non-empty convex set $\Box_h$, there exists a sequence of points $p_k \in \Box_h$ such that $\lim_{k\to\infty}\langle p_k, e_1\rangle = h(e_1)$. If we decompose $p_k = p'_k + p''_k$, we claim that p'_k belongs to a bounded set of $M'_{\boldsymbol{R}}$. Indeed, assume by contradiction that $\lim_{k\to\infty} \|p'_k\| = +\infty$. By the usual compactness argument, we may assume that there exists $p' \in M'_{\boldsymbol{R}}$ such that $\lim_{k\to\infty} p'_k/\|p'_k\| = p'$. For every $e \in \tau$, we have

$$\langle p'_k/\|p'_k\|, e\rangle = \langle p_k/\|p'_k\|, e\rangle \geq h(e)/\|p'_k\| .$$

Letting k tend to infinity, we obtain $\langle p', e\rangle \geq 0$. Furthermore,

$$\lim_{k\to\infty} \langle p'_k, e_1\rangle = \lim_{k\to\infty} \langle p_k, e_1\rangle = h(e_1) ,$$

so a similar argument gives $\langle p', e_1\rangle = 0$. Therefore $0 \neq p' \in \tau^{\vee} \cap e_1^{\perp}$. Since e_1 belongs to the relative interior of τ, we infer that $p' \in \tau^{\perp}$. This implies $p' = 0$, a contradiction. Therefore the claim holds.

We may replace $(p_k)_k$ by a subsequence so that there exists $p' \in M'_{\boldsymbol{R}}$ with $\lim_{k\to\infty} p'_k = p'$ and $\langle p', e_1\rangle = h(e_1)$. By Theorem 1.6, there exists a positive multiple j of I such the following system has a solution:

$$\begin{cases} p'_j \in M' , \\ \langle jp', e_1\rangle - \psi(e_1) < \langle p'_j, e_1\rangle \leq \langle jp', e_1\rangle , \\ \|p'_j - jp'\| < \varepsilon(\psi)/2 . \end{cases}$$

Choose k large enough so that $j\|p' - p'_k\| < \varepsilon(\psi)/2$. Since σ'' is a strongly rational convex polyhedral cone, the following system has a solution

$$\begin{cases} p''_j \in M'', \\ p''_j \in jp''_k + \sigma''^{\vee} . \end{cases}$$

Set $p_j = p'_j + p''_j \in M$. The following holds for $e \in S(\sigma)$:

$$\begin{aligned} \langle p_j, e\rangle - jh(e) + \psi(e) &= \langle p_j - jp_k, e\rangle + \psi(e) + j(\langle p_k, e\rangle - h(e)) \\ &\geq \langle p_j - jp_k, e\rangle + \psi(e) \\ &\geq \langle p'_j - jp'_k, e\rangle + \psi(e) > 0 , \end{aligned}$$

where the last inequality follows from

$$\|p'_j - jp'_k\| \leq \|p'_j - jp'\| + \|jp' - jp'_k\| < \varepsilon(\psi) .$$

Since jh is ψ-saturated, we obtain $p_j \in \square_{jh}$. In particular, $\langle p_j, e_1\rangle \geq jh(e_1)$. The opposite inequality holds from construction, and hence $\langle p_j, e_1\rangle = jh(e_1)$. We obtain $m := p_j/j \in M_{\boldsymbol{Q}} \cap \square_h$ with $\langle m, e_1\rangle = h(e_1)$.

(2) Since m is rational, we may replace $\square$ by $\square - m$. Equivalently, we replace h by $h - m$. Thus we may assume that $0 \in \square_h$ and $e_1 \in \sigma_0 := \{e \in \sigma\,;\, h(e) = 0\}$. We may decompose $N = N' \oplus N''$, with dual decomposition $M = M' \oplus M''$, such that $e_1 = e'_1 + e''_1$, $e'_1 \in N'_{\boldsymbol{Q}}$, $e''_1 \in N''_{\boldsymbol{R}}$ and $\{m'' \in M''\,;\, \langle m'', e''_1\rangle \in \boldsymbol{Q}\} = \{0\}$. If $e''_1 = 0$, then $e_1 \in N_{\boldsymbol{Q}}$ and the theorem holds for $\sigma_1 = \boldsymbol{R}_{\geq 0} \cdot e_1$ and $m = 0$.

(2a) Assume that $e''_1 \neq 0$. We claim that the following equality holds

$$\sigma_0^\vee \cap e_1^\perp = \sigma_0^\vee \cap M'_{\boldsymbol{R}} \cap {e'_1}^\perp .$$

We only have to prove the direct inclusion. Fix $m \in \sigma_0^\vee \cap e_1^\perp$. We have to show that $m'' = 0$, where $m = m' + m''$ is the decomposition in $M'_{\boldsymbol{R}} \oplus M''_{\boldsymbol{R}}$. Assume by contradiction that $m'' \neq 0$. Since $\langle m, e_1\rangle \in \boldsymbol{Q}$, we infer by Lemma 1.7 and Theorem 1.6 that there exist a positive integer k and $m_1 \in M$ such that $-\psi(e_1) < \langle m_1 - km, e_1\rangle < 0$ and $\|m_1 - km\| < \varepsilon(\psi)$. Since $\langle m, e_1\rangle = 0$, we obtain

$$-\psi(e_1) < \langle m_1, e_1\rangle < 0\,, \quad \|m_1 - km\| < \varepsilon(\psi)\,.$$

We consider the set $C = \{e \in S(\sigma)\,;\, \langle m_1, e\rangle + \psi(e) \leq 0\}$. Since $0 \in \square_h$, we have $h \leq 0$. If the set C is empty, then

$$\langle m_1, e\rangle - jh(e) + \psi(e) \geq \langle m_1, e\rangle + \psi(e) > 0 \quad \text{for all } e \in S(\sigma)\,.$$

Therefore $m_1 \in M \cap \overset{\circ}{\square}_{jh-\psi}$, and hence $m_1 \in \square_{jh}$ by saturation. In particular, $\langle m_1, e_1\rangle \geq 0$, which contradicts the choice of m_1. Therefore the set C is non-empty. Since ψ is continuous, C is also compact. If $C \cap \sigma_0 = \emptyset$, then there exists a positive integer j with

$$j > \sup_{e \in C} \frac{\langle m_1, e\rangle + \psi(e)}{h(e)}\,.$$

Then $m_1 \in M \cap \overset{\circ}{\square}_{jh-\psi}$, and saturation implies that $m_1 \in \square_{jh}$. Hence $\langle m_1, e_1\rangle \geq 0$, which contradicts the choice of m_1. Therefore there exists $e \in C \cap \sigma_0$. In particular,

$$\langle m, e\rangle < \frac{\langle m_1, e\rangle + \psi(e)}{k} \leq 0\,.$$

Therefore $\langle m, e\rangle < 0$, contradicting the assumption $e \in \sigma_0$, $m \in \sigma_0^\vee$.

(2b) The function h is continuous, being upper convex [6, Theorem 10.1]. Therefore σ_0 is a closed convex cone in $N_{\boldsymbol{R}}$. By duality (cf. [5, Theorem A.1]), (2a) is equivalent to

$$\sigma_0 + \boldsymbol{R} \cdot e'_1 + N''_{\boldsymbol{R}} = \sigma_0 + \boldsymbol{R} \cdot e_1\,.$$

In particular, there exists an open neighborhood U'' of e''_1 in $N''_{\boldsymbol{R}}$ such that $e'_1 + U'' \subset \sigma_0$. Since $\dim(U'') = \dim(N'')$, there exist $\bar{e}_1, \ldots, \bar{e}_{n+1} \in U'' \cap N''_{\boldsymbol{Q}}$, where $n = \dim(N'')$, and there exists $\lambda_i \in (0, 1)$ such that $\sum_{i=1}^{n+1} \lambda_i = 1$ and $e''_1 = \sum_{i=1}^{n+1} \lambda_i \bar{e}_i$. Let σ_1 be the rational polyhedral cone spanned by $e'_1 + \bar{e}_1, \ldots, e'_1 + \bar{e}_{n+1}$. It is clear that $\sigma_1 \subset \sigma$, $e_1 \in \operatorname{relint}(\sigma_1)$ and $h|_{\sigma_1} = 0$.

THEOREM 3.2. *Let $\sigma \subseteq N_{\boldsymbol{R}}$ be a rational convex polyhedral cone and let $\square \in \mathcal{C}(\sigma^\vee)$, with support function $h \in \mathcal{S}(\sigma)$. Assume that there exists a log discrepancy function $\psi : \sigma \to \boldsymbol{R}$ such that $\square$ is asymptotically ψ-saturated, and $h - \psi$ is upper convex. Then $\square$ is a rational convex polyhedral set.*

PROOF. We use induction on $\dim(N)$. If $\dim(N) = 1$, then $\square$ is either a point or an interval of the form $[a, b]$, $(-\infty, a]$ or $[a, +\infty)$. Its endpoints are rational by Theorem 3.1, and so $\square$ is a rational convex polyhedral set.

Assume now that $\dim(N) > 1$ and the theorem holds for smaller dimensional lattices N. We prove the theorem in three steps.

(1) Assume $0 \in \square$ and $\sigma_1 \subset \sigma$ is a rational convex polyhedral cone such that $h|_{\sigma_1} = 0$. Then there exists a rational polyhedral cone $\sigma_2 \subset \sigma$ such that $\mathrm{relint}(\sigma_1) \subset \mathrm{relint}(\sigma_2)$, and one of the following two properties holds:

(a) $\dim(\sigma_2) = \dim(\sigma_1) + 1$, and $h|_{\sigma_2} = 0$.

(b) $\dim(\sigma_2) = \dim(\sigma)$ and there exist finitely many rational points $m_1, \ldots, m_n \in M_{\boldsymbol{Q}} \cap \square$ such that for every $e \in \sigma_2$ there exists some i with $\langle m_i, e\rangle = h(e)$.

PROOF. Let $N' = N/(N \cap (\sigma_1 - \sigma_1))$, with dual lattice $M' = M \cap \sigma_1^\perp$. If $\dim(\sigma_1) = \dim(N)$, we are in Case (1b). Assume now $\dim(\sigma_1) < \dim(N)$, so that $0 < \dim(N') < \dim(N)$. With the notation in Lemma 2.9, we have a projection homomorphism

$$\pi_{\boldsymbol{Z}} \colon N \to N', \quad \sigma' = \pi(\sigma),$$

the support function $h' \colon \sigma' \to \boldsymbol{R}$ of $\square \cap M'_{\boldsymbol{R}}$ and the log discrepancy functions $\psi'_k \colon \sigma' \to \boldsymbol{R}$ for $k \geq 1$. By Lemma 2.9, $\square_{h'}$ is asymptotically ψ'_k-saturated and $kh' - \psi'_k$ is upper convex. The inductive assumption implies that there exists a finite set $\{m'_i\}_{i \in I} \subset M'_{\boldsymbol{Q}} \cap \square$ such that for every $e' \in \sigma'$, $h'(e') = \langle m'_i, e'\rangle$ for some $i \in I$. We distinguish two cases, depending on whether the convex set $\square_{h'} \subseteq M'_{\boldsymbol{R}}$ is maximal dimensional or not.

(a) Assume $\dim(\square_{h'}) < \dim(M')$. Equivalently, the lattice $N'' = N' \cap \square_{h'}{}^\perp$ is nonzero. Let $\psi''_k = \psi'_k|_{N''}$ and M'' the dual lattice of N''. By Theorem 2.6, $M'' \cap \mathring{\square}_{-\psi''_k} = \{0\}$. Furthermore, $kh - \psi_k$ is upper convex and $h|_{N''_{\boldsymbol{R}}}$ is linear. Hence $-\psi''_k$ is upper convex. Therefore Theorem 2.7 applies, and hence there exists $0 \neq e'_k \in N''$ such that

$$\psi''_k(e'_k) + \psi''_k(-e'_k) \leq C,$$

where C is a positive constant depending only on $\dim(N'')$. By Lemma 2.3, the e'_k's belong to a compact set, and hence we may assume that $e'_k = e'$ for infinitely many k's. Then there exist $e^+_k, e^-_k \in \sigma$ such that $\pi(e^+_k) = e'$, $\pi(e^-_k) = -e'$ and

$$k[h'(e') - h(e^+_k) + h'(-e') - h(e^-_k)] + \psi(e^+_k) + \psi(e^-_k) \leq C + 1.$$

In particular,

$$\psi(e^+_k) + \psi(e^-_k) \leq C + 1.$$

By Lemma 2.3, the sequences $(e^+_k)_k$ and $(e^-_k)_k$ belong to a compact set, so we may assume that the limits $e^- = \lim_{k\to\infty} e^-_k$, $e^+ = \lim_{k\to\infty} e^+_k$ exist. It is clear that $e^+, e^- \in \sigma$ and

$\pi(e^+) = e', \pi(e^-) = -e'$. The above inequality and the positivity of ψ implies

$$h'(e') - h(e_k^+) + h'(-e') - h(e_k^-) \le (C+1)/k\,.$$

Letting k tend to infinity, we obtain $h'(e') = h(e^+)$, $h'(-e') = h(e^-)$. Since $e' \in N''$, we have $h'(e') = h'(-e') = 0$.

We claim that we may assume that $e^+, e^- \in N_{\boldsymbol{Q}}$. Indeed, since $\pi(e^+) \in N'_{\boldsymbol{Q}}$ and σ_1 is rational, there exists $f \in \sigma_1$ such that $e^+ + f \in \sigma_1 \cap N$. Then $h(e^+ + f) \ge h(e^+) + h(f) = 0$, and hence $h(e^+ + f) = 0$. Also, $\pi(e^+ + f) = e'$, so that we may replace e^+ by $e^+ + f$. A similar argument applies to e^-.

The rational convex polyhedral cone $\sigma_2 = \sigma_1 + \boldsymbol{R}_{\ge 0} e^+ + \boldsymbol{R}_{\ge 0} e^-$ satisfies (1a).

(b) Assume $\dim(\Box_{h'}) = \dim(M')$. In this case, the ample fan $\Delta_{h'}$ of h' is a fan in N' with $|\Delta_{h'}| = \sigma'$.

(b1) For every $e' \in \Delta_{h'}(1)$, there exists $e \in \sigma \cap N$ such that $\pi(e) = e'$ and $h(e) = h'(e')$.

Indeed, since h' is rational piecewise linear and asymptotically ψ'_k-saturated, we obtain by Theorem 2.6 that $\psi'_k(e') \le 1$ for all $k \ge 1$. Therefore there exists $e_k \in \sigma$ such that $\pi(e_k) = e'$ and

$$kh'(e') - (kh - \psi)(e_k) \le 2\,.$$

In particular, we obtain $\psi(e_k) \le 2$. By Lemma 2.3, the sequence $(e_k)_k$ belongs to a bounded set, so that we may assume that the limit $e = \lim_{k \to \infty} e_k$ exists. We clearly have $\pi(e) = e'$. The positivity of ψ implies that

$$h'(e') - h(e_k) \le 2/k\,.$$

Letting k tend to infinity, we obtain $h'(e') - h(e) \le 0$, and hence $h'(e') - h(e) = 0$. The rationality of e is obtained the same way as in the proof of (a) above.

(b2) Let τ be a maximal dimensional cone of σ', spanned by $e'_1, \ldots, e'_r \in \Delta_{h'}(1)$. There exists $i \in I$ such that $h'(e') = \langle m_i, e' \rangle$ for every $e' \in \sigma'$. By (b1), there exist $e_j \in \sigma \cap N_{\boldsymbol{Q}}$ such that $\pi(e_j) = e'_j$ and $h(e_j) = h'(e'_j)$ for $1 \le j \le r$. Therefore $h(e) = \langle m_i, e' \rangle$ for every $e \in \sigma_1 + \sum_{j=1}^p \boldsymbol{R}_{\ge 0} e_i$. The cone $\sigma_1 + \sum_{j=1}^p \boldsymbol{R}_{\ge 0} e_i \subset \sigma$ has the same dimension as σ. The union of all these cones, taken after all maximal cones τ in Δ, contains a cone σ_2 satisfying (b1) with respect to $\{m'_i\}_{i \in I}$.

(2) Every non-zero point $e \in \sigma$ has an open polyhedral neighborhood on which h is rational, piecewise linear.

Indeed, fix e as above. By Theorem 3.1, there exists $m_0 \in M_{\boldsymbol{Q}} \cap \Box_h$ and there exists a rational convex polyhedral cone $\sigma_0 \subset \sigma$ such that $e \in \operatorname{relint}(\sigma_0)$ and $h(e) = \langle m_0, e \rangle$ for every $e \in \sigma_0$.

We may replace $\Box_h$ by its rational translate $\Box_h - m_0$, so that we may assume that $m_0 = 0$. In particular, $0 \in \Box_h$ and $h|_{\sigma_0} = 0$. By (1), either the claim holds, or there exists a $(\dim(\sigma_0)+1)$-dimensional rational polyhedral cone $\sigma_1 \subseteq \sigma$ such that $\operatorname{relint}(\sigma_0) \subset \operatorname{relint}(\sigma_1)$ and $h|_{\sigma_1} = 0$. By (1) again, either the claim holds, or there exists a $(\dim(\sigma_1)+2)$-dimensional cone $\sigma_2 \subseteq \sigma$ such that $\mathrm{relint}(\sigma_1) \subset \mathrm{relint}(\sigma_2)$ and $h|\sigma_2 = 0$. We repeat this procedure for σ_2 and so on. This procedure clearly stops in a finite number of steps, hence the claim holds.

(3) Fix a norm $\|\cdot\|$ on $N_{\boldsymbol{R}}$ and set $S(\sigma) = \{e \in \sigma\,;\ \|e\| = 1\}$. For each point $e \in S(\sigma)$, we consider the pair $(\sigma(e)\,;\ \{m_i(e)\}_{i\in I(e)})$ constructed in (2). We obtain an open covering

$$S(\sigma) = \bigcup_{e\in S(\sigma)} S(\sigma) \cap \mathrm{relint}(\sigma(e))\,.$$

Since $S(\sigma)$ is compact, it may be covered by the relative interiors of the cones corresponding to finitely many points $e_1, \dots, e_k$. Let K be the convex hull of the finitely many rational points $\{m_i(e_1)\}_{i\in I(e_1)} \cup \cdots \cup \{m_i(e_k)\}_{i\in I(e_k)}$. Then $\square = K + \sigma^\vee$. Therefore $\square$ is a rational convex polyhedral set.

4. Toric FGA algebras. We will prove Theorem 0.1 in this section. First, we recall the definition of asymptotic saturation of an algebra with respect to a log variety, due to Shokurov [7, Section 4.32]. Let (X, B) be a log pair with Kawamata log terminal singularities, and $\pi\colon X \to S$ a proper surjective morphism with connected fibers, where S is affine. Let D be an $\boldsymbol{R}$-divisor on X and $\mathcal{L} \subseteq \bigoplus_{i=0}^\infty \pi_*\mathcal{O}_X(iD)$ a graded $\mathcal{O}_S$-subalgebra with $\mathcal{L}_0 = \mathcal{O}_S$ and $\mathcal{L}_i \neq 0$ for some $i > 0$. For $i \geq 0$, let $\bar{\mathcal{L}}_i$ be the integral closure of $\mathcal{L}_i$ in $\boldsymbol{C}(X)$. The $\mathcal{O}_S$-algebra $\bar{\mathcal{L}} = \bigoplus_{i=0}^\infty \bar{\mathcal{L}}_i$ is called the integral closure of $\mathcal{L}$ in $\boldsymbol{C}(X)$ [7, Example 4.8 and Proposition 4.15]. For $\mathcal{L}_i \neq 0$, there exists a birational morphism $\mu_i\colon X_i \to X$ and a $\pi\circ\mu_i$-free divisor M_i such that $\bar{\mathcal{L}}_i = (\pi\circ\mu_i)_*\mathcal{O}_{X_i}(M_i)$, X_i is nonsingular and $\mathrm{Supp}(K_{X_i} - \mu_i^*(K_X+B)) \cup \mathrm{Supp}(M_i)$ is a simple normal crossings divisor. Then $\mathcal{L}$ is called *asymptotically saturated with respect to* $(X/S, B)$ if there exists a positive integer I such that the following inclusions hold:

$$(\pi\circ\mu_i)_*\mathcal{O}_{X_i}(\lceil K_{X_i} - \mu_i^*(K_X + B) + (j/i)M_i\rceil) \subseteq (\pi\circ\mu_j)_*\mathcal{O}_{X_j}(M_j) \quad \text{for all } I|i, j\,.$$

EXAMPLE 4.1. Assume that B is effective and D is $\boldsymbol{Q}$-Cartier. Then the $\mathcal{O}_S$-algebra $\bigoplus_{i=0}^\infty \pi_*\mathcal{O}_X(iD)$ is asymptotically saturated with respect to $(X/S, B)$.

For the rest of this section, we consider the toric case of the above set-up. We have $X = T_N\,\mathrm{emb}(\Delta)$, $S = T_{\bar{N}}(\bar{\sigma})$, and π corresponds to a map of fans $\varphi_{\boldsymbol{Z}}\colon (N, \Delta) \to (\bar{N}, \bar{\sigma})$ such that $|\Delta| = \varphi^{-1}(\bar{\sigma})$ is a rational convex set. Let $B = \sum_{e\in\Delta(1)} b_e V(e)$, where $\Delta(1)$ is the set of primitive vectors of the one dimensional cones of Δ. The log canonical divisor $K_X + B$ is represented by a function $\psi\colon |\Delta| \to \boldsymbol{R}$ such that ψ is Δ-linear and $\psi(e) = 1 - b_e$ for every $e \in \Delta(1)$. Since (X, B) has Kawamata log terminal singularities, ψ is a log discrepancy function. Let i be a positive multiple of I. Since $\mathcal{L}$ is torus invariant, there exist $m_{i,1}, \dots, m_{i,n_i} \in M$ such that $\chi^{m_{i,1}}, \dots, \chi^{m_{i,n_i}}$ generate the $\mathcal{O}_S$-module $\mathcal{L}_i$. Define $h_i\colon |\Delta| \to \boldsymbol{R}$ by $h_i(e) = \min_{1\leq j\leq n}\langle m_{i,j}, e\rangle$. The support function h_i is independent of the choice of generators, and $\bar{\mathcal{L}}_i = \bigoplus_{m\in M\cap\square_{h_i}} \boldsymbol{C}\cdot\chi^m$.

LEMMA 4.2. *The asymptotic saturation of $\mathcal{L}$ with respect to $(X/S, B)$ means that $M \cap \overset{\circ}{\square}_{(j/i)h_i - \psi} \subset \square_{h_j}$ for all $I|i, j$.*

PROOF. Choose a refinement Δ_i of the fan Δ so that Δ_i is a simple fan and h_i is Δ_i-linear. This corresponds to a toric resolution of singularities $\mu_i\colon X_i = T_N \operatorname{emb}(\Delta_i) \to X$ such that $M_i = \sum_{e\in\Delta_i(1)} -h_i(e)V(e)$ is a $\pi\circ\mu_i$-free divisor. Since X_i is nonsingular, the union of its invariant prime divisors $\sum_{e\in\Delta_i(1)} V(e)$ has simple normal crosssings. In this set-up, the asymptotic saturation property of $\mathcal{L}$ with respect to $(X/S, B)$ means that

$$H^0(X_i, \lceil K_{X_i} - \mu_i^*(K+B) + (j/i)M_i\rceil) \subseteq H^0(X_j, M_j) \quad \text{for } I|i, j\,.$$

Let $m \in M$. Then $\chi^m \in H^0(X_i, \lceil K_{X_i} - \mu_i^*(K+B) + (j/i)M_i\rceil)$ if and only if $\langle m, e\rangle + \lceil -1 + \psi(e) - (j/i)h_i(e)\rceil \geq 0$ for every $e \in \Delta_i(1)$. Since $\langle m, e\rangle \in \boldsymbol{Z}$, this is equivalent to $\langle m, e\rangle > (j/i)h_i(e) - \psi(e)$ for every $e \in \Delta_i(1)$. Since ψ and h_i are Δ_i-linear, the latter is equivalent to $\langle m, e\rangle > (j/i)h_i(e) - \psi(e)$ for every $e \in |\Delta| \setminus 0$.

On the other hand, $\chi^m \in H^0(X_j, M_j)$ if and only if $m \in \square_{h_j}$. This proves the claim.

LEMMA 4.3. *The function $h = \lim_{i\to\infty} h_i/i\colon |\Delta| \to \boldsymbol{R}$ is a well-defined positively homogeneous, upper convex function.*

PROOF. We can write $D = \sum_{e\in\Delta(1)} d_e V(e)$. Let $\tilde{h}\colon |\Delta| \to \boldsymbol{R}$ be the support function of the convex set $\{m \in M_{\boldsymbol{R}}\,;\ \langle m, e\rangle \geq -d_e \quad \text{for all } e \in \Delta(1)\}$. Since $\mathcal{L}_i \subseteq H^0(X, iD)$, we obtain $h_i \geq i\tilde{h}$. On the other hand, the property $\mathcal{L}_i \cdot \mathcal{L}_j \subseteq \mathcal{L}_{i+j}$ implies $h_i + h_j \geq h_{i+j}$. Then it is easy to see that for every $e \in |\Delta|$, the sequence $(1/i)h_i(e)$ is bounded from below and converges to its infimum. Being a limit of positively homogeneous upper convex functions, h satisfies these two properties too. Note that $h_i \geq ih$ for every i.

LEMMA 4.4. *The asymptotic saturation of $\mathcal{L}$ with respect to $(X/S, B)$ is equivalent to*

$$M \cap \overset{\circ}{\square}_{jh-\psi} \subset \square_{h_j} \quad \textit{for } I|j\,.$$

PROOF. Fix $I|j$, choose a norm $\|\cdot\|$ on $N_{\boldsymbol{R}}$ and set $S(|\Delta|) = \{e \in |\Delta|\,;\ \|e\| = 1\}$. Let $m \in M \cap \overset{\circ}{\square}_{jh-\psi}$. This means that the function $f\colon S(|\Delta|) \to \boldsymbol{R}$, $f(e) = \langle m, e\rangle - jh(e) + \psi(e)$ takes only positive values. The functions $(1/i)h_i$ are upper convex, so that they converge uniformly to h on the compact set $S(|\Delta|)$, by Theorem 1.1. Therefore there exists some i such that the function $f - ((j/i)h_i - jh)|_{S(|\Delta|)}$ takes only positive values. This means that $m \in M \cap \overset{\circ}{\square}_{(j/i)h_i-\psi}$. By Lemma 4.2, we obtain $m \in \square_{h_j}$.

The converse is clear by Lemma 4.2, since $h_i \geq ih$.

PROOF OF THEOREM 0.1. The statement is local over the base, so we may assume that S is affine. We use the above notation.

The function $-\psi$ is upper convex since $-(K_X + B)$ is nef. Therefore $h - \psi$ is upper convex. By Lemma 4.4 and the inclusions $\square_{h_j} \subset \square_{jh}$, we infer that $\square_h$ is asymptotically saturated with respect to ψ.

(1) The hypothesis of Theorem 3.2 is satisfied, and so $\square_h$ is a rational polyhedral set. In particular, there exists a positive integer $I|n$ such that $\square_{nh}$ is the convex hull of its lattice points. We have $M \cap \square_{nh} \subseteq M \cap \overset{\circ}{\square}_{nh-\psi} \subset \square_{h_n}$, and hence $\square_{nh}$, the convex hull of $M \cap \square_{nh}$, is included in $\square_{h_n}$. Therefore $h_n \geq nh$. The opposite inclusion holds by construction, and

hence $nh = h_n$. Since $kh_n \geq h_{kn} \geq knh$ for $k \geq 1$, we obtain $h_{kn} = kh_n$ for every $k \geq 1$. This means that

$$\bigoplus_{k=0}^{\infty} \bar{\mathcal{L}}_{kn} = \bigoplus_{k=0}^{\infty} (\pi \circ \mu_n)_* \mathcal{O}_{X_n}(kM_n)\,.$$

The right hand side is finitely generated, since M_n is a $\pi \circ \mu_n$-free divisor. Therefore $\bigoplus_{k=0}^{\infty} \bar{\mathcal{L}}_{kn}$ is finitely generated, and so $\bar{\mathcal{L}}$ is finitely generated. The extension $\mathcal{L} \subseteq \bar{\mathcal{L}}$ is integral, so we conclude that $\mathcal{L}$ is finitely generated.

(2) By (1), $\square_h \in \mathcal{C}(|\Delta|^\vee)$ is a rational convex polyhedral set and we have an S-isomorphism

$$\mathrm{Proj}(\bar{\mathcal{L}}) \simeq \mathrm{Proj}\left(\bigoplus_{i=0}^{\infty} \bigoplus_{m \in M \cap i\square_h} \boldsymbol{C} \cdot \chi^m\right).$$

The right hand side is the torus embedding of the ample fan $\Delta_{\square_h}$. Since $\square_h$ is asymptotically ψ-saturated and $h - \psi$ is upper convex, Theorem 2.8 applies. Therefore $\Delta_{\square_h}$ belongs to a finite set of fans associated to $(X/S, B)$. This proves (2).

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RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES
KYOTO UNIVERSITY
KYOTO 606–8502
JAPAN

E-mail address: ambro@kurims.kyoto-u.ac.jp