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COMPLETION OF REAL FANS AND ZARISKI-RIEMANN SPACES

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Abstract. Given a real fan in a real space consisting of real convex polyhedral cones, we construct a complete real fan which contains the fan, by two completely different methods. The first one is purely combinatorial and a proof of a related version was sketched earlier by Ewald. The second one is based on Nagata's method of imbedding an abstract variety into a complete variety. For the second method, we introduce the theory of Zariski-Riemann space of a fan.

Introduction. A fan in a real space is defined as a cell complex consisting of polyhedral cones with the apex at the origin. A finite fan Σ is said to be complete if the union of cones in Σ is the whole space. The theory of toric varieties says that, to each finite fan consisting of rational polyhedral cones, is associated a toric variety, and the fan is complete if and only if the toric variety is complete (see, for example, Ewald [E], Fulton [F2], Oda [O1]). Nagata's compactification theorem says that any algebraic variety can be embedded in a complete algebraic variety [N1, Theorem 4.3]. This theorem was generalized for normal algebraic varieties with algebraic group actions by Sumihiro [S1], i.e., the equivariant completion theorem. By using Sumihiro's theorem, we can complete a rational fan Σ as follows Let X be the toric variety associated to Σ. Since X is a normal variety with torus action, there exists an equivariant completion $\bar{X}$. Since $\bar{X}$ is a complete toric variety, it corresponds to a complete fan $\bar{\Sigma}$. Then $\bar{\Sigma}$ is a completion of Σ.

Since it is a quite simple problem on convex polyhedral sets in a real space, we would like to avoid this roundabout proof. In this paper, we give two different direct proofs which are valid for not necessarily rational fans. The first proof given in Section 1 is purely combinatorial and was sketched in [E, Theorem 2.8] in the case of a rational fan.

The second one is done by using Nagata's method applied for fans. In Nagata's proof, the Zariski-Riemann space, i.e., the topological space of all valuation rings of the function field plays an important role. The Zariski-Riemann space was introduced originally by Zariski [Z1], [Z2] for the theory of local uniformization of algebraic varieties. In Section 2, we define the Zariski-Riemann spaces for rational fans. We discuss on blowups of not necessarily rational fans in Section 3. Using the results of these sections, the existence of the completion is first proved for rational fans in Section 4. In the last section, the definition of the Zariski-Riemann spaces is generalized for $\boldsymbol{k}$-fans for any subfield $\boldsymbol{k}$ of $\boldsymbol{R}$. Theorem 5.4 claims that any

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finite $\boldsymbol{k}$-fan is embedded in a complete $\boldsymbol{k}$-fan. The theorem applied for $\boldsymbol{k} = \boldsymbol{R}$ is the second proof of the completion theorem for real fans.

1. Combinatorial proof.

NOTATION. Given a set E of vectors in $\boldsymbol{R}^n$ we denote by $\operatorname{pos} E$ the set of all linear combinations of elements of E with non-negative coefficients, and call it the *positive hull* of E. If E is finite, we say $\operatorname{pos} E$ is a (real) polyhedral cone σ. The dimension $\dim \sigma$ of σ is defined to be the dimension of the linear hull $\operatorname{lin} \sigma$ of σ. If σ has linearly independent generators, we call it a *simplicial* cone. If σ has rational generators, we call σ *rational*. By a face of σ we mean an intersection $\sigma \cap H$, where H is a (linear) hyperplane such that σ is totally contained in one of the two closed half-spaces bounded by H. If $\{0\}$ is a face of σ, we call it the *apex* of σ.

A collection Σ of (real) polyhedral cones with apex $\{0\}$ is said to be a (real) fan if it is a cell complex, that is, (i) each face of a cone of Σ is also in Σ, (ii) the intersection of two cones of Σ is a common face of the cones. We call Σ *rational* if all cones are rational. If Σ and Σ' are fans in $\boldsymbol{R}^n$ and $\Sigma \subset \Sigma'$, we say Σ is a *subfan* of Σ'. By the *star* $\operatorname{st}(\sigma, \Sigma)$ of a cone $\sigma \in \Sigma$ in Σ we mean the set of all cones $\tau \in \Sigma$ such that $\sigma \subset \tau$. The *support* $|\Sigma|$ of Σ is the union of the cones in Σ. If Σ is finite and $|\Sigma|$ equals $\boldsymbol{R}^n$, we say Σ is *complete*, and a completion of any of its subfans. If a finite fan Σ is not complete, we call the collection of cones of Σ which lie in the (topological) boundary of $|\Sigma|$ the *boundary* $\operatorname{bd} \Sigma$ of Σ. Clearly, $\operatorname{bd} \Sigma$ is again a fan. Given $\varepsilon > 0$, the *ε-neighborhood* of Σ is defined as the union of all 1-cones $\operatorname{pos}\{a\}$ where a is a unit vector representing a point of distance less than ε from $|\Sigma|$.

As a specific type of cones we consider the following. If ρ is a 1-dimensional cone not contained in the linear hull of a cone σ, we call $\sigma.\rho = \operatorname{pos}(\sigma \cup \rho)$ a *pyramid* with apex ρ over the basis σ. Clearly, $\dim(\sigma.\rho) = 1 + \dim \sigma$. A pyramid over a pyramid is said to be a *twofold pyramid* or a *2-pyramid*, and a pyramid over a $(k-1)$-pyramid inductively a *k-fold pyramid* or a *k-pyramid*. σ is considered a 0-fold pyramid over itself. k-fold pyramids can be written as

$$\sigma.\rho_1.\cdots.\rho_k = \sigma.\tau ,$$

where $\tau = \rho_1.\cdots.\rho_k$ is a simplicial cone. If P is a polytope or a polyhedral set and $p^1, \dots, p^k$ are vectors (representing points) such that $\sigma = \operatorname{pos} P$ and $\rho_i = \operatorname{pos} p^i$, $i = 1, \dots, k$, then $\sigma.\tau = \operatorname{pos}(P.p^1.\cdots.p^k)$, where $P.p^1.\cdots.p^k$ is an ordinary k-fold pyramid (unbounded if P is not a polytope).

1.1. Main result. All fans in this section are assumed to be finite.

THEOREM 1.1. *Every real fan Σ can be completed.*

In [E, Theorem 2.8], we have sketched the proof of this theorem for rational fans. In this section, we present an explicit proof in the general case. The second proof of this theorem is given in Section 5 (cf. Theorem 5.4).

PROOF OF THEOREM 1.1. For the purpose of our proof it is useful to show a somewhat stronger version of Theorem 1.1:

THEOREM 1.2. *Given a fan Σ and an $\varepsilon > 0$, there exists a fan Σ_0 and a complete fan Σ' such that the following are satisfied*:

(1) $\Sigma \subset \Sigma_0 \subset \Sigma'$.

(2) $\Sigma_0 \setminus \Sigma$ *consists of the multifold pyramids joining the cones of Σ to simplicial cones in* bd Σ_0, *and of the faces of such pyramids.*

(3) *If $\sigma \in \Sigma'$ intersects the cells of Σ only in* 0, *then σ is a simplicial cone.*

(4) $|\Sigma| \setminus \{0\}$ *lies in the interior of* $|\Sigma_0|$.

(5) $|\Sigma_0|$ *is contained in the ε-neighborhood of Σ.*

We apply induction on n. For $n = 1$, either $\Sigma = \{\{0\}\}$ or $\Sigma = \{\{0\}, \mathrm{pos}\{1\}\}$ or $\Sigma = \{\{0\}, \mathrm{pos}\{-1\}\}$ or $\Sigma = \{\{0\}, \mathrm{pos}\{1\}, \mathrm{pos}\{-1\}\}$, the last fan being the completion of all the others. If $n = 2$, let S be the unit circle. We may assume $\{0\}$ not to be the only cone of Σ, any complete fan being a completion of Σ in this case. So Σ splits S into finitely many (closed) circular arcs. Let C be one of these arcs and p, q its end points. We choose points p', q' in the relative interior of C so that the distances between p, p' and q, q', respectively, are less than $\varepsilon/2$, and that the cones $\mathrm{pos}\{p, p'\}$, $\mathrm{pos}\{q, q'\}$ intersect only in $\{0\}$. We extend Σ by adding $\mathrm{pos}\{p, p'\}$, $\mathrm{pos}\{p'\}$, $\mathrm{pos}\{q, q'\}$, $\mathrm{pos}\{q'\}$. Doing this for all arcs (and assuming ε small enough to begin with), we obtain a fan Σ_0. Now $S \setminus |\Sigma_0|$ consists of finitely many arcs. If one of the arcs has length π or more, we split it into two arcs of length less than π. We add to Σ the closed angular regions determined by the arcs and their boundary 1-cones and obtain a complete fan Σ'. It is readily verified that Σ, Σ_0 and Σ' satisfy (1) through (5).

Let $n > 2$. Again we may assume that $\{0\}$ is not the only cone of Σ. So let ρ be a 1-cone of Σ for which $\mathrm{st}(\rho, \Sigma)$ is not complete (if none of such exists, Σ is already complete). Let H be the (affine) tangent hyperplane of the unit sphere S at $a = \rho \cap S$. Then H intersects the cones of Σ either not at all or in convex polytopes (if bounded) or so-called polyhedral sets (if unbounded). In Figure 1 we illustrate the case $n = 3$ (heavy lines and hatched regions). Let d be the smallest distance that a has from the cones of $\Sigma \setminus \mathrm{st}(\rho, \Sigma)$. Clearly $d > 0$. We consider the $(n-2)$-sphere S_a of radius at most $d/2$ and center a in H. For the moment we regard a as the origin of the $(n-1)$-space H. By pos_a let us denote the positive hull with respect to this origin.

$$\Sigma_a := \{\mathrm{pos}_a(H \cap \sigma)\ ;\ \sigma \in \mathrm{st}(\rho, \Sigma)\}$$

is then a fan in H. It represents the quotient fan Σ/ρ (up to a translation; compare Ewald [E, p. 81, Definition 3.3]).

Now we apply the inductive assumption to Σ_a and obtain for any $\varepsilon_a > 0$ fans $\Sigma_{a,0}$, Σ'_a satisfying (1)–(5) (with terms indexed by a, and $d/2$ considered as unit length). We wish to construct an extension of Σ by using $\Sigma_{a,0}$, Σ'_a. Since the latter fans collide, in general, with cones of $\Sigma \setminus \mathrm{st}(\rho, \Sigma)$, we first construct a map which assigns to each cone of Σ'_a a polytope or polyhedral set contained in the cone:

I. Let $\sigma_a \in \Sigma_a$, hence $\sigma_a = \mathrm{pos}_a(H \cap \sigma)$ for some $\sigma \in \mathrm{st}(\rho, \Sigma)$. Then we assign

$$\sigma_a \mapsto \phi_a(\sigma_a) = H \cap \sigma\ .$$

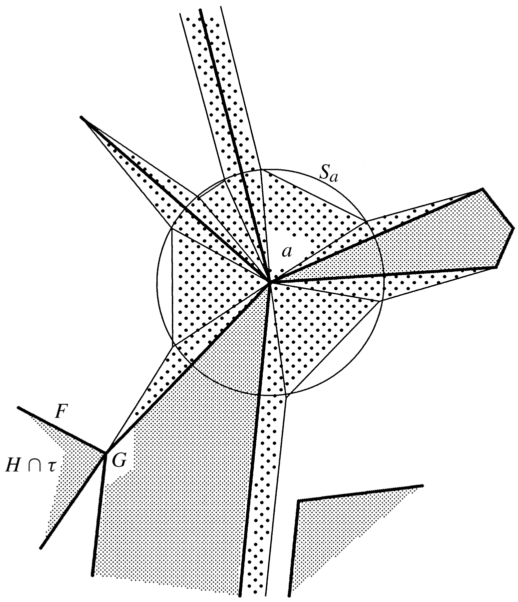

FIGURE 1.

II. Let $\sigma_a \in \Sigma_{a,0} \setminus \Sigma_a$ be a multifold pyramid $\sigma_{a,o}.\rho_{a,1}.\cdots.\rho_{a,k}$, where $\sigma_{a,o} \in \Sigma_a$, $\sigma_{a,o} = \mathrm{pos}_a(H \cap \sigma_o)$, and $\rho_{a,i} = \mathrm{pos}_a\{p_{a,i}\}$, $p_{a,i} \in S_a$, $i = 1, \ldots, k$. We assign

$$\sigma_a \mapsto \phi_a(\sigma_a) = \mathrm{clconv}((H \cap \sigma_o) \cup \{p_{a,1}\} \cup \cdots \cup \{p_{a,k}\}),$$

where "cl" means the "topological closure" (needed if $H \cap \sigma_o$ is unbounded).

III. Let $\sigma_a \in \Sigma_{a,0} \setminus \Sigma_a$ and $\sigma_a \cap |\Sigma_a| = \{a\}$ or $\sigma_a \in \Sigma'_a \setminus \Sigma_{a,0}$. Then $\sigma_a = \mathrm{pos}_a\{p^1, \ldots, p^r\}$, where $p^1, \ldots, p^r$ are in S_a and linearly independent (with respect to H as a linear space). We set

$$\sigma_a \mapsto \phi_a(\sigma_a) = \mathrm{conv}\{a, p^1, \ldots, p^r\} \quad \text{(an } (r+1)\text{-simplex)}.$$

In Figure 1 the dotted regions illustrate the $\phi_a(\sigma_a)$ of type II or III.

IV. $\phi_a(\{a\}) = \{a\}$.

LEMMA 1.3. *ϕ_a as defined by* I–IV *maps Σ'_a onto a cell complex A consisting of polytopes and polyhedral sets in H. It has the following properties*:

(a) *ϕ_a is bijective and preserves inclusions.*

(b) $\mathrm{pos}_a\, \phi_a(\sigma_a) = \sigma_a$ *for all* $\sigma_a \in \Sigma'_a$.

(c) $\varepsilon_a > 0$ *can be chosen so that for any* $\tau \in \Sigma \setminus \mathrm{st}(\rho, \Sigma)$ *we have*: $\tau \cap |A|$ *is empty or contained in* $\phi_a(\sigma_a)$ *for some* $\sigma_a \in \Sigma_a$.

PROOF. (a) and (b) readily follow from the definitions I–IV. In order to show (c) we recall an elementary fact from convex polytope theory: (*) If F, G are (closed) polytopes or polyhedral sets in $\boldsymbol{R}^n$ such that $F \cap G$ is empty, then F and G have positive distance, that is, there exists a $d > 0$ such that each point of F has at least distance d from G. Suppose (c) is false. Then there exists a $\tau \in \Sigma \setminus \mathrm{st}(\rho, \Sigma)$ and an x in $\tau \cap |A|$ but not in $\phi_a(\sigma_a)$ for any $\sigma_a \in \Sigma_a$. Since the $\phi_a(\sigma_a)$ of type III are contained in the ball B_a, x cannot lie in one of such. So x lies in a k-fold pyramid $\phi_a(\sigma_{a,0}.p^1.\cdots.p^k)$ over $\phi_a(\sigma_{a,0})$, where $\sigma_{a,0} \in \Sigma_a$ but x does not lie in the basis $\phi_a(\sigma_{a,0})$ of the pyramid.

Let F be the smallest face of $\tau \cap H$ which contains x (in its relative interior). If $F \cap \phi_a(\sigma_{a,0}) = \emptyset$, then F has, by (*), positive distance from $\phi_a(\sigma_{a,0})$, and we choose ε_a to be at most half this distance. Then x cannot lie in the above pyramid. So let $F \cap \phi_a(\sigma_{a,0})$ be nonempty. Since F and $\phi_a(\sigma_{a,0})$ are intersections of cones of Σ and H, $F \cap \phi_a(\sigma_{a,0})$ is a common face G of F and $\phi_a(\sigma_{a,0})$. Let $\dim G = m$. Then G, a, and x span an $(m+2)$-dimensional affine space in which the hyperplanes spanned by G, a and G, x, respectively, have an angle $\alpha > 0$ (see the illustration in Figure 1). Since x lies in the k-fold pyramid $\phi_a(\sigma_{a,0}).p^1.\cdots.p^k$ over $\phi_a(\sigma_{a,0})$, the angle α could be made arbitrarily small by choosing ε_a small enough, a contradiction to $\alpha > 0$ being given. So let ε_a be chosen appropriately. Since our arguments apply to finitely many faces, we may select the smallest ε_a among those which occur as a common bound. This proves Lemma 1.3. □

Now we define the following map ψ_a on A:

I′. If $\sigma \in \Sigma$ and $H \cap \sigma \in A$, we assign

$$H \cap \sigma \mapsto \psi_a(H \cap \sigma) = \sigma .$$

II′. If $\phi_a(\sigma_a) = \mathrm{clconv}((H \cap \sigma_o) \cup \{p_{a,1}\} \cup \cdots \cup \{p_{a,k}\})$ according to II, we consider the point $p_{a,i}$ as vector $q_i = a + p_{a,i}$ in $\boldsymbol{R}^n$ and assign for $\rho_i = \mathrm{pos}\{q_i\}$, $i = 1, \ldots, k$

$$\phi_a(\sigma_a) \mapsto \psi_a(\phi_a(\sigma_a)) = \sigma_o.\rho_1.\cdots.\rho_k .$$

III′. If $\phi_a(\sigma_a) = \mathrm{conv}\{a, p^1, \ldots, p^r\}$ according to III, we consider again the points p^i as vectors $q^i = a + p^i$ in $\boldsymbol{R}^n$ and assign for $\rho^i = \mathrm{pos}\{q^i\}$, $i = 1, \ldots, r$ (and $\rho = \mathrm{pos}\, a$)

$$\phi_a(\sigma_a) \mapsto \psi_a(\phi_a(\sigma_a)) = \rho.\rho^1.\cdots.\rho^r .$$

IV′.

$$\{a\} \mapsto \psi(\{a\}) = \rho .$$

LEMMA 1.4. *If we add to Σ all cones $\psi_a(\phi_a(\sigma_a))$ and their faces for $\sigma_a \in \Sigma'_a \setminus \Sigma_a$, then we obtain a set $\Sigma^{(1)}$ which is a fan, provided ε_a is chosen small enough.*

PROOF. By construction, ψ_a clearly is bijective. We must show that for sufficiently small ε_a two cones τ, τ' of $\Sigma^{(1)}$ intersect in a common face of τ, τ' which belongs to $\Sigma^{(1)}$. Since $\Sigma \subset \Sigma^{(1)}$, this is true if τ, τ' both lie in Σ. Let $\tau \in \Sigma$, $\tau' = \sigma_o.\rho_1.\cdots.\rho_k$ of type II′. If $\sigma_o \cap H$ is bounded, we have $\tau' = \mathrm{pos}((\sigma_o \cap H) \cup \{q_1, \ldots, q_k\})$, and by Lemma 1.3, (c) $\tau' \cap |\Sigma| = \sigma_o$ for sufficiently small ε_a, so $\tau \cap \tau' = \tau \cap \sigma_o$ is a common face of τ and τ'.

If $\sigma_o \cap H$ is unbounded, we consider an affine hyperplane H' which does not contain 0 such that $\sigma_o \cap H'$ is nonempty and bounded (H' exists since 0 is the apex of σ_o). For sufficiently small ε_a the cones $\rho_1, \ldots, \rho_k$ intersect H' in points $q'_1, \ldots, q'_k$, respectively, so that $\tau' = \mathrm{pos}((\sigma_o \cap H') \cup \{q_1', \dots, q_k'\})$. We may again apply the arguments of Lemma 1.3, (c) so as to obtain $\tau' \cap |\Sigma| = \sigma_o$ for sufficiently small ε_a.

For all the other choices of τ, τ' analogous arguments apply. Since only finitely many restrictions are imposed on ε_a, Lemma 1.4 follows. □

Now we apply to $\Sigma^{(1)}$ the same procedure of extension as we applied to Σ, choosing as ρ a 1-cone ρ_1 of the "old" fan Σ (if there is a ρ_1 other than ρ for which Σ/ρ_1 is incomplete). We denote the new fan by $\Sigma^{(2)}$. Continuing in this way, we find after a finite number i of steps a fan $\Sigma^{(i)} =: \Sigma^0$ such that Σ^0/ρ_j is complete for all 1-cones $\rho_j = \rho_1, \dots, \rho_i$. We assert:

LEMMA 1.5. *Given $\varepsilon > 0$, Σ^0 can be chosen so that* (2), (4) *and* (5) *in Theorem* 1.2 *are satisfied for Σ^0 instead of Σ_0.*

PROOF. First we show that (2) in Theorem 1.2 holds for $\Sigma^{(1)}$ instead of Σ^0. In fact, by the definition of ϕ_a the points $p_{a,1}, \dots, p_{a,k}$ in II and the points $p_1, \dots, p_r$ in III lie on the boundary of $|A|$ (A as in Lemma 1.3). Hence, by the definition of ψ_a, we see that $\rho_{a,1}, \dots, \rho_{a,k}$ (see II$'$) and $\rho^1, \dots, \rho^r$ (see III$'$) lie in bd $\Sigma^{(1)}$. So if σ is an element of $\Sigma^{(1)}$, we find $\sigma = \sigma_0 \cdot \tau_0$, where $\sigma_0 \in \Sigma$ and τ_0 either equal some $\rho_1 . \cdots . \rho_k$ (if σ_0 is different from ρ_1) or some $\rho^1 . \cdots . \rho^r$ (if $\sigma_0 = \rho_1$). In both cases τ_0 is a simplicial cone in bd $\Sigma^{(1)}$.

According to the construction of $\Sigma^{(2)}$ we see, analogously, that for $\sigma \in \Sigma^{(2)} \setminus \Sigma^{(1)}$ we have $\sigma = \sigma_1 . \tau_1$, where $\sigma_1 \in \Sigma^{(1)}$ and τ_1 is a simplicial cone in bd $\Sigma^{(2)}$. Since either $\sigma_1 \in \Sigma$ or $\sigma_1 = \sigma_0 . \tau_0$ for $\sigma_0 \in \Sigma$ and τ_0 is a simplicial cone in bd $\Sigma^{(1)}$, we obtain $\sigma = \sigma_0 . \tau_0 . \tau_1$. According to Lemma 1.4 applied to $\Sigma^{(1)}$, $\Sigma^{(2)}$ instead of Σ, $\Sigma^{(1)}$, respectively, the choice of a sufficiently small $\varepsilon_{a,1}$ (instead of ε_a) guarantees that $\tau_0 \in$ bd $\Sigma^{(2)}$. Therefore, $\tau_0 . \tau_1 \in$ bd $\Sigma^{(2)}$ so that (2) holds for Σ, $\Sigma^{(2)}$ instead of Σ, Σ_0, respectively. Continuing in this way we find that (2) is also satisfied if we replace $\Sigma^{(2)}$ successively by $\Sigma^{(3)}, \dots, \Sigma^{(i)} = \Sigma^0$. Let hereby $\varepsilon_{a,2}, \dots, \varepsilon_{a,i}$ replace $\varepsilon_{a,1}$.

(4) is readily implied by the construction of Σ^0. In order to obtain (5) we choose $\varepsilon_{a,1}, \dots, \varepsilon_{a,i}$ all to be smaller than the given $\varepsilon > 0$. Since $p_{a,1}, \dots, p_{a,k}$ in II, $p^1, \dots, p^r$ in III, and their analogs in the constructions of $\Sigma^{(2)}, \dots, \Sigma^{(i)}$ all lie in the ε-neighborhood of Σ, the same is true for τ_0, τ_1 and their analogs in $\Sigma^{(2)}, \dots, \Sigma^{(i)} = \Sigma^0$. This implies (5) for Σ^0. □

If $|\Sigma^0|$ is contained in a (linear closed) half-space, we add to Σ^0 an n-dimensional simplicial cone σ and its faces so that $\sigma \cap |\Sigma^0| = \{0\}$ and $\mathrm{pos}(\sigma \cup |\Sigma^0|) = \boldsymbol{R}^n$. The extended fan we denote again by Σ^0.

In order to construct Σ_0 and Σ' we consider the set $P = \mathrm{cl}(\boldsymbol{R}^n \setminus |\Sigma^0|)$. P and $|\Sigma^0|$ have a common boundary which carries a subfan Σ^{00} of Σ^0 and is a union $F_1 \cup \dots \cup F_m$ of $(n-1)$-dimensional cones of Σ^{00}. Let

$$H_i := \mathrm{lin}\, F_i\,, \quad i = 1, \dots, m$$

be the linear hyperplanes spanned by F_i, and let H_i^+, H_i^- be the closed half-spaces bounded by H_i.

LEMMA 1.6. *Σ^0 can be mapped isomorphically onto a fan in such a way that $|\Sigma|$ remains pointwise fixed, and that all H_i are different. We denote the new fan again by Σ^0.*

PROOF. Each cone in $\Sigma^0 \setminus \Sigma$ is either a multifold pyramid with basis in Σ and apex-1-cones not in Σ or a simplicial cone with generating 1-cones not in Σ. Replacing the 1-cones by 1-cones in sufficiently small neighborhoods does not change the structure of Σ^0 and leaves Σ unchanged. The new 1-cones may be chosen so that no two of the hyperplanes $H_1, \ldots, H_m$ coincide. This proves Lemma 1.6. □

$H_1, \ldots, H_m$ split $\boldsymbol{R}^n$ into a system M of polyhedral n-cones with apex 0, each of which is an intersection of half-spaces H_i^+, H_i^-.

LEMMA 1.7. *If $\sigma \in M$, then σ is either totally contained in P or in $|\Sigma^0|$.*

PROOF. Suppose σ contains a point $x \in \operatorname{int} P$ and a point $y \in \operatorname{int}|\Sigma^0|$. Then we assert: The line segment $[x, y]$ intersects at least one F_i, $i = 1, \ldots, m$. Indeed, this follows from a generalized version of the Jordan Curve Theorem; but we can see it directly: $[x, y]$ intersects finitely many n-cones of Σ^0. Among these the one closest to x contains the point $z \in [x, y] \cap |\Sigma^0|$ closest to x on its boundary. Since the boundary of $|\Sigma^0|$ is covered by $F_1, \ldots, F_m$, z lies on one of the F_i. Then H_i separates x and y, a contradiction. □

By Lemma 1.7, P is the union of the cones of a subset $M_0 \subset M$. Although $\boldsymbol{R}^n$ is covered by Σ^0 and M_0, the union of Σ^0, M_0 and the faces of the cones of M_0 do not, in general, provide a fan, since the common boundary of $|\Sigma^0|$ and P is covered differently by cones of Σ^0 and faces of cones of M_0. However, Lemma 1.6 and Lemma 1.7 imply:

LEMMA 1.8. *If a face μ of a cone of M_0 is contained in $|\Sigma^0|$, it is contained in an F_i, $i \in \{1, \ldots, m\}$.*

So each F_i is the union of the $(n-1)$-faces of cones of M_0. This remains true if we refine M_0 as follows.

LEMMA 1.9. *The fan $\Sigma(M_0)$ consisting of M_0 and the faces of M_0 may be turned into a simplicial fan Σ_1 having the same* 1*-cones as $\Sigma(M_0)$ by splitting the cones of $\Sigma(M_0)$.*

PROOF. This follows from a combinatorial theorem (see [E, III, Theorem 2.6]). □

Now we adjust Σ^0 to Σ as follows. Each cone $\sigma \in \Sigma^0 \setminus \Sigma$ which is not contained in the boundary of $|\Sigma(M_0)|$ is a k-fold pyramid $\tau.\rho_1.\cdots.\rho_k = \tau.\tau'$, where $\tau' = \rho_1.\cdots.\rho_k$ is a simplicial cone and the basis τ lies in Σ^0. In Σ_1, τ' is split into simplicial cones $\tau_1, \ldots, \tau_s$, hence σ is split into simplicial cones $\tau.\tau_1, \ldots, \tau.\tau_s$. This turns Σ^0 into a fan Σ_0.

LEMMA 1.10. *Lemma* 1.5 *is true for Σ_0 instead of Σ^0.*

PROOF. This readily follows from $|\Sigma^0| = |\Sigma_0|$ and the construction of Σ_0. □

The fan $\Sigma_0 \cup M_0$ is complete; we denote it by Σ'. So Σ_0, Σ' satisfy all the properties (1) through (5) and the proof of Theorem 1.2, hence also the proof of Theorem 1.1, is completed.

2. The Zariski-Riemann space of a rational fan. Let $r \geq 0$ be an integer, and N a free $\boldsymbol{Z}$-module of rank r. $N_{\boldsymbol{R}} := N \otimes_{\boldsymbol{Z}} \boldsymbol{R}$ is an r-dimensional real space with the lattice N. We consider cones and fans in $N_{\boldsymbol{R}}$ from this section on. Namely, a subset $C \subset N_{\boldsymbol{R}} := N \otimes \boldsymbol{R}$ is said to be a *convex polyhedral cone* if there exists a finite subset $\{y_1, \ldots, y_s\} \subset N_{\boldsymbol{R}}$ with

$$C = \mathrm{pos}\{y_1, \ldots, y_s\} = \boldsymbol{R}_0 y_1 + \cdots + \boldsymbol{R}_0 y_s ,$$

where $\boldsymbol{R}_0$ is the set of nonnegative real numbers. The cone C is said to be *rational* if we can choose $y_1, \ldots, y_s$ in N, and *strongly convex* if $C \cap (-C) = \{0\}$.

Let $M := \mathrm{Hom}_{\boldsymbol{Z}}(N, \boldsymbol{Z})$ and $M_{\boldsymbol{R}} := M \otimes \boldsymbol{R}$. There exists a natural perfect pairing $\langle\ ,\ \rangle : M_{\boldsymbol{R}} \times N_{\boldsymbol{R}} \to \boldsymbol{R}$. A subset $C' \subset C$ is a *face* and written $C' \prec C$ if there exists $x \in M_{\boldsymbol{R}}$ with $C \subset (x \geq 0)$ and $C' = C \cap (x = 0)$, where we denote $(x \geq 0) = \{y \in N_{\boldsymbol{R}}\ ;\ \langle x, y\rangle \geq 0\}$ and $(x = 0) = \{y \in N_{\boldsymbol{R}}\ ;\ \langle x, y\rangle = 0\}$.

From this section on, we prefer to use letters $X, Y, \ldots$ for fans rather than Greek capitals. Recall that a nonempty set X of strongly convex rational polyhedral cones is said to be a *fan* if

(i) $\sigma \in X$ and $\eta \prec \sigma$ imply $\eta \in X$, and

(ii) if $\sigma, \tau \in X$, then $\sigma \cap \tau$ is a common face of σ and τ.

The condition (ii) can be replaced by the following "separability condition" (ii') (cf. [F2, 1.2, (12)]).

(ii$'$) For $\sigma, \tau \in X$, there exists $x \in M_{\boldsymbol{R}}$ with $\sigma \subset (x \geq 0)$, $\tau \subset (x \leq 0)$ and $\sigma \cap (x = 0) = \tau \cap (x = 0)$.

The set $F(\pi)$ of all faces of a strongly convex rational polyhedral cone π is a fan with the unique maximal element π. Such a fan is called an *affine fan*.

For each $\sigma \in X$, the dual cone $\sigma^\vee \subset M_{\boldsymbol{R}}$ is defined by

$$\sigma^\vee := \{x \in M_{\boldsymbol{R}}\ ;\ \langle x, u\rangle \geq 0 \text{ for any } u \in \sigma\} .$$

This is an r-dimensional polyhedral cone. Since $M \cap \sigma^\vee$ is a finitely generated additive semigroup (cf. [O1, Prop.1.1]), the semigroup ring $\boldsymbol{C}[M \cap \sigma^\vee]$ over the complex number field is an affine ring and the quotient field is equal to that of the group ring $\boldsymbol{C}[M]$. The toric variety $X_{\boldsymbol{C}}$ associated to a fan X is defined to be the union of the affine toric varieties $\mathrm{Spec}\, \boldsymbol{C}[M \cap \sigma^\vee]$ for $\sigma \in X$. The function field of $X_{\boldsymbol{C}}$ is the quotient field of $\boldsymbol{C}[M]$, and the algebraic torus $T_N := \mathrm{Spec}\, \boldsymbol{C}[M]$ acts on $X_{\boldsymbol{C}}$. The toric variety $X_{\boldsymbol{C}}$ is of finite type, i.e., it is an algebraic variety if and only if the fan X is finite.

The topology of a fan X is defined as follows. A subset $U \subset X$ is defined to be open if $\sigma \in U$ and $\eta \prec \sigma$ imply $\eta \in U$. Namely, U is open if and only if it is empty or a subfan of X. For each point $x \in X_{\boldsymbol{C}}$, we define $\phi(x)$ to be the minimal $\sigma \in X$ with $x \in \mathrm{Spec}\, \boldsymbol{C}[M \cap \sigma^\vee]$. Then the map $\phi : X_{\boldsymbol{C}} \to X$ is continuous.

A collection of convex polyhedral cones not necessarily rational is called a *real fan* if it satisfies (i) and (ii). Usual fans which define toric varieties are sometimes called *rational fans*.

The support $|X|$ of a fan X is defined to be $\bigcup_{\sigma \in X} \sigma$, and X is said to be *complete* if it is finite and $|X| = N_{\boldsymbol{R}}$. For a rational finite fan X, the toric variety $X_{\boldsymbol{C}}$ is complete if and only if X is complete. If X is a subfan of a complete rational fan $\bar{X}$, then $X_{\boldsymbol{C}}$ is an open subvariety of $\bar{X}_{\boldsymbol{C}}$, i.e., $\bar{X}_{\boldsymbol{C}}$ is a completion of $X_{\boldsymbol{C}}$.

A subring R of a field K is said to be a *valuation ring* if it satisfies the condition:

$$1/x \in R \quad \text{for any } \ x \in K \setminus R\,.$$

In particular, $R = K$ is a valuation ring.

Let $(R, \mathfrak{m})$ and $(R', \mathfrak{m}')$ be not necessarily Noetherian local rings with a common quotient field K. We say R' *dominates* R and write $R \leq R'$ if $R \subset R'$ and $\mathfrak{m} = \mathfrak{m}' \cap R$.

It is known that, for any local ring $(R, \mathfrak{m})$ with the quotient ring K, there exists a valuation ring $(R', \mathfrak{m}')$ which dominates $(R, \mathfrak{m})$. When $(R, \mathfrak{m})$ is a local ring of an algebraic variety, R' can be regarded as the limit of a transfinite sequence of blowups of R.

Let X be an algebraic variety over a field k, i.e., a reduced and irreducible separated scheme of finite type over k, and K the function field. We say a valuation ring R of K *dominates* a point x of the scheme X if it dominates the local ring $\mathcal{O}_x$. We denote by $\mathrm{ZR}(X)$ the set of all valuation rings of K which dominates a point of X (cf. [ZS, Chap. VI, §17]). $\mathrm{ZR}(X)$ is called the Zariski-Riemann space of X. We denote by $\mathrm{ZR}(K)$ the set of all valuation rings of K which contains the base field k.

The following theorem is known as the valuative criterion of properness of varieties (cf. [H, Thm. 4.7]).

THEOREM 2.1. *An algebraic variety X with the function field K is complete if and only if* $\mathrm{ZR}(X) = \mathrm{ZR}(K)$.

For an algebraic variety X, the topology of $\mathrm{ZR}(X)$ is defined as follows (cf. [N1, §1]). For a proper birational morphism $X' \to X$ and a closed subset $Y' \subset X'$, let F be the set of all valuation rings in $\mathrm{ZR}(X)$ which dominate points of Y'. We define the set of all such F as a basis of the closed sets of $\mathrm{ZR}(X)$. This topology is equal to the topology defined by the open basis consisting of the following $E(B)$'s. Let B be an integral domain of finite type over k with the quotient field K, and $E(B)$ the set of all valuation rings in $\mathrm{ZR}(X)$ which contain B.

The following theorem was used in Nagata's compactification theorem.

THEOREM 2.2 ([ZS, Thm. 40], [N1, Prop. 1.1]). *The space* $\mathrm{ZR}(X)$ *is quasi-compact for any algebraic variety X.*

We define the Zariski-Riemann space for a rational fan. In this case, we replace the field K by the dual module $M \simeq \boldsymbol{Z}^r$ of N.

A relation $\leq$ on M is said to be an *additive preorder* if it satisfies the following conditions:

(1) For any $x, y \in M$, either $x \leq y$ or $y \leq x$ is satisfied.
(2) $x \leq y$ and $y \leq z$ imply $x \leq z$.
(3) If $x \leq y$, then $x + z \leq y + z$ for every $z \in M$.

Note that we do not assume the anti-symmetry: $x \leq y$ and $y \leq x$ imply $x = y$. It is easy to see that $x \leq y$ and $x' \leq y'$ imply $x + x' \leq y + y'$, and $0 \leq nx$ for an integer $n > 0$ implies $0 \leq x$.

DEFINITION 2.3. We define *the Zariski-Riemann space* $\mathrm{ZR}(M)$ to be the set of all additive preorders of M.

We are going to define the topology of $\mathrm{ZR}(M)$. Each element v of $\mathrm{ZR}(M)$ is denoted by $\leq_v$ when it is used as a relational operator. We denote $x =_v y$ if $x \leq_v y$ and $y \leq_v x$, while $x <_v y$ if $x \leq_v y$ and not $y \leq_v x$. $L(v) := \{x \in M \ ; \ 0 \leq_v x\}$ is a substitute for the valuation ring in our case. We see easily that $L(v)$ is a subsemigroup satisfying $-x \in L(v)$ for all $x \in M \setminus L(v)$. We define $L^0(v) := \{x \in M \ ; \ x =_v 0\}$, which is equal to the $\boldsymbol{Z}$-submodule $L(v) \cap (-L(v))$ of M. $M/L^0(v)$ is equal to the quotient of M by the equivalence relation $x =_v y$, and is a free $\boldsymbol{Z}$-module.

We denote by $\eta(M)$ the trivial preorder of M, i.e., $L^0(\eta(M)) = M$.

$$\phi_M : \mathrm{ZR}(M) \setminus \{\eta(M)\} \to (N_{\boldsymbol{R}} \setminus \{0\})/\boldsymbol{R}_+$$

is defined as follows. For $v \in \mathrm{ZR}(M) \setminus \{\eta(M)\}$, let C_v be the convex closure of $L(v)$ in $M_{\boldsymbol{R}}$. Then the closure $\bar{C}_v$ is a closed half space (cf. Lemma 5.1). Hence there exists $x \in N_{\boldsymbol{R}} \setminus \{0\}$ with $\bar{C}_v = (x \geq 0)$. We define $\phi_M(v)$ to be the image of x in $(N_{\boldsymbol{R}} \setminus \{0\})/\boldsymbol{R}_+$.

We set $S_N := (N_{\boldsymbol{R}} \setminus \{0\})/\boldsymbol{R}_+$. Then $\mathrm{ZR}(M)$ has a recursive structure as follows. For each $x \in N_{\boldsymbol{R}} \setminus \{0\}$, let $\bar{x}$ be its image in S_N. Let $M(x)_{\boldsymbol{R}}$ be the largest rational subspace of $M_{\boldsymbol{R}}$ contained in $(x = 0)$, and let $M(x) := M(x)_{\boldsymbol{R}} \cap M$. Then $\phi_M^{-1}(\bar{x})$ is identified with $\mathrm{ZR}(M(x))$ by identifying each $v \in \phi_M^{-1}(\bar{x})$ with its restriction to $M(x)$.

Let C be a rational polyhedral cone in $N_{\boldsymbol{R}}$ which is not necessarily strongly convex. Then there exists a finite subset $\{x_1, \ldots, x_s\} \subset M$ with

$$C = (x_1 \geq 0) \cap \cdots \cap (x_s \geq 0)$$

(cf. [O1, Thm. A.2]). We define a subset $\|C\|$ of $\mathrm{ZR}(M)$ by

$$\begin{aligned} \|C\| &:= \{v \in \mathrm{ZR}(M) \ ; \ 0 \leq_v x_i, i = 1, \ldots, s\} \\ &= \{v \in \mathrm{ZR}(M) \ ; \ M \cap C^\vee \subset L(v)\}. \end{aligned}$$

For any rational polyhedral cones C_1, C_2, we see easily that $M \cap C_1^\vee, M \cap C_2^\vee \subset L(v)$ implies $M \cap (C_1^\vee + C_2^\vee) \subset L(v)$. Since $C_1^\vee + C_2^\vee = (C_1 \cap C_2)^\vee$, we have $\|C_1\| \cap \|C_2\| = \|C_1 \cap C_2\|$. We take the set of all such $\|C\|$'s as the open basis of $\mathrm{ZR}(M)$. Since the set of all finite subsets of M is countable, the topology of $\mathrm{ZR}(M)$ defined by this open basis satisfies the second countability axiom.

Let $\mathrm{ZR}_0(M)$ be the set of the elements in $\mathrm{ZR}(M)$ which have the anti-symmetry property, i.e., the set of additive orders of M. If we identify M with the set of monomials of a Laurent polynomial ring, $\mathrm{ZR}_0(M)$ is equal to the space introduced by Kuroda [K] (see also [S3]). Kuroda [K] introduced this space in order to prove the infinity of the SAGBI bases of some invariant rings.

We omit the proof of the following proposition which we do not use in this paper.

PROPOSITION 2.4. $\mathrm{ZR}_0(M)$ *is closed in* $\mathrm{ZR}(M)$ *and the induced topology of* $\mathrm{ZR}_0(M)$ *is equal to that of Kuroda.*

The Zariski-Riemann space $\mathrm{ZR}(X)$ of a fan X is defined by

$$\mathrm{ZR}(X) := \bigcup_{\sigma\in\Delta} \|\sigma\| .$$

THEOREM 2.5. *The Zariski-Riemann space* $\mathrm{ZR}(M)$ *is quasi-compact.* $\mathrm{ZR}(X)$ *is quasi-compact for any finite fan* X *of* $M_{\boldsymbol{R}}$*. Here "quasi-compact" means "compact but not necessarily Hausdorff".*

PROOF. We follow the method of Zariski-Samuel [ZS, Thm. 40]. For every $m \in M\backslash\{0\}$, we set $\hat{S}^0_m := \{-1, 0, 1\}$. For $v \in \mathrm{ZR}(M)$ and $m \in M$, we define $v(m) \in \hat{S}^0_m$ to be -1 if $m <_v 0$, 0 if $m =_v 0$ and 1 if $0 <_v m$. Since $v \in \mathrm{ZR}(M)$ is determined by the set of m's with $0 \leq_v m$, we regard v as the map from M to $\{-1, 0, 1\}$, and we get an embedding

$$\mathrm{ZR}(M) \subset \prod_{m\in M\backslash\{0\}} \hat{S}^0_m .$$

The weak topology of $\hat{S}^0_m$ is defined by setting $\{\emptyset, \{0\}, \{0, 1\}, \{-1, 0, 1\}\}$ as the set of open subsets. Since the set of finite intersections of $\{v \,;\, v(m) = 0, 1\}$ is an open basis of $\mathrm{ZR}(M)$, the topology of $\mathrm{ZR}(M)$ is equal to the relative topology of the product topology of $\prod_{m\in M} \hat{S}^0_m$. Now we introduce the discrete topology on $\hat{S}^0_m$. Then the product space is compact by Tychonoff's theorem. $\mathrm{ZR}(M)$ is a closed subset of the compact product space. Actually, it is defined by the equalities

$$v(m) = -1 \quad \text{or} \quad v(m') = -1 \quad \text{or} \quad v(m + m') = 0,\ 1 , \tag{1}$$

$$v(m) = 0,\ 1 \quad \text{or} \quad v(-m) = 1 \tag{2}$$

and

$$v(m) = 0,\ -1 \quad \text{or} \quad v(-m) = -1 \tag{3}$$

for all $m, m' \in M$. Hence $\mathrm{ZR}(M)$ is compact in the strong topology, hence so is it in the weak topology.

In order to show the compactness of $\mathrm{ZR}(X)$ for a finite fan X, it suffices to show that of each $\|\sigma\|$. We can show the compactness of $\|\sigma\|$ by taking a generator $\{x_1, \ldots, x_s\} \subset M$ of the cone $\sigma^\vee$ and adding the equalities

$$v(x_i) = 0,\ 1 \quad (i = 1, \ldots, s)$$

to those of (1), (2) and (3). □

Let v be an element of $\mathrm{ZR}(M)$. We will express v by a sequence of elements in $N_{\boldsymbol{R}}$. We set $v_0 := v$ and $M^0(v) := M$. If $v_0 \neq \eta(M^0(v))$, then let $M^1(v)$ be the intersection $M \cap (x_0 = 0) \subset M_{\boldsymbol{R}}$ for $x_0 := \phi_M(v_0)$, and v_1 the restriction of v_0 to $M^1(v)$. Inductively, if $v_{i-1} \neq \eta(M^{i-1}(v))$, we define $M^i(v)$ to be the intersection $M^{i-1}(v) \cap (x_{i-1} = 0) \subset M^{i-1}(v)_{\boldsymbol{R}}$ for $x_{i-1} := \phi_{M^{i-1}(v)}(v_{i-1})$, and v_i the restriction of v_{i-1} to $M^i(v)$. Since the rank of M is r, there exists a nonnegative integer $s \leq r$ with $v_s = \eta(M^s(v))$. We call s the rank of v and denote it by $\mathrm{rank}(v)$. The rank of $\eta(M)$ is defined to be 0. This is an analog of the rank defined for a valuation ring.

Note that the preorder v is recovered from the integer s and the sequence $(x_0, \ldots, x_{s-1})$. Actually, if we take a representative $y_i \in N_{\boldsymbol{R}}$ of x_i for each i, then $m \in L(v)$ if and only if either there exists $0 \leq j \leq s-1$ such that $\langle m, y_k \rangle = 0$ for $0 \leq k < j$ and $\langle m, y_j \rangle > 0$, or $\langle m, y_k \rangle = 0$ for all $0 \leq k \leq s-1$. This is also equivalent to the condition $\langle m, \sum_{i=0}^{s-1} \varepsilon^i y_i \rangle \geq 0$ for a sufficiently small positive real number ε. Since we often use this sequence $(y_0, \ldots, y_{s-1})$ of points in $N_{\boldsymbol{R}}$, we call it a *defining sequence* of v. By construction, $y_0, \ldots, y_{j-1}$ are 0 on $M^j(v)_{\boldsymbol{R}}$ and y_j is not identically 0 on this linear subspace. Since $\{M^j(v)_{\boldsymbol{R}}\}$ is a shrinking sequence of vector spaces, $\{y_0, \ldots, y_{s-1}\}$ is linearly independent.

When $v \neq \eta(M)$, the *first generalization* $v' \in \mathrm{ZR}(M)$ of v is defined by

$$x \leq_{v'} y \iff x \leq_v y \text{ or } y - x \in M^{\mathrm{rank}(v)-1}(v)\,.$$

If $\mathrm{rank}(v) = s$ and $(y_0, \ldots, y_{s-1})$ is a defining sequence of v, then $(y_0, \ldots, y_{s-2})$ is a defining sequence of v'. Hence, we get $\eta(M)$ by s-times repetition of the first generalization starting from v. For $v, w \in \mathrm{ZR}(M)$, w is said to be a *generalization* of v if $L(v) \subset L(w)$. This is equivalent to the condition that we get w from v by a finite repetition of the first generalization.

If $\mathrm{rank}(v) = s$, then $M^s(v)$ is equal to the $\boldsymbol{Z}$-submodule $L^0(v)$ of M. We say that an element v of $\mathrm{ZR}(M)$ *dominates* a cone C of $N_{\boldsymbol{R}}$ if $M \cap C^\vee \subset L(v)$ and $M \cap C^\vee \cap L^0(v) = M \cap C^\perp$, where $C^\perp := \{x \in M_{\boldsymbol{R}} \; ; \; \langle x, y \rangle = 0 \text{ for all } y \in C\}$. This situation is described by the sequence $(M^0(v), \ldots, M^s(v))$ and the defining sequence $(y_0, \ldots, y_{s-1})$ of v as follows. The additive preorder v dominates C if and only if

$$M^i(v) \cap C^\vee \subset (y_i \geq 0)$$

for each $i = 0, \ldots, s-1$, and

$$M^s(v) \cap C^\vee = M \cap C^\perp\,.$$

LEMMA 2.6. *Let $(y_0, \ldots, y_{s-1})$ be a defining sequence of $v \in \mathrm{ZR}(M)$. For a positive real number ε, we set $z_\varepsilon := \sum_{i=0}^{s-1} \varepsilon^i y_i$. Then the following conditions on a rational polyhedral cone C are equivalent.*

(1) *v dominates C.*

(2) *There exists $\varepsilon_0 > 0$ such that $z_\varepsilon \in \mathrm{rel.\,int}\, C$ for every $0 < \varepsilon \leq \varepsilon_0$.*

(3) *There exists a sequence $\{\varepsilon_j\}$ of positive real numbers with $\lim_{j \to \infty} \varepsilon_j = 0$ and $z_{\varepsilon_j} \in \mathrm{rel.\,int}\, C$ for every j.*

PROOF. We show (1) $\Rightarrow$ (2). We take $m_1, \ldots, m_t \in M \cap (C^\vee \setminus C^\perp)$ and $m_{t+1}, \ldots, m_u \in M \cap C^\perp$ so that they generate the semigroup $M \cap C^\vee$. Since $C^\perp$ is a rational subspace, it is generated by $\{m_{t+1}, \ldots, m_u\}$. For each m_i with $1 \leq i \leq t$, there exists $1 \leq j \leq s-1$ such that

$$\langle m_i, y_0 \rangle = \cdots = \langle m_i, y_{j-1} \rangle = 0\,, \quad \langle m_i, y_j \rangle > 0\,.$$

Hence, $\langle m_i, z_\varepsilon\rangle > 0$ for a sufficiently small $\varepsilon > 0$ for $i = 1, \ldots, t$. Since $\langle m_i, z_\varepsilon\rangle = 0$ for every $t+1 \leq i \leq u$, we have $C^\vee \subset (z_\varepsilon \geq 0)$ and $C^\vee \cap (z_\varepsilon = 0) = C^\perp$ for such ε. Hence $z_\varepsilon \in \operatorname{rel.int} C$. (2) $\Rightarrow$ (3) is obvious.

We show (3) $\Rightarrow$ (1). The condition implies $M^i(v)_{\boldsymbol{R}} \cap C^\vee \subset (z_{\varepsilon_j} \geq 0)$ for $i = 0, \ldots, s-1$. Since $y_0 = \cdots = y_{i-1} = 0$ on $M^i(v)_{\boldsymbol{R}}$, the limit of the linear functions $\varepsilon_j^{-i} z_{\varepsilon_j}$ of $M^i(v)_{\boldsymbol{R}}$ is equal to y_i. Hence $M^i(v)_{\boldsymbol{R}} \cap C^\vee \subset (y_i \geq 0)$ for every i. This implies $M \cap C^\vee \subset L(v)$. Let $\varepsilon := \varepsilon_1$. Then clearly, $z_\varepsilon = 0$ on $M^s(v)_{\boldsymbol{R}}$. Hence

$$M \cap C^\vee \cap L^0(v) \subset M \cap C^\vee \cap (z_\varepsilon = 0) = M \cap C^\perp .$$

On the other hand, $C^\perp \subset (z_{\varepsilon_j} = 0)$ for $j = 1, \ldots, s$ imply that $y_0, \ldots, y_{s-1}$ are zero on $C^\perp$, and hence $M \cap C^\perp \subset L^0(v)$. Hence, v dominates C. □

The following proposition is an analog of the valuative criterion of separatedness of an algebraic prevariety.

PROPOSITION 2.7. *For a fan X of $N_{\boldsymbol{R}}$, an element $v \in \mathrm{ZR}(M)$ dominates at most one cone of X. Conversely, if X is a union of affine fans and any $v \in \mathrm{ZR}(M)$ dominates at most one cone of X, then X is a fan.*

PROOF. If σ and τ are distinct cones of the fan X, then $\operatorname{rel.int}\sigma \cap \operatorname{rel.int}\tau = \emptyset$. If v dominates σ, then it does not dominates τ since the condition (2) of Lemma 2.6 is satisfied for $C = \sigma$

Now we prove the second part. It suffices to show that $\sigma \cap \tau$ is a face of σ for any $\sigma, \tau \in X$. Any point y in the relative interior of the cone $\sigma \cap \tau$ is contained in the relative interior of a face σ_1 of σ and in that of a face τ_1 of τ. If we take $v \in \mathrm{ZR}(M)$ with $L(v) = M \cap (y \geq 0)$, then v dominates σ_1 and τ_1, and hence $\sigma_1 = \tau_1$ by assumption. By $\tau_1 \subset \tau$, we have $\sigma_1 \subset \sigma \cap \tau$. On the other hand, the defining element $x \in M_{\boldsymbol{R}}$ of the face $\sigma_1 \subset \sigma$ defines a face of $\sigma \cap \tau$. This face is equal to $\sigma \cap \tau$ itself, since it contains y in the relative interior. We know $\sigma \cap \tau \subset \sigma_1$, since $\sigma_1 = \sigma \cap (x = 0)$. Hence $\sigma \cap \tau = \sigma_1$, and $\sigma \cap \tau$ is a face of σ. □

DEFINITION 2.8. We denote by $\mathrm{dom}(\sigma)$ the set of elements of $\mathrm{ZR}(M)$ which dominate the strongly convex rational polyhedral cone σ.

If we take a point y in a strongly convex rational polyhedral cone σ, then the rank one element $v \in \mathrm{ZR}(M)$ with $L(v) = M \cap (y \geq 0)$ dominates σ. In particular, $\mathrm{dom}(\sigma)$ is not empty.

LEMMA 2.9. *Let C be a rational polyhedral cone of $N_{\boldsymbol{R}}$, and v an element of $\mathrm{ZR}(M)$. Then $M \cap C^\vee \subset L(v)$ if and only if v dominates a face of C. In particular, if π is a strongly convex rational polyhedral cone, then we have the equality*

$$\|\pi\| = \mathrm{ZR}(F(\pi)) = \bigcup_{\sigma \in F(\pi)} \mathrm{dom}(\sigma) .$$

PROOF. If v dominates a face C_1 of C, then

$$M \cap C^\vee \subset M \cap C_1^\vee \subset L(v)\,.$$

Conversely, suppose $M \cap C^\vee \subset L(v)$. We see easily by induction that $C^\vee \cap M^i(v)_{\boldsymbol{R}}$ is a face of $C^\vee$ for $i = 0, \ldots, \mathrm{rank}(v)$. In particular, $C^\vee \cap L^0(v)_{\boldsymbol{R}}$ is a face of $C^\vee$. Hence there exists a face $C_1 \prec C$ with $C^\vee \cap L^0(v)_{\boldsymbol{R}} = C^\vee \cap C_1^\perp$ (cf. [O1, Prop. A.6]). Since $C_1^\perp \subset L^0(v)_{\boldsymbol{R}}$ and $C_1^\vee = C^\vee + C_1^\perp$ by [O1, Cor. A.7], we have $M \cap C_1^\vee \subset L(v)$ and $M \cap C_1^\vee \cap L(v) = M \cap C_1^\perp$. Hence v dominates C_1.

The equalities are now obvious. □

LEMMA 2.10. *Let C be a rational polyhedral cone of $N_{\boldsymbol{R}}$, and v an element of* $\mathrm{ZR}(M)$ *which dominates C. If w is a generalization of v, then w dominates a face of C. The dimension of C is at least* $\mathrm{rank}(v)$.

PROOF. We have $L(v) \subset L(w)$ since w is a generalization of v. Since $M \cap C^\vee \subset L(w)$, w dominates a face C' of C by Lemma 2.9. Let $\mathrm{rank}(v) = s$ and let $(y_0, \ldots, y_{s-1})$ be a defining sequence of v. By Lemma 2.6, $z_\varepsilon := \sum_{i=0}^{s-1} \varepsilon^i y_i$ satisfies $z_\varepsilon \in \mathrm{rel.\,int}\, C$ for a sufficiently small $\varepsilon > 0$. Since $y_0, \ldots, y_{s-1}$ are linearly independent, z_ε's for s distinct ε's are also linearly independent by Vandermonde's equality. Hence the dimension of C is at least s. □

REMARK 2.11. C is also a face of itself. Hence w might dominates C in Lemma 2.10.

For a fan X, we define

$$\mathrm{ZR}(X)^1 := \{v \in \mathrm{ZR}(X)\ ;\ \mathrm{rank}(v) = 1\}\,.$$

PROPOSITION 2.12. *For finite fans X, Y, the following conditions are equivalent.*

(1) $\mathrm{ZR}(X) \subset \mathrm{ZR}(Y)$.

(2) $\mathrm{ZR}(X)^1 \subset \mathrm{ZR}(Y)^1$.

(3) $|X| \subset |Y|$.

PROOF. (1) $\Rightarrow$ (2) is obvious.

For (2) $\Rightarrow$ (3), let $x \in \sigma \in X$. Since $|Y|$ contains 0, we assume $x \neq 0$. Let v be the preorder of rank one with $L(v) = M \cap (x \geq 0)$. Since $M \cap \sigma^\vee \subset M \cap (x \geq 0)$, we have $v \in \mathrm{ZR}(X)^1$. Since $v \in \mathrm{ZR}(Y)^1$ by (2), we have $M \cap \tau^\vee \subset L(v) = M \cap (x \geq 0)$ for a rational cone $\tau \in Y$. Then $\tau^\vee \subset (x \geq 0)$ and $x \in \tau \subset |Y|$.

We show (3) $\Rightarrow$ (1). Suppose $v \in \mathrm{ZR}(X)$ dominates $\sigma \in X$. Let $\mathrm{rank}(v) = s$ and $(y_0, \ldots, y_{s-1})$ a defining sequence of v. Then by Lemma 2.6, $z_\varepsilon := \sum_{i=0}^{s-1} \varepsilon^i y_i \in \mathrm{rel.\,int}\,\sigma$ for sufficiently small $\varepsilon > 0$. Since $z_\varepsilon \in |X| \subset |Y|$ and Y is finite, there exist $\tau \in Y$ and a convergent sequence $\{\varepsilon_j\}$ with the limit 0 and $z_{\varepsilon_j} \in \mathrm{rel.\,int}\,\tau$. Hence $v \in \mathrm{dom}\,\tau \subset \mathrm{ZR}(Y)$ by Lemma 2.6. □

Since the proposition is also true even if we exchange X and Y, we have the following corollary.

COROLLARY 2.13. *For finite fans X, Y, the following conditions are equivalent.*

(1) $\mathrm{ZR}(X) = \mathrm{ZR}(Y)$.
(2) $\mathrm{ZR}(X)^1 = \mathrm{ZR}(Y)^1$.
(3) $|X| = |Y|$.

3. Blowups of fans. Fans in this section are not necessarily rational and cones are finitely generated convex polyhedral cones unless otherwise mentioned.

Let D be a cone of $M_{\boldsymbol{R}}$. A nonempty convex subset P of $M_{\boldsymbol{R}}$ is said to be D-convex if it has the D-ideal property, i.e., if

$$x \in P,\ \ y \in D \Rightarrow x + y \in P\ .$$

For a subset S of $M_{\boldsymbol{R}}$, we set

$$S^{\vee} := \{y \in N_{\boldsymbol{R}}\ ;\ \langle x, y\rangle \geq 0 \text{ for all } x \in S\}\ .$$

For the convex hull $\mathrm{conv}(S)$ and the convex cone $\mathrm{Cone}(S)$ generated by S, we see easily the equalities

$$S^{\vee} = \mathrm{conv}(S)^{\vee} = \mathrm{Cone}(S)^{\vee}\ .$$

Here $\mathrm{Cone}(S)$ is not necessarily finitely generated if S is not a finite set. For subsets S, T of $M_{\boldsymbol{R}}$, clearly we have

$$(S \cup T)^{\vee} = S^{\vee} \cap T^{\vee}\ .$$

For a cone C, a $C^{\vee}$-convex set P generated by a finite set S is called a *$C^{\vee}$-convex polyhedron*. When C is rational, P is said to be *rational* if S consists of finite rational points.

Let C be a cone of $N_{\boldsymbol{R}}$. For an r-dimensional $C^{\vee}$-convex polyhedron P,

$$\mathrm{Fan}(P) := \{(P - x)^{\vee}\ ;\ x \in P\}$$

is a finite real fan with support C. If P is rational, $\mathrm{Fan}(P)$ is also a rational fan. As is well-known, the relationship between P and $\mathrm{Fan}(P)$ is as follows. A subset $Q \subset P$ is called a *face* of P if there exist an element $u \in N_{\boldsymbol{R}}$ and a real number a with $P \subset (u \geq a)$ and $Q = P \cap (u = a)$, where $(u \geq a) = \{x \in M_{\boldsymbol{R}}\ ;\ \langle x, u\rangle \geq a\}$ and $(u = a) = \{x \in M_{\boldsymbol{R}}\ ;\ \langle x, u\rangle = a\}$. Each element x of P is contained in the relative interior of a unique face of P, and the cone $(P - x)^{\vee}$ is determined by the face. By this correspondence, $\mathrm{Fan}(P)$ is in bijective correspondence with the set of faces of P. If $\sigma \in \mathrm{Fan}(P)$ corresponds to a face Q of P, then we have the equality $\dim \sigma + \dim Q = r$. If another cone $\tau \in \mathrm{Fan}(P)$ corresponds to a face R, then $R \subset Q$ if and only if $\sigma \prec \tau$. It follows that σ and τ are faces of a common $\rho \in \mathrm{Fan}(P)$ if and only if $Q \cap R \neq \emptyset$.

We define $\mathrm{Fan}(P)$ similarly for P of dimension less than r. In this case, $\mathrm{Fan}(P)$ consists of cones which are not strongly convex. The support of $\mathrm{Fan}(P)$ is also C.

In case $C = N_{\boldsymbol{R}}$, P is a convex polytope. If $\dim P = r$, then $\mathrm{Fan}(P)$ is a complete fan. We call $\mathrm{Fan}(P)$ the *projective* (real) fan defined by P (cf. [OP, p. 383, Remark]). It is common to call it a *polytopal* fan, but we adopt this terminology instead for the convenience to translate Nagata's proof. If P is rational, then $\mathrm{Fan}(P)$ defines a projective toric variety.

Let π be a strongly convex rational polyhedral cone. Then $M\cap\pi^\vee$ is a finitely generated semigroup with the unit 0. If a subset S of $m_0 + M\cap\pi^\vee$ for some $m_0 \in M$ satisfies the "semigroup ideal" condition

$$m \in S,\ m' \in M\cap\pi^\vee \Rightarrow m+m' \in S\,, \tag{4}$$

then there exist a finite number elements $m_1, \dots, m_s \in S$ with

$$S = \bigcup_{i=1}^{s}(m_i + M\cap\pi^\vee)\,. \tag{5}$$

This fact is checked as follows. For an arbitrary field k, we consider the semigroup ring

$$k[M\cap\pi^\vee] := \bigoplus_{m\in M\cap\pi^\vee} k\boldsymbol{e}(m)\,.$$

Then the vector subspace I generated by $\{\boldsymbol{e}(m-m_0)\ ;\ m\in S\}$ is an ideal. Since $k[M\cap\pi^\vee]$ is Noetherian, we can find a finite set of generators $\{\boldsymbol{e}(m_1-m_0), \dots, \boldsymbol{e}(m_s-m_0)\}$ of the ideal. Then $m_1, \dots, m_s$ satisfies the condition.

The convex hull of the above S is the $\pi^\vee$-convex set generated by $m_1, \dots, m_s \in S$.

3.1. The blowup of a fan at a closed subset. Let π be a strongly convex rational polyhedral cone. For a closed proper subset Y of the rational affine fan $F(\pi)$, the blowup $\mathrm{Bl}^M_Y(F(\pi))$ of $F(\pi)$ along Y is defined by using the lattice M as follows.

We set $\mathcal{S}(\pi) := M\cap\pi^\vee$, and $\mathcal{S}(\pi;\sigma) := M\cap\pi^\vee\cap\sigma^\perp$ for each $\sigma\in F(\pi)$. The set $P(\pi, Y)$ is defined to be the convex hull of

$$S = \mathcal{S}(\pi)\setminus\bigcup_{\sigma\in Y}\mathcal{S}(\pi;\sigma)\,.$$

S is nonempty, since the zero cone $\mathbf{0} = \{0\}$ is not in Y, and $P(\pi, Y)$ is a $\pi^\vee$-convex polyhedron, since S satisfies (4). Then $\mathrm{Bl}^M_Y(F(\pi)) := \mathrm{Fan}(P(\pi, Y))$ is a finite fan with support π.

The morphism of toric varieties

$$\mathrm{Bl}^M_Y(F(\pi))_{\boldsymbol{C}} \to F(\pi)_{\boldsymbol{C}}$$

corresponding to this subdivision is equal to the normalization of the blowup of $F(\pi)_{\boldsymbol{C}}$ along the reduced closed subvariety $Y_{\boldsymbol{C}}$.

Let X, Y be rational fans of $N_{\boldsymbol{R}}$. If each $\sigma\in X$ is contained in some ρ in Y, there exists a birational morphism $X_C \to Y_{\boldsymbol{C}}$ of the toric varieties. Then we say that the fan X *dominates* Y and write as $f : X\to Y$. This f also represents the map which sends each $\sigma\in X$ to the minimal cone in Y which contains σ.

For a $\pi^\vee$-convex polyhedron P generated by a finite subset of $M_{\boldsymbol{Q}}$, $\mathrm{Fan}(P)$ is a subdivision of $F(\pi)$, and the corresponding morphism of toric varieties is the natural morphism $\mathrm{Proj}\, B \to \mathrm{Spec}\,\boldsymbol{C}[M\cap\pi^\vee]$ defined for the graded ring

$$B := \bigoplus_{n=0}^{\infty}[M\cap nP]_{\boldsymbol{C}}\,,$$

where, for a subset F of M, we denote by $F_{\boldsymbol{C}}$ the vector subspace $\bigoplus_{m\in F}\boldsymbol{C}\boldsymbol{e}(m)$ of $\boldsymbol{C}[M]$. We understand $0P = \pi^{\vee}$.

3.2. General blowups of fans. Let X be a fan. We consider a set $I = \{I_\sigma\ ;\ \sigma \in X\}$ of subsets of $M_{\boldsymbol{R}}$ such that each I_σ is $\sigma^{\vee}$-convex and the equality $I_\sigma = I_\tau + \sigma^{\vee}$ holds for any $\sigma, \tau \in X$ with the relation $\sigma \prec \tau$. Then

$$\mathrm{Fan}_X(I) := \bigcup_{\sigma\in X} \mathrm{Fan}(I_\sigma)$$

is a subdivision of the fan X. If everything is rational, this subdivision corresponds to the normalization of the blowup of a toric variety along a fractional ideal. Hence we use similar terminology for fans. Namely, we call $I = \{I_\sigma\ ;\ \sigma \in X\}$ a *polyhedral fractional ideal* of X, and $\mathrm{Fan}_X(I)$ the *blowup* of X along I. We call I a *polyhedral ideal* if $I_\sigma \subset \sigma^{\vee}$ for every $\sigma \in X$. For a polyhedral ideal I, we define the *support* of I by $\{\sigma \in X\ ;\ I_\sigma \neq \sigma^{\vee}\}$. We say that I is *unitary* at σ if $I_\sigma = \sigma^{\vee}$. Namely, I is unitary on the open subset $X \setminus Y$ if Y is the support of I.

3.3. The composite of blowups.

THEOREM 3.1. *Let X be a finite fan and $I = \{I_\sigma\ ;\ \sigma \in X\}$ a polyhedral fractional ideal. We set $X' := \mathrm{Fan}_X(I)$. Let $I' = \{I'_\rho\ ;\ \rho \in X'\}$ be a polyhedral fractional ideal of X'. For a positive real number a, we define an ideal $J = \{J_\sigma\ ;\ \sigma \in X\}$ of X by*

$$J_\sigma = \bigcap_{\substack{\rho\in X'\\ \rho\subset\sigma}} (aI_\sigma + I'_\rho)$$

for $\sigma \in X$. Then, there exists a positive real number a_0 such that we have the equality $\mathrm{Fan}_X(J) = \mathrm{Fan}_{X'}(I')$ for any $a \geq a_0$. In particular, the fan $\mathrm{Fan}_X(J)$ does not depend on the choice of $a \geq a_0$. If I and I' are polyhedral ideals, then so is J.

First, we prove the following lemma.

LEMMA 3.2. *Let C be a polyhedral cone of $N_{\boldsymbol{R}}$, and $P \subset M_{\boldsymbol{R}}$ an r-dimensional $C^{\vee}$-convex polyhedron. We denote by $X_P := \mathrm{Fan}(P)$ the fan defined by P. Let $K = (K_\sigma)$ be a polyhedral fractional ideal of X_P. We define a $C^{\vee}$-convex set Q by*

$$Q = \bigcap_{\sigma\in X_P} (aP + K_\sigma)$$

for a positive real number a. Then there exists a positive real number a_1 such that the fan $X_Q := \mathrm{Fan}(Q)$ is equal to $\mathrm{Fan}_{X_P}(K)$ for any $a \geq a_1$. In particular, the fan X_Q does not depend on the choice of $a \geq a_1$.

PROOF. For each $\sigma \in X_P$, we take an element $y_\sigma \in P$ with $\sigma = (P - y_\sigma)^{\vee}$. Since $a(P - y_\sigma) \subset \sigma^{\vee}$ and $K_\sigma = K_\sigma + \sigma^{\vee}$, we have

$$aP + K_\sigma = ay_\sigma + a(P - y_\sigma) + K_\sigma = ay_\sigma + K_\sigma$$

for each σ. This implies that the support of the fan $\mathrm{Fan}(aP + K_\sigma)$ is σ, and the support of the fan X_Q defined by the intersection Q of these convex sets is C, i.e., the support of the fan X_P.

The support of $\mathrm{Fan}_{X_P}(K)$ is also C, since it is a subdivision of X_P. Hence, for the equality $X_Q = \mathrm{Fan}_{X_P}(K)$, it suffices to show that every $\eta \in \mathrm{Fan}_{X_P}(K)$ is a member of X_Q.

For each η, there exist $\sigma \in X_P$ and $z \in K_\sigma$ with $\eta = (K_\sigma - z)^\vee$. Let τ be an arbitrary element of X_P. Then the cones σ and τ are separated by $y_\tau - y_\sigma$. Actually, $\sigma \subset (y_\tau - y_\sigma \geq 0)$, since $y_\tau - y_\sigma \in P - y_\sigma$, while $\tau \subset (y_\tau - y_\sigma \leq 0)$, since $y_\sigma - y_\tau \in P - y_\tau$. We also have the equality $\sigma \cap (y_\tau - y_\sigma = 0) = \tau \cap (y_\tau - y_\sigma = 0)$, since the restrictions of $(P - y_\sigma)^\vee$ and $(P - y_\tau)^\vee$ to $(y_\tau - y_\sigma = 0)$ are equal. Hence $\boldsymbol{R}_0(y_\tau - y_\sigma) + K_\tau = K_\rho$ for $\rho = \tau \cap \sigma$. Since $K_\sigma \subset K_\rho$, the convex set $a(y_\tau - y_\sigma) + K_\tau$ contains a neighborhood U of z in K_σ for sufficiently large a. Then $ay_\sigma + z$ is an element of $ay_\tau + K_\tau$, and

$$((ay_\tau + K_\tau) - (ay_\sigma + z))^\vee \subset (U - z)^\vee = (K_\sigma - z)^\vee = \eta\,. \tag{6}$$

If we take the real number a sufficiently large for all $\tau \in X_P$, then $ay_\sigma + z$ is in Q. Since $(Q - (ay_\sigma + z))^\vee$ is equal to the sum of the first terms of (6) for all $\tau \in X_P$, it is equal to η for such a. Hence η is in X_Q.

Since $\mathrm{Fan}_{X_P}(K)$ is a finite fan, every $\eta \in \mathrm{Fan}_{X_P}(K)$ is in X_Q for sufficiently large $a \geq 0$. □

PROOF OF THEOREM 3.1. $\mathrm{Fan}_X(J)$ and $\mathrm{Fan}_{X'}(I')$ are subdivisions of X by definitions. We apply Lemma 3.2 to $P = I_\sigma$ and K, which is defined to be the restriction of J to $\{\rho \in X' \;;\; \rho \subset \sigma\}$. Then we know that $\mathrm{Fan}_X(J)$ and $\mathrm{Fan}_{X'}(I')$ are equal on the cone σ for sufficiently large $a \geq 0$. Since X is finite, there exists $a_0 \geq 0$ such that $\mathrm{Fan}_X(J) = \mathrm{Fan}_{X'}(I')$ for every $a \geq a_0$.

The last assertion of the theorem is clear from the first part.

3.4. Sums and intersections of ideals. Let $I = \{I_\sigma \;;\; \sigma \in X\}$ and $J = \{J_\sigma \;;\; \sigma \in X\}$ be polyhedral fractional ideals of X. The sum $I + J$ of these ideals is defined to be $\{I_\sigma + J_\sigma \;;\; \sigma \in X\}$. This is an analog of the product of fractional ideals of an integral scheme. Since $I_\sigma \cap J_\sigma$ is a $\sigma^\vee$-convex polyhedron, $I \cap J := \{I_\sigma \cap J_\sigma \;;\; \sigma \in X\}$ is also a fractional ideal of X.

For finite fans X, X', we define the *join* by

$$J(X, X') := \{\sigma \cap \tau \;;\; \sigma \in X, \tau \in X'\}\,.$$

Then $J(X, X')$ is also a finite fan, and dominates both X and X'. If a fan Y dominates both X and X', then the join $J(X, X')$ is also dominated by Y. The equality

$$\mathrm{ZR}(J(X, X')) = \mathrm{ZR}(X) \cap \mathrm{ZR}(X')$$

is checked easily.

PROPOSITION 3.3. *For polyhedral fractional ideals I, J of X, we have the equality*

$$\mathrm{Fan}_X(I + J) = J(\mathrm{Fan}_X(I), \mathrm{Fan}_X(J))\,.$$

PROOF. Since both fans are subdivisions of X, it suffices to show that they define the same subdivision on each $\sigma \in X$. We set $P = I_\sigma$ and $Q = J_\sigma$. For $\rho \in \mathrm{Fan}_X(I + J)$

contained in σ, there exists $z \in P + Q$ with $\rho = (P + Q - z)^{\vee}$. For $x \in P$ and $y \in Q$ with $z = x + y$, we have the equalities

$$(P + Q - z)^{\vee} = ((P - x) + (Q - y))^{\vee} = (P - x)^{\vee} \cap (Q - y)^{\vee} .$$

Hence $\rho \in J(\mathrm{Fan}_X(I), \mathrm{Fan}_X(J))$. By the same equalities, we know that each element of $J(\mathrm{Fan}_X(I), \mathrm{Fan}_X(J))$ is a member of $\mathrm{Fan}_X(I + J)$. □

For a polyhedral fractional ideal I, the polyhedral fractional ideal I^{-1} is defined by

$$I_{\sigma}^{-1} := \{x \in M_{\boldsymbol{R}} \ ; \ I_{\sigma} + x \subset \sigma^{\vee}\} .$$

Then $I + I^{-1}$ is a polyhedral ideal of X, since $I_{\sigma} + I_{\sigma}^{-1} \subset \sigma^{\vee}$ for every σ. The fan $\mathrm{Fan}_X(I + I^{-1})$ is a subdivision of $\mathrm{Fan}_X(I)$ by Proposition 3.3.

3.5. The maximal extension of an ideal. Let X be a fan, and U a subfan of X. For a polyhedral ideal I of U, there exists the largest polyhedral ideal I' of X with $I'|U = I$. It is obtained by setting

$$I'_{\sigma} := \sigma^{\vee} \cap \bigcap_{\eta \in F(\sigma) \cap U} I_{\eta}$$

for each $\sigma \in X$.

3.6. The primary decomposition. Let σ be an element of a fan X. A polyhedral ideal I of the affine fan $F(\sigma)$ is said to be *primary* if $\sigma^{\vee} \setminus I_{\sigma}$ is nonempty and $\sigma^{\perp}$-bounded, where we say that a subset $S \subset M_{\boldsymbol{R}}$ is *$\sigma^{\perp}$-bounded* if $S = S + \sigma^{\perp}$ and if the image of S in $M_{\boldsymbol{R}}/\sigma^{\perp}$ is bounded. In this case, the equality $I_{\eta} = \eta^{\vee}$ holds for any $\eta \in F(\sigma) \setminus \{\sigma\}$. Conversely, I is primary if it satisfies this condition. The ideal I is primary if and only if, for any $m \in M \cap (\sigma^{\vee} \setminus \sigma^{\perp})$, there exists a positive integer c with $cm \in I_{\sigma}$. The maximal extension of a primary polyhedral ideal I to X is called the *primary polyhedral ideal* of X at σ. If we denote also by I the the extended ideal, I_{ρ} is unitary unless $\sigma \prec \rho$. Namely, the support of the primary ideal is contained in the closure of $\{\sigma\}$ in X.

PROPOSITION 3.4. *Let I be a polyhedral ideal of X with support Y. Then there exists a set $\{I^{\sigma} \ ; \ \sigma \in X\}$ of polyhedral ideals of X, such that each I^{σ} is unitary on X or primary at σ and*

$$I = \bigcap_{\sigma \in X} I^{\sigma} = \bigcap_{\sigma \in Y} I^{\sigma} . \tag{7}$$

Here the right-hand side is essentially a finite intersection for each $\rho \in X$ even if X is not finite.

PROOF. Proposition 3.4 is equivalent to the assertion that there exists $\{P_{\sigma} \ ; \ \sigma \in X\}$ such that $\sigma^{\vee} \setminus P_{\sigma}$ is $\sigma^{\perp}$-bounded for every σ and

$$I_{\rho} = \bigcap_{\sigma \in F(\rho)} P_{\sigma}$$

for every $\rho \in X$. The construction of P_σ is done inductively from low dimensional cones. We set $P_\sigma = \sigma^\vee$ for σ outside Y. Assume that P_η is determined for every η in $F(\sigma) \setminus \{\sigma\}$. Then

$$I_\sigma \setminus \bigcap_{\eta \in F(\sigma)\setminus\{\sigma\}} P_\eta$$

is $\sigma^\perp$-bounded. Let $Q_1, \ldots, Q_s$ be the $\sigma^\perp$-bounded faces of codimension one of I_σ. We take $y_1, \ldots, y_s \in N_{\boldsymbol{R}}$ and $c_1, \ldots, c_s \in \boldsymbol{R}$ with

$$I_\sigma \subset (y_i \geq c_i)\ , \quad Q_i = I_\sigma \cap (y_i = c_i)\ , \quad i = 1, \ldots, s\ .$$

Then we have $y_1, \ldots, y_s \in \operatorname{rel.int} \sigma$. Hence

$$P_\sigma := \sigma^\vee \cap \bigcap_{i=1}^{s} (y_i \geq c_i)$$

satisfies the condition. Since I^σ is trivial for $\sigma \in X \setminus Y$, we get the last equality of (7).

The last assertion follows from the fact that $I^\sigma_\rho = \rho^\vee$ for $\sigma \notin F(\rho)$, □

3.7. Local blowups. Let I be a polyhedral ideal of a fan X, and U an open subset of X. For the support Y of I, we have a primary decomposition

$$I = \bigcap_{\sigma \in Y} I^\sigma\ ,$$

by Proposition 3.4. If we set

$$I' = \bigcap_{\sigma \in Y \cap U} I^\sigma\ ,$$

then I' and I are equal on U. On the other hand, if $\rho \in X$ is not contained in the closure of $Y \cap U$, then $I'_\rho = \rho^\vee$. Hence the blowup $\operatorname{Fan}_X(I')$ is equal to $\operatorname{Fan}_X(I)$ on U and to X on $X \setminus \overline{Y \cap U}$. This localization of the blowup is not possible in general for a polyhedral fractional ideal.

3.8. Some lemmas. Let $Q \subset M_{\boldsymbol{R}}$ be a rational convex polytope, i.e., a convex closure of a finite set of rational points. Then, for any polyhedral cone σ, $I(Q)_\sigma := Q + \sigma^\vee$ is a $\sigma^\vee$-convex subset.

Let X be a fan. Then $I(Q, X) := \{I(Q)_\sigma\ ;\ \sigma \in X\}$ is a polyhedral fractional ideal. Hence $\operatorname{Fan}_X(I(Q, X))$ is a subdivision of X. If Q is r-dimensional, then $\operatorname{Fan}(Q)$ is projective and $\operatorname{Fan}_X(I(Q, X))$ is the join $J(\operatorname{Fan}(Q), X)$ of $\operatorname{Fan}(Q)$ and X. In particular, $\operatorname{Fan}_X(I(Q, X))$ dominates the projective fan $\operatorname{Fan}(Q)$.

For the polyhedral ideal $I(Q, X)+I(Q, X)^{-1}$ of X, the fan $\operatorname{Fan}_X(I(Q, X)+I(Q, X)^{-1})$ is a subdivision of $\operatorname{Fan}_X(I(Q, X))$. In particular, it dominates both X and $\operatorname{Fan}(Q)$. Since $(I(Q, X) + I(Q, X)^{-1})_\gamma = \gamma^\vee$ for γ with $\dim \gamma \leq 1$, the support of this ideal consists of cones of dimension at least two. If σ is contained in both X and $\operatorname{Fan}(P)$, then the polyhedral ideals $I(Q, X)$ and $I(Q, X)^{-1}$ are unitary at σ. Hence the blowup

$$\operatorname{Fan}_X(I(Q, X) + I(Q, X)^{-1}) \to X \tag{8}$$

does not subdivide the cone σ. Since this is a blowup along a polyhedral ideal, local blowups are possible for any subfan of X.

LEMMA 3.5. *Suppose that $v \in \mathrm{ZR}(M)$ dominates τ and the first generalization w of v dominates σ. We define*

$$P := \mathrm{conv}\{m \in M \cap \tau^\vee \ ; \ m_0 \leq_v m\}$$

for an element $m_0 \in M \cap (\tau^\vee \cap \sigma^\perp)$. Then P defines a primary polyhedral ideal of $F(\tau)$.

PROOF. Clearly, P is a $\tau^\vee$-convex subset of $\tau^\vee$. It suffices to show that, for any element $m \in M \cap (\tau^\vee \setminus \tau^\perp)$, cm is in P for a sufficiently large integer c.

If $m \in M \cap (\tau^\vee \setminus \sigma^\perp)$, then $m_0 <_w m$, since $m_0 \in M \cap \sigma^\perp$. Hence $m_0 \leq_v m$ and $m \in P$.

Suppose $m \in M \cap \tau^\vee \cap (\sigma^\perp \setminus \tau^\perp)$. Since v dominates τ, $M \cap \tau^\vee \setminus \tau^\perp$ is a subset of $L(v) \setminus L^0(v)$. Hence $0 <_v m$. Let s be the rank of v, and $(y_0, \ldots, y_{s-1})$ a defining sequence of it. Then linear functions $y_0, \ldots, y_{s-2}$ are zero on $M \cap \sigma^\perp$, and

$$x_1 \leq_v x_2 \ \Leftrightarrow \ \langle x_1, y_{s-1}\rangle \leq \langle x_2, y_{s-1}\rangle$$

for $x_1, x_2 \in M \cap \sigma^\perp$. We have $\langle m, y_{s-1}\rangle > 0$, since $m \in M \cap \sigma^\perp$ and $0 <_v m$. Hence we have $\langle m_0, y_{s-1}\rangle \leq \langle cm, y_{s-1}\rangle$ for sufficiently large c. Then $m_0 \leq_v cm$ and $cm \in P$. □

LEMMA 3.6. *Let X be a fan and U an open subset of it. Let Y_1, Y_2 be closed subsets of U with $Y_1 \cap Y_2 = \emptyset$. For a polyhedral ideal I of X with the support $Y := \bar{Y}_1 \cap \bar{Y}_2$, let X' be the blowup of X at I. Then, if we regard U an open subset of X', then the closures of Y_1 and Y_2 in X' are disjoint.*

PROOF. Since Y does not intersect U, this blowup leaves U unchanged. It suffices to show that $\sigma \in Y_1$ and $\tau \in Y_2$ cannot be faces of a common element of X'. Since X is covered by affine fans, we may assume that $X = F(\pi)$ and that σ and τ are faces of π. Since σ and τ are outside Y, $I_\pi \cap \sigma^\perp$ and $I_\pi \cap \tau^\perp$ are nonempty. Let $\rho \in F(\pi)$ be the minimal face of π which contains σ and τ. Then I_π does not intersect $\pi^\vee \cap \rho^\perp$, since $\rho \in Y$. Since the intersection of $\pi^\vee \cap \sigma^\perp$ and $\pi^\vee \cap \tau^\perp$ is $\pi^\vee \cap \rho^\perp$, we have

$$(I_\pi \cap \sigma^\perp) \cap (I_\pi \cap \tau^\perp) = \emptyset\,.$$

Hence there is no cone in $X' = \mathrm{Fan}(I_\pi)$ which contains both σ and τ. □

4. Completions of fans. In this section, we assume that fans are rational, and we prove the following theorem. A similar theorem for not necessarily rational fans will be proved in the next section as the second proof of Theorem 1.1.

THEOREM 4.1. *Let X be a finite* (*rational*) *fan. Then there exists a complete fan X' such that X is a subfan of X'.*

Any affine fan has a completion by the following lemma.

LEMMA 4.2. *For a rational polyhedral cone σ, there exists a projective fan X which contains σ as an element.*

PROOF. First, we consider the case $\sigma = \mathbf{0}$. Let $\{m_1, \ldots, m_r\}$ and $\{n_1, \ldots, n_r\}$ be mutually dual basis of M and N. We set $n_0 := -(n_1 + \cdots + n_r)$. The fan Π_r is defined to be the set of cones generated by proper subsets of $\{n_0, n_1, \ldots, n_r\}$. This is equal to the fan $\mathrm{Fan}(P)$ for the convex closure P of $\{0, m_1, \ldots, m_r\}$. This is a complete fan and of course contains $\mathbf{0}$. The associated toric variety of this fan is the r-dimensional projective space (cf. [O1, p. 96]).

In case $\dim \sigma = r$, we take $n \in N \cap \operatorname{int} \sigma$ and set

$$Y := F(\sigma) \cup \{\eta + \boldsymbol{R}_0(-n) \ ; \ \eta \in F(\sigma) \setminus \{\sigma\}\} .$$

Then Y is equal to the projective fan $\mathrm{Fan}(P)$ for the r-dimensional convex polytope $P = \{x \in \sigma^\vee \ ; \ \langle x, n\rangle \leq 1\}$, and $\sigma \in Y$.

In the general case, let $s = \dim \sigma$. We take a decomposition $N = N' \oplus N''$ such that σ is a maximal dimensional cone in $N'_{\boldsymbol{R}}$. Then the product fan of a complete fan of $N'_{\boldsymbol{R}}$ which contains σ and Π_{r-s} of $N''_{\boldsymbol{R}}$ for a basis satisfies the condition. □

THEOREM 4.3. *The following conditions on a fan X are equivalent.*

(1) *X is complete, i.e., X is finite and $|X| = N_{\boldsymbol{R}}$.*

(2) *The equality* $\mathrm{ZR}(X) = \mathrm{ZR}(M)$ *holds.*

(3) *The equality* $\bigcup_{\sigma \in X} \operatorname{dom} \sigma = \mathrm{ZR}(M)$ *holds.*

PROOF. (2) and (3) are equivalent since $\mathrm{ZR}(X) = \bigcup_{\sigma \in X} \operatorname{dom} \sigma$ by Lemma 2.9.

Suppose (2) holds. Since $\{\|\sigma\| \ ; \ \sigma \in X\}$ is an open covering of $\mathrm{ZR}(M)$, there exists a finite subfan $X' \subset X$ with $\bigcup_{\sigma \in X'} \|\sigma\| = \mathrm{ZR}(M)$ by Theorem 2.5. Since $\mathrm{dom}(\sigma)$'s are nonempty and disjoint for $\sigma \in X$, we have $X' = X$. Hence X is finite. Take an arbitrary element $y \in N_{\boldsymbol{R}}$ and consider $v \in \mathrm{ZR}(M)^1$ with $L(v) = M \cap (y \geq 0)$. By assumption and Lemma 2.9, there exists a cone $\sigma \in X$ with $M \cap \sigma^\vee \subset L(v) \subset (y \geq 0)$. Then $y \in \sigma \subset |X|$. Hence $|X| = N_{\boldsymbol{R}}$ and X is complete.

Suppose (1) holds and take an arbitrary element $v \in \mathrm{ZR}(M)$. Let s be the rank of v and $(y_0, \ldots, y_{s-1})$ a defining sequence of v. Since $|X| = N_{\boldsymbol{R}}$, $z_\varepsilon := \sum_{i=0}^{s-1} \varepsilon^i y_i$ is contained in the relative interior of a cone of X for every $\varepsilon > 0$. Since X is a finite fan, there exist $\tau \in X$ and a sequence $\{\varepsilon_j\}$ of positive real numbers with the limit 0 such that all z_{ε_j}'s are contained in $\operatorname{rel.int} \tau$. Then $v \in \operatorname{dom} \tau$ by Lemma 2.6. This implies (3). □

For finite fans X, X', we denote by $X \cap X'$ the set of cones contained in both X and X'. Clearly, $X \cap X'$ is a subfan of both X and X'. We set

$$D_{X,X'} := \{\sigma \in X \ ; \ \text{there exists } \tau \in X' \text{ such that } \sigma \subset \tau\} .$$

This is the maximal subfan of X which dominates X'. Clearly, we have

$$D_{X,X'} \cap D_{X',X} = X \cap X' .$$

The fan

$$J(X, X') := \{\sigma \cap \tau \ ; \ \sigma \in X, \tau \in X'\}$$

defined in Section 3 is finite and dominates X and X'.

We say X' to be *quasi-dominant* over X if $J(X, X')$ is a subfan of X'. Then $J(X, X') = D_{X',X}$. If $\mathrm{ZR}(X) \subset \mathrm{ZR}(X')$ and X' is quasi-dominant over X, then $J(X, X')$ is a subfan of X' and is a subdivision of X. In particular, if a complete fan X^* is quasi-dominant over X, then a subfan of X^* is a subdivision of X.

More generally, we define the join $J(X_1, \dots, X_s)$ for a finite number of finite fans $X_1, \dots, X_s$ $(s \geq 1)$. Namely, we inductively define $J(X_1) := X_1$ and

$$J(X_1, \dots, X_i) = J(J(X_1, \dots, X_{i-1}), X_i)$$

for $i = 2, \dots, s$. It is easy to see that $J(X_1, \dots, X_s)$ is independent of the order of $X_1, \dots, X_s$.

As we mentioned in Section 3, $J(X, Y)$ is equal to $\mathrm{Fan}(P + Q)$ if $X = \mathrm{Fan}(P)$ and $Y = \mathrm{Fan}(Q)$ for convex polyhedra P and Q.

The following theorem is "Chow's lemma" for a fan.

THEOREM 4.4. *For a finite fan X, there exists a projective fan X^* such that a subfan X' of X^* is a subdivision of X, i.e., X^* is quasi-dominant over X.*

PROOF. Let $\{\sigma_1, \dots, \sigma_s\}$ be the set of maximal elements of X. By Lemma 4.2, there exists a projective fan X_i which contains σ_i as an element for each i. Then $X^* := J(X_1, \dots, X_s)$ is a projective fan. Clearly, each σ_i is a union of cones in X^*. Hence the set $X' \subset X^*$ of cones contained in one of σ_i's is a subdivision of X. □

THEOREM 4.5. *Let X_1, X_2 be finite fans and v an element of* $\mathrm{ZR}(X_1) \cap \mathrm{ZR}(X_2)$. *Then there exists a fan X_v with the following properties*: (1) *X_v is the blowup at a polyhedral ideal I of X_1.* (2) *The ideal I of* (1) *is unitary at D_{X_1,X_2} and hence $D_{X_1,X_2} \subset X_v$.* (3) *If v dominates $\tau_v \in X_v$ and $\tau_2 \in X_2$, then $\tau_v \subset \tau_2$.*

PROOF. We prove the theorem by induction on the rank of v. If $v = \eta(M)$, then v dominates $\mathbf{0}$ and $X_v := X_1$ satisfies the condition.

Assume that the rank is at least one. Let $\tau_1 \in X_1$ and $\tau_2 \in X_2$ be the cones dominated by v. If $\tau_1 \subset \tau_2$, then $X_v := X_1$ is enough. Hence we assume that τ_1 is not contained in τ_2. In particular, $\tau_1 \notin D_{X_1,X_2}$.

Let w be the first generalization of v. By Lemma 2.10, w dominates a face σ_1 of τ_1 and a face σ_2 of τ_2. Since $\mathrm{rank}(w) = \mathrm{rank}(v) - 1$, we can apply the induction assumption for w. Hence, by replacing X_1 by X_w, we may assume $\sigma_1 \subset \sigma_2$. Here, recall that the composite of blowups is a blowup by Theorem 3.1. Assume that the semigroup $M \cap \tau_2^\vee$ is generated by $m_1, \dots, m_s$. Since $\sigma_1 \subset \sigma_2 \subset \tau_2$, we have

$$M \cap \tau_2^\vee \subset M \cap \sigma_2^\vee \subset M \cap \sigma_1^\vee .$$

Hence, there exists $m_0 \in M \cap \mathrm{rel.\,int}(\tau_1^\vee \cap \sigma_1^\perp)$ with

$$m_1, \dots, m_s \in -m_0 + M \cap \tau_1^\vee$$

[O1, Prop. 1.3].

Let s be the rank of v and $(y_0, \dots, y_{s-1})$ a defining sequence of v. Then $P := \mathrm{conv}\{m \in M \cap \tau_1^\vee\ ;\ m_0 \leq_v m\}$ defines a primary polyhedral ideal of $F(\tau_1)$ by Lemma 3.5. Let $I(P)$ be the resulting primary polyhedral ideal of X_1, and X_v the blowup of X_1 at $I(P)$. Since $I(P)$ is unitary at cones which do not contain τ_1, it is unitary on D_{X_1,X_2} and $D_{X_1,X_2} \subset X_v$ is satisfied. Since $m_1, \ldots, m_s \in L(v)$ and $m_1 + m_0, \ldots, m_s + m_0 \in M \cap \tau_1^\vee$, we have $m_1, \ldots, m_s \in \boldsymbol{R}(P - m_0)$. The cone $\boldsymbol{R}(P - m_0)^\vee$ of X_v is contained in the cone τ_2, since $\{m_1, \ldots, m_s\}$ generates $\tau_2^\vee$. Since $M \cap \boldsymbol{R}(P - m_0) \subset L(v)$, v dominates a face τ_v of $\boldsymbol{R}(P - m_0)^\vee$ by Lemma 2.9. Clearly, $\tau_v \in X_v$ and $\tau_v \subset \tau_2$. □

THEOREM 4.6. *Let X, X' be finite fans. Then there exists a fan X^* with the following properties*: (1) *X^* is a blowup of X along a polyhedral ideal I.* (2) *I is unitary on $D_{X,X'}$ and $D_{X,X'} \subset X^*$.* (3) *X^* is quasi-dominant over X'.*

PROOF. For each element $v \in \mathrm{ZR}(J(X, X'))$, Theorem 4.5 says that there exists a blowup X_v of X at a polyhedral ideal which is unitary on $D_{X,X'}$ and the cone $\sigma_v \in X_v$ dominated by v is contained in some $\tau \in X'$. Since $\|\sigma_v\|$ is an open neighborhood of v, there exist $v_1, \ldots, v_s \in \mathrm{ZR}(J(X, X'))$ with

$$\mathrm{ZR}(J(X, X')) = \bigcup_{i=1}^{s} \|\sigma_{v_i}\|$$

by the compactness of $\mathrm{ZR}(J(X, X'))$. We set $X^* := J(X_{v_1}, \ldots, X_{v_s})$. Then X^* is a blowup of X at a polyhedral ideal which is unitary on $D_{X,X'}$ by Theorems 3.1 and 4.5. If $v \in \mathrm{ZR}(J(X, X'))$ dominates $\sigma \in X^*$, then σ is contained in some σ_{v_i} and hence in some $\tau \in X'$. Hence X^* is quasi-dominant over X'. □

THEOREM 4.7. *Let X be a finite fan and v an element of* $\mathrm{ZR}(M)$. *Then there exists a finite fan X' which contains X and satisfies $v \in \mathrm{ZR}(X')$. Furthermore, we can take X' so that $X' \setminus X$ is contained in a projective fan.*

PROOF. We prove the first assertion of the theorem by induction on the rank of v.

If $v \in \mathrm{ZR}(X)$, then $X' := X$ is enough. Hence we assume $v \notin \mathrm{ZR}(X)$. In particular, the rank is at least one. Let w be the first generalization of v. By the induction assumption, there exists a finite fan X'' which contains X and contains w in its Zariski-Riemann space. Hence by replacing X by X'', we may assume $w \in \mathrm{ZR}(X)$. Then w dominates a cone $\sigma \in X$.

Let X^* be a projective fan which contains σ. By using Theorem 4.6, we replace X^* by its blowup so that X^* is quasi-dominant over X. Since X^* is complete, a subfan $X_1 \subset X^*$ is a subdivision of X. Let $Z^* := X_1 \setminus (X \cap X^*)$, and let $\overline{Z^*}$ be its closure in X^*. Since X^* is complete, v dominates a cone ρ of X^*.

(1) First, we consider the case $\rho \notin \overline{Z^*}$. In this case, it suffices to show that $X' := X \cup (X^* \setminus \overline{Z^*})$ is a fan. Since $X^* \setminus \overline{Z^*}$ is a fan, it suffices to show that it satisfies the last condition of Proposition 2.7. Assume that $u \in \mathrm{ZR}(M)$ dominates $\tau \in X$ and $\eta \in X^* \setminus \overline{Z^*}$. Then, $\eta \in X_1$ since $u \in \mathrm{ZR}(X) = \mathrm{ZR}(X_1)$. Since $\eta \notin Z^* \subset \overline{Z^*}$, we have $\eta \in X \cap X^*$. Since X is a fan, η is equal to τ.

(2) Now, assume $\rho \in \overline{Z^*}$. $\rho \in X^* \setminus X_1$ by the assumption $v \notin \mathrm{ZR}(X)$. Since the generalization w of v dominates σ, we know that σ is a face of ρ. We take an element

$m_0 \in M \cap \mathrm{rel.\,int}(\rho^\vee \cap \sigma^\perp)$, and set

$$P := \mathrm{conv}\{m \in M \cap \rho^\vee\ ;\ m_0 \leq_v m\}\,.$$

Then P defines a polyhedral ideal of X^* primary at ρ. Let X^{**} be the blowup of X^* along this ideal. Note that this blowup does not change any cones which do not contain ρ. In particular, X is invariant by the blowup. Since $M \cap \boldsymbol{R}(P - m_0) \subset L(v)$ and $\boldsymbol{R}(P - m_0)^\vee$ is an element of X^{**}, v dominates a face τ of $\boldsymbol{R}(P - m_0)^\vee$. For the closure $\overline{Z^{**}}$ of $Z^{**} := Z^*$ in X^{**}, we will show $\tau \notin \overline{Z^{**}}$. Suppose that a face η of τ was contained in Z^*. Since $\eta \subset \rho$ and $\eta, \rho \in X^*$, η is a face of ρ. Since $\eta \in Z^*$ and $\sigma \in X \cap X^*$, η is not a face of σ. Hence m_0 is not in $\rho^\vee \cap \eta^\perp$ by [O1, Prop. A.6]. Since η corresponds to the face $P \cap \eta^\perp$ of P and since m_0 is not in $P \cap \eta^\perp \subset \rho^\vee \cap \eta^\perp$, η is not a face of $\boldsymbol{R}(P - m_0)^\vee$. Hence η is not a face of τ contrary to the assumption. Since $X \cap X^* = X \cap X^{**}$, we are reduced to the case (1) by replacing X^* by X^{**}.

For the last assertion, we take $\tau \in X'$ which is dominated by v. If we replace X' by $X \cup F(\tau)$, then $X' \setminus X$ is contained in a projective fan which contains τ (cf. Lemma 4.2). □

THEOREM 4.8. *Let X_1, X_2 be finite fans, and let $X := X_1 \cap X_2$. If $X_1 \setminus X$ is a subset of a projective fan X^*, then there exists a finite fan X_3 which contains X and satisfies* $\mathrm{ZR}(X_3) = \mathrm{ZR}(X_1) \cup \mathrm{ZR}(X_2)$.

PROOF. We show later that we can replace X_1, X_2, X^* by their subdivisions without shrinking $X = X_1 \cap X_2$ so that they satisfy the following conditions:

(1) X_1 is quasi-dominant over X_2, i.e., $U_1 = J(X_1, X_2)$ is an open subset of X_1.

(2) U_1 is a subdivision of an open subset $U_2 \subset X_2$.

(3) Let $W_1 := U_1 \setminus X$ and $Y := X \setminus (X \cap X^*)$, and let W_2 be the image of W_1 in U_2. Then the closures $\overline{W_2}$ and $Y_2 := \bar{Y}$ in X_2 are disjoint.

(4) X^* is quasi-dominant over X_2.

By the property (4), $U^* := J(X_2, X^*)$ is an open subset of X^*, and the natural map $\psi : U^* \to X_2$ is a subdivision. (1) and (3) imply $U_2 = X \cup W_2$ and Y is closed in U_2.

Since X^* is projective, there exists a blowup of X_2 of type (8) in Section 3.8. Namely, there exists a polyhedral ideal I of X_2 such that I is unitary on $X \cap X^*$ and the blowup of X_2 along I is a subdivision of U^*. Let I' be the maximal extension of the restriction of the ideal I to the open subset $X_2 \setminus (Y_2 \cup \overline{W_2})$ of X_2. I' is unitary on the open set U_2, since $U_2 \subset (X \cap X^*) \cup Y_2 \cup \overline{W_2}$. Let U_2^* be the blowup of X_2 along I', and $\phi : U_2^* \to X_2$ the subdivision map. Consider the two subdivision maps $U_2^* \setminus \phi^{-1}(Y_2) \to X_2 \setminus Y_2$ and $U^* \setminus \psi^{-1}(Y_2) \to X_2 \setminus Y_2$. Then we see that the former is a subdivision of the latter on $X_2 \setminus (Y_2 \cup \overline{W_2})$ and, on the contrary, the latter is a subdivision of the former on $U_2 \setminus Y$. Hence the restriction of the join $J(U_2^*, U^*)$ to the open set $X_2 \setminus Y_2$ can be patched with the restriction of U_2^* to the open set $X_2 \setminus \overline{W_2}$, and they form a subdivision V_2 of X_2. Since $U_1 \setminus Y \subset X^*$, we see that U^* and hence V_2 is equal to $U_1 \setminus Y$ over $U_2 \setminus Y$. On the other hand, since $X \subset X_2 \setminus \overline{W_2}$ and I' is unitary on $X \subset U_2$, X is contained in V_2. Hence the fan V_2 over X_2 is equal to U_1

over $U_2 = (U_2 \setminus Y) \cup X$, and we can patch it with X_1. The resulting fan X_3 satisfies the conditions of the theorem.

Now we prove that we can subdivide X_1 and X_2 so that the conditions (1) and (2) are satisfied. By Theorem 4.6, there exists a blowup X_2' of X_2 which is quasi-dominant over X_1 and contains D_{X_2,X_1}. Then

$$U_2 := \{\sigma \in X_2' \; ; \text{ there exists } \tau \in X_1 \text{ such that } \sigma \subset \tau\}$$

is an open subset of X_2' and is equal to $J(X_1, X_2')$. Again by Theorem 4.6, there exists a blowup X_1' of X_1 quasi-dominant over X_2' and $D_{X_1,X_2'} \subset X_1'$. We set

$$U_1 := \{\tau \in X_1' \; ; \text{ there exists } \rho \in X_2' \text{ such that } \tau \subset \rho\} .$$

Since $\mathrm{ZR}(U_1) = \mathrm{ZR}(X_1) \cap \mathrm{ZR}(X_2') = \mathrm{ZR}(U_2)$, U_1 is a subdivision of U_2. We replace X_1 by X_1' and X_2 by X_2'. Then the new $X = X_1 \cap X_2$ contains the original X. Since X_1' is a blowup of the original X_1 along a polyhedral ideal I which is unitary on X, we retain the relation $X_1 \setminus X \subset X^*$ if we replace X^* by its blowup at the maximal extension of $I|(X_1 \cap X^*)$.

Next, we make them satisfy (3) keeping (1) and (2). Since $W_1 \subset X_1 \setminus X \subset X_1 \cap X^*$, W_1 and Y are disjoint closed subsets of U_1. If $V := \overline{W_2} \cap Y_2$ is not empty, then this is a closed subset of X_2 contained in $X_2 \setminus U_2$. By Lemma 3.6, the closures of W_1 and Y are disjoint in the blowup X_2' of X_2 at V. (3) is satisfied if we replace X_2 by X_2'. Since the center of the blowup is outside U_2, X_1 is still quasi-dominant over X_2.

Finally, we make (4) satisfied. By Theorem 4.6, we can make X^* quasi-dominant over X_2 by a blowup. Here we can take the center of the blowup outside $X_1 \cap X^*$, since $v \in \mathrm{ZR}(M)$ dominates no cones in X_2 if it dominates a point σ in an open subset $X_1 \cap X^*$ of X^* and if $\sigma \in X_1 \setminus U_1$. Hence, there is no change in the relation $X_1 \setminus X \subset X^*$. The conditions (1) and (2) are independent of X^*. (3) is also satisfied, since the new Y is contained in the original Y. □

PROOF OF THEOREM 4.1. Let X be a finite fan. For any $v \in \mathrm{ZR}(M)$, there exists a fan X_v such that $X \subset X_v$, $v \in \mathrm{ZR}(X_v)$ and $X_v \setminus X$ is contained in a projective fan by Theorem 4.7. Since $\mathrm{ZR}(M)$ is compact by Theorem 2.5, there exist finite elements $v_1, \ldots, v_s \in \mathrm{ZR}(M)$ with

$$\mathrm{ZR}(M) = \mathrm{ZR}(X_{v_1}) \cup \cdots \cup \mathrm{ZR}(X_{v_s}) .$$

By applying Theorem 4.8, we construct inductively a sequence of finite fans $X_1 = X_{v_1}, X_2, \ldots, X_s$ with

$$\mathrm{ZR}(X_i) = \mathrm{ZR}(X_{v_1}) \cup \cdots \cup \mathrm{ZR}(X_{v_i})$$

and $X_{i-1} \cap X_{v_i} \subset X_i$ for $i = 2, \ldots, s$. Then $X' := X_s$ is complete by Theorem 4.3 and contains X.

5. Compactifications of real fans. In this section, we will prove the compactification theorem for real fans. Indeed, we prove it for $\boldsymbol{k}$-fans for an arbitrary subfield $\boldsymbol{k}$ of $\boldsymbol{R}$. In the case of rational fans, we used the lattice M for blowups. However, we cannot use it in the general case.

Let M and N be free $\boldsymbol{Z}$-modules of rank $r \geq 0$ as in the previous sections. We fix a subfield $\boldsymbol{k}$ of $\boldsymbol{R}$. Let $N_{\boldsymbol{k}} := N \otimes \boldsymbol{k}$ and $M_{\boldsymbol{k}} := M \otimes \boldsymbol{k}$. A cone σ in $N_{\boldsymbol{R}}$ is said to be a $\boldsymbol{k}$-cone if it is generated by a finite subset of $N_{\boldsymbol{k}}$. A real fan X is said to be a $\boldsymbol{k}$-fan if every $\sigma \in X$ is a $\boldsymbol{k}$-cone. In particular, $\boldsymbol{R}$-fans are real fans and $\boldsymbol{Q}$-fans are usual fans. Although our theory does not depend on the field $\boldsymbol{k}$, it is an interesting problem to find the properties of $\boldsymbol{k}$-fans which depend on the field $\boldsymbol{k}$.

We set $\boldsymbol{k}_0 := \{a \in \boldsymbol{k} \ ; \ a \geq 0\}$. Let $C \subset M_{\boldsymbol{R}}$ be the cone generated by a finite subset $\{x_1, \ldots, x_s\}$ of $M_{\boldsymbol{k}}$. Then $M_{\boldsymbol{k}} \cap C = \boldsymbol{k}_0 x_1 + \cdots + \boldsymbol{k}_0 x_s$ as easily shown by Carathéodory's theorem.

A preorder $\leq$ on $M_{\boldsymbol{k}}$ is said to be $\boldsymbol{k}$-*additive* if the following conditions are satisfied.

(1) For any $x, y \in M_{\boldsymbol{k}}$, either $x \leq y$ or $y \leq x$ is satisfied.

(2) $x \leq y$ and $y \leq z$ imply $x \leq z$.

(3) If $x \leq y$, then $x + z \leq y + z$ for every z.

(4) If $x \leq y$, then $ax \leq ay$ for every $a \in \boldsymbol{k}_0$.

We define the Zariski-Riemann space $\mathrm{ZR}(M_{\boldsymbol{k}})$ as the set of all $\boldsymbol{k}$-additive preorders of $M_{\boldsymbol{k}}$. We define the weakest topology on $\mathrm{ZR}(M_{\boldsymbol{k}})$ such that $\{v \in \mathrm{ZR}(M_{\boldsymbol{k}}) \ ; \ 0 \leq_v x\}$ is open for every $x \in M_{\boldsymbol{k}}$.

We define $L_{\boldsymbol{k}}(v) := \{x \in M_{\boldsymbol{k}} \ ; \ 0 \leq_v x\}$ and $L^0_{\boldsymbol{k}}(v) := \{x \in M_{\boldsymbol{k}} \ ; \ 0 =_v x\}$ for $v \in \mathrm{ZR}(M_{\boldsymbol{k}})$, where we write $x =_v y$ if $x \leq_v y$ and $y \leq_v x$. If a cone $C \subset M_{\boldsymbol{R}}$ is generated by $\{x_1, \ldots, x_s\} \subset L^0_{\boldsymbol{k}}(v)$, then it is easy to see from the above conditions that $M_{\boldsymbol{k}} \cap C$ is contained in $L^0_{\boldsymbol{k}}(v)$. We set $\|\sigma\|_{\boldsymbol{k}} := \{v \in \mathrm{ZR}(M_{\boldsymbol{k}}) \ ; \ M_{\boldsymbol{k}} \cap \sigma^\vee \subset L_{\boldsymbol{k}}(v)\}$ for a convex polyhedral $\boldsymbol{k}$-cone σ. Then the set of all subsets $\|\sigma\|_{\boldsymbol{k}}$ forms an open basis of the topology of $\mathrm{ZR}(M_{\boldsymbol{k}})$. For a $\boldsymbol{k}$-fan X, $\mathrm{ZR}(X)$ is defined as the the union of $\|\sigma\|_{\boldsymbol{k}}$ for $\sigma \in X$. The compactness of $\mathrm{ZR}(X)$ for a finite $\boldsymbol{k}$-fan X is proved similarly to Theorem 2.5.

We denote by η the trivial preorder in $\mathrm{ZR}(M_{\boldsymbol{k}})$ with $L_{\boldsymbol{k}}(\eta) = M_{\boldsymbol{k}}$.

LEMMA 5.1. *Let v be an element of* $\mathrm{ZR}(M_{\boldsymbol{k}}) \setminus \{\eta\}$. *Then the closure of* $\mathrm{conv}(L_{\boldsymbol{k}}(v))$ *in $M_{\boldsymbol{R}}$ is equal to that of $L_{\boldsymbol{k}}(v)$, and is a closed half space, i.e.,* $(x_0 \geq 0)$ *for an element $x_0 \in N_R$.*

PROOF. It suffices to show the convexity of the closure $\overline{L_{\boldsymbol{k}}(v)} \subset M_{\boldsymbol{R}}$ for the first part. Let x, y be elements of $\overline{L_{\boldsymbol{k}}(v)}$. Then there exist sequences $\{x_i\}$, $\{y_i\}$ in $L_{\boldsymbol{k}}(v)$ converging to x and y, respectively. We know $tx_i + (1-t)y_i \in L_{\boldsymbol{k}}(v)$ for $t \in \boldsymbol{k}$ with $0 \leq t \leq 1$ for all i. This implies that all the segments $\overline{x_i y_i}$ are in $\overline{L_{\boldsymbol{k}}(v)}$, and the limit segment $\overline{xy}$ is also in the closed set. Hence $\overline{L_{\boldsymbol{k}}(v)}$ is convex and equal to the closure of $\mathrm{conv}(L_{\boldsymbol{k}}(v))$.

We have $M_{\boldsymbol{k}} \cap (\mathrm{conv}(L_{\boldsymbol{k}}(v))) = L_{\boldsymbol{k}}(v)$ by Carathéodory's theorem. Since $v \neq \eta$, we know $\mathrm{conv}(L_{\boldsymbol{k}}(v)) \neq M_{\boldsymbol{R}}$ by this equality. Since $\overline{L_{\boldsymbol{k}}(v)}$ is the closure of this convex set by the first part, it is not equal to $M_{\boldsymbol{R}}$. Since $L_{\boldsymbol{k}}(v) \cup (-L_{\boldsymbol{k}}(v)) = M_{\boldsymbol{k}}$, we have $\overline{L_{\boldsymbol{k}}(v)} \cup (-\overline{L_{\boldsymbol{k}}(v)}) = M_{\boldsymbol{R}}$. Hence the closed convex cone $\overline{L_{\boldsymbol{k}}(v)}$ is a closed half space. □

By this lemma, the ranks and the defining sequences are defined for elements in $\mathrm{ZR}(M_{\boldsymbol{k}})$ as in Section 2. We say that an element $v \in \mathrm{ZR}(M_{\boldsymbol{k}})$ dominates a $\boldsymbol{k}$-cone σ if $M_{\boldsymbol{k}} \cap \sigma^\vee \subset L_{\boldsymbol{k}}(v)$ and $M_{\boldsymbol{k}} \cap \sigma^\vee \cap L^0_{\boldsymbol{k}}(v) = M_{\boldsymbol{k}} \cap \sigma^\perp$.

Theorem 4.5 modified for $\boldsymbol{k}$-fans is proved similarly. However, we need a modification for the part in which we used the lattice M. It suffices to show the following lemma.

LEMMA 5.2. *Assume that $v \in \mathrm{ZR}(M_{\boldsymbol{k}})$ dominates $\boldsymbol{k}$-cones τ_1, τ_2, and the first generalization w of v dominates a face σ_1 of τ_1 and a face σ_2 of τ_2. Furthermore, we assume that σ_1 is contained in σ_2. Then there exists a polyhedral ideal I of $F(\tau_1)$ primary at τ_1 with the following property. Let X' be the blowup of $F(\tau_1)$ along I. Then the cone $\tau_v \in X'$ dominated by v is contained in τ_2.*

PROOF. Let $\{x_1, \dots, x_t\} \subset M_{\boldsymbol{k}}$ be a set of generators of $\tau_2^\vee$. We may assume that $x_1, \dots, x_l$ are contained in $\tau_2^\vee \setminus \tau_2^\perp$ and $x_{l+1}, \dots, x_t$ are in $\tau_2^\perp$. Let s be the rank of v, and $(y_0, \dots, y_{s-1})$ a defining sequence of v. Then we have

$$L^0_{\boldsymbol{k}}(v) = \{x \in M_{\boldsymbol{k}} \,;\, \langle x, y_i\rangle = 0 \text{ for } i = 0, \dots, s-1\}\,.$$

Since v dominates τ_1, τ_2, the linear spaces $\tau_1^\perp$ and $\tau_2^\perp$ are contained in $L^0_{\boldsymbol{k}}(v)_{\boldsymbol{R}}$. Hence $L^0_{\boldsymbol{k}}(v)^\perp_{\boldsymbol{R}}$ is contained in $N(\tau_1)_{\boldsymbol{R}} := \tau_1 + (-\tau_1)$ and $N(\tau_2)_{\boldsymbol{R}} := \tau_2 + (-\tau_2)$. We take $x_0 \in M_{\boldsymbol{k}} \cap \mathrm{rel.\,int}(\tau_1^\vee \cap \sigma_1^\perp)$ so that $x_0 + x_i \in \tau_1^\vee$ for every $1 \le i \le t$. This is possible by the relation

$$\tau_2^\vee \subset \sigma_2^\vee \subset \sigma_1^\vee = \tau_1^\vee + (-\,\mathrm{rel.\,int}(\tau_1^\vee \cap \sigma_1^\perp))\,.$$

If we take $\varepsilon > 0$ sufficiently small, then $z = z_\varepsilon := \sum_{j=0}^{s-1} \varepsilon^j y_j$ is contained in $\mathrm{rel.\,int}\,\tau_1 \cap \mathrm{rel.\,int}\,\tau_2$ and $\langle x_i, z\rangle > 0$ for all $1 \le i \le l$. Now we take elements a, b of $\boldsymbol{k}$ such that $0 < a < b$, $\langle x_0, z\rangle < a$ and $\langle x_0 + x_i, z\rangle > b$ for $i = 1, \dots, l$. Since $z \in \mathrm{rel.\,int}\,\tau_1$, we see that $\tau_1^\vee \cap (z \le b)$ is $\tau_1^\perp$-bounded, where $(z \le b) = \{x \in M_{\boldsymbol{R}} \,;\, \langle x, z\rangle \le b\}$. We take a point z_0 in $N_{\boldsymbol{k}} \cap L^0_{\boldsymbol{k}}(v)^\perp_{\boldsymbol{R}}$ sufficiently near to $z \in L^0_{\boldsymbol{k}}(v)^\perp_{\boldsymbol{R}}$. Then z_0 is contained in $\mathrm{rel.\,int}\,\tau_1 \cap \mathrm{rel.\,int}\,\tau_2$ and satisfies $\langle x_0, z_0\rangle < a$ and

$$\tau_1^\vee \cap (z \ge b) \subset \tau_1^\vee \cap (z_0 \ge (a+b)/2) \subset \tau_1^\vee \cap (z > a)\,.$$

Let P be the convex hull of the union of

$$\tau_1^\vee \cap (z_0 \ge (a+b)/2) \quad\text{and}\quad \tau_1^\vee \cap (\{x_0\} + L^0_{\boldsymbol{k}}(v)_{\boldsymbol{R}})\,.$$

Note that $x_0 + x_i$ is contained in the set on the left hand side for $i = 1, \dots, l$ and in the set on the right hand side for $i = l+1, \dots, t$. Then P is a $\tau_1^\perp$-convex subset contained in τ_1, and $\tau_1^\vee \setminus P$ is τ_1-bounded, since it is contained in $\tau_1^\vee \cap (z \le b)$. Hence P defines a polyhedral ideal $I(P)$ primary at τ_1.

The fan X' obtained by the blowup of $F(\tau_1)$ along this ideal contains $\rho := (P - x_0)^\vee$ as an element. Then $\tau_v \subset \tau_2$ since $P \setminus \{x_0\}$ contains $\{x_0 + x_1, \dots, x_0 + x_t\}$, and hence $M_{\boldsymbol{k}} \cap \rho^\vee \subset L_{\boldsymbol{k}}(v)$. Hence v dominates a face τ_v of ρ. □

In order to prove the theorem analogous to Theorem 4.7 for $\boldsymbol{k}$-fans, it suffices to show the following lemma.

LEMMA 5.3. *Assume that $v \in \mathrm{ZR}(M_{\boldsymbol{k}})$ dominates a cone τ, and the first generalization w of v dominates a face σ of τ. Then there exists a polyhedral ideal I of $F(\tau)$ primary at τ with the following properties. For the blowup X of $F(\tau)$ along I, the cone $\tau_v \in X$ dominated by v satisfies $\tau_v \setminus \mathrm{rel.\,int}\,\tau \subset \sigma$.*

PROOF. Let $(y_0, \dots, y_{s-1})$ be a defining sequence of v. We take $\varepsilon > 0$ sufficiently small so that $z = z_\varepsilon$ is in $\operatorname{rel.int} \tau$. We take an arbitrary $x_0 \in M_{\boldsymbol{k}} \cap \operatorname{rel.int}(\tau^\vee \cap \sigma^\perp)$, $a \in \boldsymbol{k}$ greater than $\langle x_0, z\rangle$ and $b \in \boldsymbol{k}$ greater than a. As in the proof of Lemma 5.2, we take z_0 of $N_{\boldsymbol{k}} \cap L^0_{\boldsymbol{k}}(v)^\perp_{\boldsymbol{R}}$ sufficiently near z so that

$$\tau^\vee \cap (z \geq b) \subset \tau^\vee \cap (z_0 \geq (a+b)/2) \subset \tau^\vee \cap (z > a)\,.$$

We consider the primary polyhedral ideal $I(P)$ defined by the convex closure P of the union of $\tau^\vee \cap (z_0 \geq (a+b)/2)$ and $\tau^\vee \cap (\{x_0\} + \tau^\perp)$. We will show $(P - x_0)^\vee \setminus \operatorname{rel.int} \tau \subset \sigma$. Let u be a point of $(P - x_0)^\vee$ and ρ the minimal face of τ which contains u. If $u \notin \operatorname{rel.int} \tau$, then $\rho \neq \tau$ and $P \cap \rho^\perp$ is nonempty. Let y be an element in it. Then since $\langle y, u\rangle = 0$, $\langle y - x_0, u\rangle \geq 0$ and $\langle x_0, u\rangle \geq 0$, we have $\langle x_0, u\rangle = 0$. Hence $x_0 \in \rho^\perp$ and hence ρ is a face of σ. Since $M_{\boldsymbol{k}} \cap (P - x_0) \subset L_{\boldsymbol{k}}(v)$ by the construction of P, v dominates a face τ_v of $(P - x_0)^\vee$. Then the conditions of the lemma is satisfied for τ_v. □

Now the following theorem is proved similarly to Theorem 4.1.

THEOREM 5.4. *For an arbitrary finite **k**-fan X, there exists a complete finite **k**-fan X' with $X \subset X'$.*

If we set $\boldsymbol{k} := \boldsymbol{R}$, then we get the second proof of Theorem 1.1.

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