Tohoku Math. J.
53 (2001), 491–510

HYPERELLIPTIC VARIETIES

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(Received November 24, 1999)

Abstract. A hyperelliptic variety is by definition a complex projective variety, not isomorphic to an abelian variety, which admits an abelian variety as a finite étale covering. The main contribution of this paper is a classification of hyperelliptic threefolds.

Introduction. A *hyperelliptic surface* (in the sense of [BPV], [GH]) is a compact complex surface, not isomorphic to an abelian surface, which admits a finite étale covering by an abelian surface. These surfaces were classified by Enriques–Severi and Bagnera-de Franchis in their fundamental papers, for which they received the Bordin prize of the French Academy of Sciences in 1907 and 1908, respectively. There are exactly five one-dimensional and two two-dimensional families of such surfaces. It seems reasonable to define more generally a *hyperelliptic variety of dimension* n to be a complex projective variety, not isomorphic to an abelian variety, but admitting an abelian variety as a finite étale covering. It is the aim of this paper to classify hyperelliptic threefolds and to give many examples of hyperelliptic varieties in any dimensions.

The starting point of Enriques-Severi and Bagnera-de Franchis is a theorem saying that for any hyperelliptic surface S there is an abelian surface A admitting a finite group of biholomorphic maps Γ acting fixed point freely on A, such that S is isomorphic to a desingularization of A/Γ. Theorem 1.1 below implies that this result is valid for hyperelliptic varieties in any dimension n. Hence in order to classify hyperelliptic varieties it suffices to classify the pairs (A, Γ) with an abelian variety A and a finite group Γ acting holomorphically and fixed point freely on A.

In [UY] Uchida and Yoshihara showed with a very elegant group theoretical proof that in the threefold case any such group Γ is either cyclic of order 2, 3, 4, 5, 6, 8, 10, 12 or abelian of type $(2, 2)$, $(2, 4)$, $(2, 6)$, $(2, 12)$, $(3, 3)$, $(3, 6)$, $(4, 4)$, $(6, 6)$ or the dihedral group D_4 of order 8. Moreover they gave examples for these threefolds.

In this paper it is shown that the dihedral group D_4 does not occur in this list. So, if we call the finite group Γ *associated to* the hyperelliptic variety, we can say that any group Γ associated to a hyperelliptic threefold is abelian (see Theorem 6.1). For the remaining groups mentioned above we construct families of hyperelliptic varieties of dimension $n \geq 3$ associated to these groups, which in the threefold case comprise all such families. To be more precise, any biholomorphic map $f : A \to A$ of an abelian variety A can be uniquely written as $f = t_x \circ g$ with an automorphism g and a translation t_x of A. The elements g form a finite

2000 *Mathematics Subject Classification.* Primary 14J30; Secondary 14J50.

group of automorphisms G. We may assume that the groups Γ and G are isomorphic. In the threefold case we compute the list of all pairs (A, G), A an abelian threefold and G a finite group of automorphisms for which there exists a finite group Γ of biholomorphic maps of A acting fixed point freely on A and such that $\Gamma \simeq G$ under the canonical map $f = t_x \circ g \mapsto g$. In order to determine Γ one has to write down only the corresponding groups of translations. This is easy for any pair (A, G), but since there are many groups of such translations, the list would just be too long. So we omit writing down Γ.

The contents of the paper is as follows: In Section 1 we generalize the above mentioned theorem of Enriques-Severi and Bagnera-de Franchis to arbitrary dimensions. Section 2 gives some preliminary properties of hyperelliptic varieties. Sections 3 and 4 classify cyclic and abelian hyperelliptic varieties. In Section 5 all hyperelliptic varieties of type (2, 2) are constructed, and finally in Section 6 we complete classification of hyperelliptic threefolds.

Notation: If g is an endomorphism of an abelian variety $A = V/\Delta$, $\rho_a(g) : V \to V$ denotes the analytic representation of g. If K denotes an algebraic subgroup of A, K^0 denotes its connected component of 0, which is an abelian subvariety of A. Finally $A(n)$ denotes the group of n-division points of A for any integer $n \geq 2$.

I am grateful to the Mathematical Institute of Tohoku University for its hospitality during the preparation of this paper.

1. The Theorem of Enriques-Severi-Bagnera-de Franchis. In [ES] Enriques and Severi and in [BdF] Bagnera and de Franchis proved independently the following theorem: Let S be a smooth complex projective surface S, not rational and not an elliptic surface, admitting an abelian surface A as a finite cover. Then there exists a group Γ of biholomorphic maps of A onto A such that S is a desingularization of A/Γ. In this section this theorem will be generalized to varieties of arbitrary dimension in the following form:

THEOREM 1.1. *Let X be a compact normal complex space such that there exists a complex torus T and finite holomorphic map $\pi : T \to X$ of degree d ramified at most in codimension ≥ 2. Then there is a finite group Γ of biholomorphic maps of T onto T such that*

$$X \simeq T/\Gamma .$$

REMARK 1.2. Bagnera and de Franchis show that if the surface S is not rational and not elliptic, the map $\pi : A \to S$ is automatically ramified at most in finitely many points. So Theorem 1.1 may be considered as a direct generalization of the Theorem of Enriques-Severi-Bagnera-de Franchis.

For the proof we need some preliminaries. First of all, without loss of generality we may assume that $\pi : T \to X$ does not factorize via an isogeny $f : T \to T'$ of complex tori. Let $T = V/\Lambda$ with a complex vector space V of dimension n and a lattice $\Lambda \subseteq V$. Fix a point $x_1 \in X$ which is not a ramification point of π, let $t_1, \dots, t_d$ be its preimages in T and consider representatives $v_1, \dots, v_d$ of t_i in V. Choose open neighbourhoods X_1 of x_1 in X, disjoint open neighbourhoods T_i of t_i and V_i of v_i in V, such that

(i) $\pi|T_i \to X_1$ is biholomorphic, and

(ii) $p|V_i \to T_i$ is biholomorphic for all i.

For any open set $U \subset V$ let q_U denote the composite

$$q_U := \pi \circ p|U : U \to X .$$

We mostly consider open sets U such that q_U is biholomorphic onto its image such that its inverse $q_U^{-1} : q_U(U) \to U$ is well-defined. For abbreviation we write $q_i = q_{V_i}$ and $q = q_V$.

LEMMA 1.3. *The map $q_i^{-1} \circ q_1 : V_1 \to V_i$ extends to a biholomorphic map $\varphi_i : V \to V$.*

PROOF. Let $B \subset X$ be the ramification locus of π and $A = q^{-1}(B)$, such that $q : V - A \to X - B$ is étale. Let ℓ be a path in $V - A$ starting at v_1 and $U_0, \ldots, U_t$ any chain of overlapping open sets with the following properties:

(i) $U_0, \ldots, U_t$ are balls centered at ℓ.

(ii) $U_0 \subset V_1$ with center at v_1, U_t is centered at the endpoint of ℓ.

(iii) Let $X_i := \mathrm{Im}(q_i)$. Then $q_{V_i \cup V_{i+1}} : V_i \cup V_{i+1} \to X_i \cup X_{i+1}$ is biholomorphic.

Let $W_0 := q_i^{-1}(X_i) \subseteq V_i$. Inductively one sees for $j = 1, \ldots, t$ that there is a unique open set W_j in V such that

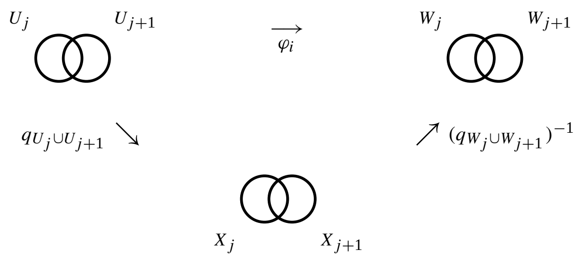
(i) $W_j \cap W_{j+1} \neq \emptyset$, and

(ii) $q_{W_i} : W_j \to X_j$ is biholomorphic for $j = 1, \ldots, t$.

Then we can define the holomorphic extension φ_i of $q_i^{-1} q_1 : V_1 \to V_i$ to $V_1 \cup (\bigcup_{j=1}^{t} U_j)$ to be the composite

$$U_j \xrightarrow{q_{U_j}} X_j \xrightarrow{q_{W_j}^{-1}} W_j$$

on U_j. So we have the following picture:



Doing this for every path in $V - A$ starting at v_1 and noting that $V - A$ is simply connected, since A is of codimension ≥ 2 in V, we see that $q_i^{-1} \circ q_1$ extends to a holomorphic map $\varphi_i : V - A \to V$. By Riemann's extension theorem φ_i extends holomorphically to $\varphi_i : V \to V$.

In the same way one shows that there is a holomorphic map $\psi_i : V \to V$ extending $q_1^{-1} \circ q_i : V_i \to V_1$. Since $\psi_i \varphi_i$ and $\varphi_i \psi_i$ are holomorphic extensions of the identity map on V_1 and V_i respectively, they are both the identity map on V. This implies that φ_i is biholomorphic. □

The map φ_i depends of course on the choice of the representative v_i of t_i. In particular, the maps $\varphi_1, \dots, \varphi_d$ do not necessarily form a group of biholomorphic maps $V \to V$. In order to obtain a group we consider all translations by lattice points of all maps φ_i. Define for $i = 1, \dots, d$ and all $\lambda \in \Lambda$ the biholomorphic map

$$\varphi_i^\lambda : V \to V \quad \text{by} \quad \varphi_i^\lambda(v) = \varphi_i(v) + \lambda\,,$$

i.e., $\varphi_i^\lambda = t_\lambda \circ \varphi_i$, where t_λ denotes the translation by λ.

LEMMA 1.4. *The maps φ_i^λ, $i = 1, \dots, d$, $\lambda \in \Lambda$, form a group Γ_0 of biholomorphic maps $V \to V$.*

PROOF. Consider the set $q^{-1}(x_1) = \{v_i + \lambda \mid i = 1, \dots, d,\ \lambda \in \Lambda\}$. Any biholomorphic map φ_i^λ induces a permutation of the set $q^{-1}(x_1)$. Moreover, by construction the map φ_i^λ is uniquely determined by the image of v_1 under this permutation and for every $\tilde{v} \in q^{-1}(x_1)$ there is exactly one biholomorphic map φ_i^λ such that $\varphi_i^\lambda(v_1) = \tilde{v}$. Hence, if $\varphi_i^{\lambda_1}$ and $\varphi_j^{\lambda_2}$ are two such biholomorphic maps, and if $\varphi_j^{\lambda_2} \circ \varphi_i^{\lambda_1}(v_1) = v_k + \lambda_3$, then we have

$$\varphi_j^{\lambda_2} \circ \varphi_i^{\lambda_1} = \varphi_k^{\lambda_3}\,.$$

Similarly, given $\varphi_i^{\lambda_1}$, if $v_k + \lambda_0$ is the element of $q^{-1}(x_1)$ with $\varphi_i^{\lambda_1}(v_k + \lambda_0) = v_1$, then

$$\left(\varphi_i^{\lambda_1}\right)^{-1} = \varphi_k^{\lambda_0}\,.$$

This implies the assertion. □

The maps φ_i^λ do not necessarily descend to biholomorphic maps $T \to T$. A necessary and sufficient condition for this is that $\varphi_i^\lambda(v_1 + \Lambda) \subseteq v_i + \Lambda$. But this need not be the case. However we have

LEMMA 1.5. *There is a positive integer m such that φ_i^λ descends to a biholomorphic map*

$$\overline{\varphi_i}^\lambda : V/m\Lambda \to V/m\Lambda$$

for $i = 1, \dots, d$ and all $\lambda \in \Lambda$.

PROOF. It suffices to show that for every $i = 1, \dots, d$ there is a positive integer m_i such that

$$\varphi_i(v_1 + m_i\Lambda) \subseteq v_i + m_i\Lambda\,. \tag{1.1}$$

For every $\lambda \in \Lambda$ the map φ_i induces a permutation σ_i^λ of the set $\pi^{-1}(x_1) = \{t_1, \dots, t_d\}$ defined by $p\varphi_i(v_j + \lambda) = t_{\sigma_i^\lambda(j)}$ for $j = 1, \dots, d$. If $\mathcal{S}_d$ denote the symmetric group on the set $\pi^{-1}(x_1)$, we obtain a permutation representation $\sigma_i : \Lambda \to \mathcal{S}_d$. Since $\operatorname{Im}\sigma_i$ is finite, there is a positive integer m_i such that

$$m_i\Lambda \subseteq \ker\sigma_i\,.$$

By construction m_i satisfies (1.1). □

PROOF OF THE THEOREM. The map $\Gamma_0 \to \text{Bihol}\,(V/m\Lambda, V/m\Lambda), \varphi_i^\lambda \mapsto \overline{\varphi_i}^\lambda$, is a homomorphism of groups. Let Γ denote its image. By construction Γ is a group of biholomorphic maps on $T = V/m\Lambda$. Since moreover $X = V/\Gamma_0$, we obtain $X = T/\Gamma$. □

2. Hyperelliptic varieties. A *hyperelliptic surface* is a complex projective surface, not isomorphic to an abelian surface, but admitting an abelian surface as an étale covering (see [GH], [BPV]). More generally, a *hyperelliptic variety of dimension* n is defined to be a complex projective variety, not isomorphic to an abelian variety, but admitting an abelian variety of dimension n as an étale covering. In this section some preliminaries on hyperelliptic varieties shall be given.

REMARK 2.1. (i) More generally a *hyperelliptic manifold* is defined to be a compact complex manifold, not isomorphic to a complex torus, but admitting a complex torus as an étale covering. There are no non-algebraic hyperelliptic surfaces. So, for dimension two both definitions coincide. However, for dimension three we will see examples of non-algebraic hyperelliptic manifolds in Remark 3.9. Since the classification of hyperelliptic threefolds given below only works in the algebraic case, we mainly stick to hyperelliptic varieties.

(ii) The notion of hyperelliptic varieties is not a generalization of the usual notion of a hyperelliptic curve. According to the genus formula of Riemann-Hurwitz there are no hyperelliptic varieties of dimension one in the above sense.

Let X be a hyperelliptic variety of dimension n. According to Theorem 1.1 there is an abelian variety A of dimension n and a finite group Γ acting holomorphically and fixed point freely on A such that $X \simeq A/\Gamma$. For every $\gamma \in \Gamma$ there is a unique decomposition

$$\gamma = t_a \circ g \tag{2.1}$$

with translation t_a, $a \in A$ and an automorphism g of A. This gives a map $\Gamma \to \text{Aut}(A)$, $\gamma \mapsto g$, which is easily seen to be a homomorphism. Let G denote its image in $\text{Aut}(A)$. Then there is an exact sequence

$$0 \to T \to \Gamma \to G \to 0\,,$$

where T denotes a finite group of translations. Passing to the quotient abelian variety $A' = A/T$, we may assume that $T = 0$, i.e., the map $\Gamma \to G$, $\gamma \mapsto g$, is an isomorphism. We call G the *group associated to the hyperelliptic variety* X.

For any abelian variety A there is a canonical exact sequence

$$0 \to A \xrightarrow{i} \text{Bihol}(A) \xrightarrow{p} \text{Aut}(A) \to 0\,, \tag{2.2}$$

where $\text{Bihol}(A)$ denotes the group of biholomorphic maps of A onto A, the map i is defined by $a \mapsto t_a$ and p is the canonical map derived from (2.1). Obviously (2.2) is a split exact sequence, i.e., $\text{Bihol}(A) = A \rtimes \text{Aut}(A)$. For any subgroup $G \subseteq \text{Aut}(A)$, the sequence (2.2) induces by pullback a split exact sequence

$$0 \to A \to \tilde{\Gamma} \xrightarrow{p} G \to 0\,. \tag{2.3}$$

Together with the above remarks this proves

PROPOSITION 2.2. *For any hyperelliptic variety X there is an abelian variety A, a finite group of automorphisms G of A and a section $\sigma : G \to \tilde{\Gamma}$ of* (2.3) *such that the group $\Gamma = \sigma(\tilde{\Gamma})$ acts fixed point freely on A and $X \simeq A/\Gamma$.*

It is well-known and easy to see that the set of sections $\sigma : G \to \tilde{\Gamma}$ of (2.3) is canonically in bijection to the set of cocycles $Z^1(G, A)$. The following proposition gives a criterion for the cocycle to yield a fixed point free action on A.

PROPOSITION 2.3. *Let $\sigma : G \xrightarrow{\sim} \Gamma \subseteq \tilde{\Gamma}$ be a section of* (2.3) *with corresponding cocycle $\varphi \in Z^1(G, A)$. The following conditions are equivalent*:

(i) *Γ acts fixed point freely on A.*

(ii) *The restriction of the cohomology class $\bar{\varphi} \in H^1(G, A)$ of φ to every nontrivial cyclic subgroup of G is nonzero.*

PROOF. Γ does not act fixed point freely if and only if there is a $g \in G$, $g \neq 1$ and an $a \in A$ such that $\sigma(g)(a) = a$. Since $\sigma(g) = t_{\varphi(g)} \circ g$, this means that there is a $g \in G$, $g \neq 1$ and an $a \in A$ such that $g(a) + \varphi(g) = a$ or equivalently $\varphi(g) = (1 - g)(a)$. This means that φ restricted to the cyclic subgroup generated by g is a coboundary. □

3. Cyclic hyperelliptic varieties. A hyperelliptic variety is called *cyclic* if the group G associated to it is a cyclic group. In this section we prove a theorem classifying cyclic hyperelliptic varieties and use it to determine all such varieties in low dimensions.

So, let X denote a cyclic hyperelliptic variety. We may assume that the cyclic covering $\pi : A \to X$ is minimal, that is, does not factor via an isogeny $A \to A'$ of abelian varieties. According to Proposition 2.2 there is a biholomorphic map $f : A \to A$ of order $d = \deg \pi$ such that

(i) f^ν admits no fixed point and is not a translation for $1 \leq \nu < d$, and

(ii) $X = A/\langle f \rangle$, where $\langle f \rangle$ denotes the group generated by f.

There is a unique decomposition

$$f = t_x \circ g \tag{3.1}$$

with a translation t_x and $g \in \mathrm{Aut}(A)$. This implies

$$f^\nu = t_{x+g(x)+\cdots+g^{\nu-1}(x)} \circ g^\nu \tag{3.2}$$

for all ν. In particular, g is an automorphism of order d of A and

$$\sum_{\nu=0}^{d-1} g^\nu(x) = 0\,. \tag{3.3}$$

To be more precise, $d-1$ is the smallest integer N such that $\sum_{\nu=0}^{N} g^\nu(x) = 0$, otherwise a power of f would not act fixed point freely.

(3.2) immediately implies

LEMMA 3.1. *The following conditions are equivalent*:

(i) *The group $\langle f \rangle$ acts fixed point freely on A.*

(ii) $\sum_{i=0}^{\nu-1} g^i(x) \notin \mathrm{Im}(1 - g^\nu)$ *for all* $\nu = 1, \ldots, d-1$.

We need the following lemma which is well-known and easy to see.

LEMMA 3.2. (a) *For any endomorphism α of A the addition map $\mu : (\mathrm{Ker}(\alpha))^0 \times \mathrm{Im}(\alpha) \to A$ is an isogeny.*

(b) $\alpha|\mathrm{Im}(\alpha) : \mathrm{Im}(\alpha) \to \mathrm{Im}(\alpha)$ *is an isogeny.*

Consider

$$B_1 := \mathrm{Ker}(1-g)^0 \quad \text{and} \quad B_2 := \mathrm{Im}(1-g)\,.$$

According to Lemma 3.2 the addition map $\mu : B_1 \times B_2 \to A$ is an isogeny. Moreover we have

LEMMA 3.3. (a) *B_1 and B_2 are abelian subvarieties of A of positive dimension.*

(b) *$f|B_1 : B_1 \to t_x B_1$ is a translation.*

(c) $g|B_1 = 1_{B_1}$.

(d) *$g|B_2$ is an automorphism of B_2 with finitely many fixed points.*

(e) *$B_1 \cap B_2 \subseteq \mathrm{Fix}(g|B_2)$ which is a finite set.*

(f) *$(1-g)|B_2$ is an isogeny of B_2.*

PROOF. (a) If $1-g$ is an isogeny, then f always admits fixed points according to Lemma 3.1. Hence $\dim B_1 > 0$. On the other hand, $1-g \neq 0$, otherwise f would be a translation. Hence $\dim B_2 > 0$. (b) and (c) are obvious.

(d) and (e). Suppose $x \in B_2$. There is a $y \in A$ with $x = y - g(y)$ implying $g(x) = (1-g)(g(y)) \in \mathrm{Im}(1-g) = B_2$. So $g|B_2$ is an automorphism of B_2, since g is injective as an automorphism of A. The fixed points of $g|B_2$ are just the points of the intersection $\mathrm{Ker}(1-g) \cap B_2$, which is finite. (f) is a consequence of Lemma 3.2 (b). □

Choose a decomposition $x = x_1 + x_2$ with $x_1 \in B_1$ and $x_2 \in B_2$, and define

$$\tilde{f} := t_{(x_1,x_2)} \circ (1 \times g)\,. \tag{3.4}$$

Then the following diagram is commutative

$$\begin{array}{ccc} B_1 \times B_2 & \xrightarrow{\tilde{f}} & B_1 \times B_2 \\ \mu\downarrow & & \downarrow\mu \\ A & \xrightarrow{f} & A \end{array}$$

LEMMA 3.4. *Let T denote the group of translations $t_{(y,-y)}$ of $B_1 \times B_2$ with $y \in B_1 \cap B_2$ and $G := \langle \tilde{f} \rangle \oplus T$. Then*

$$X \simeq B_1 \times B_2/G\,.$$

PROOF. It suffices to show that $t_{(y,-y)}$ and $\tilde{f}$ commute for any $y \in B_1 \cap B_2$, which is an immediate computation using $y \in \mathrm{Fix}(g|B_2)$. □

The decomposition (3.4) depends of course on the choice of the group structure, i.e., on the choice of the zero point of $B_1 \times B_2$. The following lemma uses this fact in order to normalize the decomposition.

LEMMA 3.5. *Choosing a suitable zero point of $B_1 \times B_2$, we may assume that*

$$\tilde{f} = t_{(x_1,0)} \circ (1 \times g)$$

with $x_1 \in B_1$ as above and $g \in \mathrm{Aut}(B_2)$.

PROOF. Let $\tilde{f} = t_{(x_1,x_2)} \circ (1 \times g) = (f_{ij})_{i,j=1,2} : B_1 \times B_2 \to B_1 \times B_2$ be as in (3.4). Then $f_{22} : B_2 \to B_2$ is given by $f_{22}(z) = g(z) + x_2$. Let $y_0 \in B_2$ with $(1 - g)(y_0) = x_2$. Then

$$f_{22}(y_0) = g(y_0) + x_2 = y_0 .$$

Choose $(0, y_0)$ as the new zero point of $B_1 \times B_2$ and let $\tilde{f} = t_{(x_1',x_2')} \circ (1 \times \tilde{g})$ be the decomposition of $\tilde{f}$ with respect to $(0, y_0)$. We have to compute (x_1', x_2'). But $\tilde{g} = t_{y_0} \circ g \circ t_{-y_0}$ and hence

$$\begin{aligned} t_{(x_1,0)}(1 \times \tilde{g})(z_1, z_2) &= (z_1 + x_1, g(z_2) - g(y_0) + y_0) \\ &= (z_1 + x_1, g(z_2) + x_2) \\ &= \tilde{f}(z_1, z_2) . \end{aligned}$$

□

The elements of $G = \langle \tilde{f} \rangle \oplus T$ are of the form $t_{(x_1,x_2)} \circ (1 \times g^i)$ with $0 \leq i < d$ and $x_1 \in B_1$, $x_2 \in B_2$ torsion points. To be more precise, $x_2 = 0$ if $i > 0$ and $x_2 = -x_1 \in B_1 \cap B_2$ if $i = 0$. In particular, we have a well-defined map

$$\varphi : G \to B_1, \quad t_{(x_1,x_2)} \circ (1 \times g^i) \mapsto x_1 .$$

LEMMA 3.6. *$\varphi : G \to B_1$ is an injective homomorphism of groups.*

PROOF. $t_{(x_1,x_2)}(1 \times g^j) \circ t_{(y_1,y_2)}(1 \times g^i) = t_{(x_1+y_1,x_2+g^i(y_2))}(1 \times g^{i+j})$, and hence φ is a homomorphism of groups. Any element of $\mathrm{Ker}(\varphi)$ is of the form $t_{(0,x_2)}(1 \times g^i)$. If $0 < i < d$, then this element has fixed points, and hence is not contained in G. □

COROLLARY 3.7. *The group of translations T is a finite abelian group with $\leq 2 \dim B_1 - 1$ generators.*

This follows from the fact that any finite torsion subgroups of B_1 is generated by $\leq 2 \dim B_1$ elements. □

According to Lemma 3.6 we may consider G as a subgroup of B_1. Moreover the map

$$\rho : G \to \mathrm{Bihol}(B_2), \quad x_1 \mapsto t_{x_2} g^i ,$$

if $\varphi^{-1}(x_1) = t_{(x_1,x_2)}(1 \times g^i)$, is a faithful representation. Combining everything we have:

THEOREM 3.8. *For a variety X of dimension n the following statements are equivalent*:

(i) *X is a cyclic hyperelliptic variety.*

(ii) *There are abelian varieties* B_1 *of dimension* $0 < n_1 < n$, B_2 *of dimension* $n_2 = n - n_1$, *a finite subgroup* $G = \langle x_1 \rangle \oplus T$ *of* B_1, *and a faithful representation* $\rho : G \to \mathrm{Bihol}(B_2)$ *such that*

(a) $g = \rho(x_1)$ *is an automorphism of* B_2 *of order* $d \geq 2$ *with* $\mathrm{Fix}(g)$ *finite, and*

(b) $\rho(T)$ *is a group of translations of* B_2 *by elements of* $\mathrm{Fix}(g)$.

If an action of G *on* $B_1 \times B_2$ *is defined by* $(x, (z_1, z_2)) \to (t_x(z_1), \rho(x)z_2)$, *then* $X \simeq B_1 \times B_2/G$.

PROOF. It remains to show that under condition (ii) the group G acts fixed point freely on $B_1 \times B_2$. Let $\tilde{f} : B_1 \times B_2 \to B_1 \times B_2$ be defined by $\tilde{f} = t_{(x_1,0)} \circ (1 \times g)$, and $\tilde{T} = \{(x, \rho(x)(0)) \in B_1 \times B_2 | x \in T\}$ considered as a group of translations of $B_1 \times B_2$. Then any element of $\langle \tilde{f} \rangle \oplus \tilde{T} \simeq G$ is of the form

$$\tilde{f}^i \circ t_{(x,\rho(x)(0))}$$

with $0 \leq i \leq d-1$ and $x \in T$, and we have

$$\tilde{f}^i \circ t_{(x,\rho(x)(0))}(z_1, z_2) = (z_1 + x + ix_1, g^i(z_2) + \rho(x)(0)) .$$

To see this one has to use that $\rho(x)(0) \subseteq \mathrm{Fix}(g)$. Now suppose $(b_1, b_2) \in B_1 \times B_2$ is a fixed point of $\tilde{f}^i \circ t_{(x,\rho(x)(0))}$. This implies $b_1 + x + ix_1 = b_1$ and thus $x = -ix_1 \neq 0$ contradicting the fact that $\langle x_1 \rangle \oplus T$ is a direct sum in B_1. □

Since all automorphisms of finite order of abelian varieties of dimension ≤ 3 are well-known, one can use Theorem 3.8 to give a list of all cyclic hyperelliptic varieties with $\dim(B_2) = r \leq 3$. For this one has only to give a list of all automorphisms of finite order of abelian varieties of dimension r together with their fixed point sets $\mathrm{Fix}(g)$. In order to define a hyperelliptic variety one has to give only an abelian variety B_1 an element $x_1 \in B_1$ of order $d = \mathrm{ord}(g)$, a finite subgroup $T \subset B_1$ not intersecting the group $\langle x_1 \rangle$ and an embedding $T \hookrightarrow \mathrm{Fix}(g)$. Tables 1 and 2 below give the triples $(B_2, g, \mathrm{Fix}(g))$ for all cyclic hyperelliptic varieties $\dim(B_2) = 1$ and 2. From this it is easy to work out the other data. The notation will be explained after the tables.

TABLE 1. $\dim(B_2) = 1$.

d	B_2	g	$\mathrm{Fix}(g)$
2	E	-1	$E(2)$
3	E_ρ	ρ	$\{\frac{\nu}{3}(2+\rho) \mid \nu = 0, 1, 2\}$
4	E_i	i	$\{\frac{\nu}{2}(1+i) \mid \nu = 0, 1\}$
6	E_ρ	$-\rho$	$\{0\}$

TABLE 2. $\dim(B_2) = 2$.

d	B_2	g	$\mathrm{Fix}(g)$
2	S	-1	$S(2)$
3	$\boldsymbol{C}^2/\Pi_1\boldsymbol{Z}^4$	$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$	$\left\{\begin{pmatrix} x \\ -x \end{pmatrix} \middle\vert 3\begin{pmatrix} x \\ -x \end{pmatrix} \in \Pi_1\boldsymbol{Z}\right\}$
3	$E_\rho \times E_\rho$	$\begin{pmatrix} \rho & 0 \\ 0 & \rho^j \end{pmatrix} j = 1, 2$	$\mathrm{Fix}(\rho) \times \mathrm{Fix}(\rho)$
4	$\boldsymbol{C}^2/\Pi_2\boldsymbol{Z}^4$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\left\{\begin{pmatrix} x \\ -x \end{pmatrix} \middle\vert 2\begin{pmatrix} x \\ -x \end{pmatrix} \in \Pi_2\boldsymbol{Z}\right\}$
4	$E_i \times E_i$	$\begin{pmatrix} i & 0 \\ 0 & i^j \end{pmatrix} j = 1, 3$	$\mathrm{Fix}(i) \times \mathrm{Fix}(i)$
4	$E \times E_i$	$\begin{pmatrix} -1 & 0 \\ 0 & i \end{pmatrix}$	$E(2) \times \mathrm{Fix}(i)$
5	S_5	$\begin{pmatrix} \xi_5 & 0 \\ 0 & \xi_5^3 \end{pmatrix}$	$\left\{\frac{1}{5}(1 + 2\xi_5^\nu + 3\xi_5^{2\nu} + 4\xi_5^{3\nu}) \vert \nu = 0, \dots, 4\right\}$
6	$\boldsymbol{C}^2/\Pi_1\boldsymbol{Z}^4$	$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$	$\{0\}$
6	$E_\rho \times E_\rho$	$\begin{pmatrix} -\rho & 0 \\ 0 & -\rho^j \end{pmatrix} j = 1, 2$	$\{0\}$
6	$E_\rho \times E_\rho$	$\begin{pmatrix} \rho & 0 \\ 0 & -\rho \end{pmatrix}$	$\mathrm{Fix}(\rho) \times \{0\}$
6	$E_\rho \times E_\rho$	$\begin{pmatrix} \rho & 0 \\ \pm\rho & -\rho \end{pmatrix}$	$\{0\}$
6	$E_\rho \times E$	$\begin{pmatrix} -\rho & 0 \\ 0 & -1 \end{pmatrix}$	$\{0\} \times E(2)$
6	$E_\rho \times E$	$\begin{pmatrix} \rho & 0 \\ 0 & -1 \end{pmatrix}$	$\mathrm{Fix}(\rho) \times E(2)$
8	$E_i \times E_i$	$\begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}$	$\{(x, x) \vert x \in \mathrm{Fix}(i)\}$
8	$E_{\sqrt{-2}} \times E_{\sqrt{-2}}$	$\begin{pmatrix} \sqrt{-2} & -1 \\ -1 & 0 \end{pmatrix}$	$\{\frac{i}{2}, 0\}$
10	S_5	$\begin{pmatrix} \xi_{10} & 0 \\ 0 & \xi_{10}^3 \end{pmatrix}$	$\{0\}$
12	$E_i \times E_i$	$\begin{pmatrix} 0 & i \\ -i & -i \end{pmatrix}$	$\{0\}$
12	$E_\rho \times E_\rho$	$\begin{pmatrix} 0 & \rho \\ -\rho & 0 \end{pmatrix}$	$\{0\}$
12	$E_i \times E_\rho$	$\begin{pmatrix} i & 0 \\ 0 & \rho \end{pmatrix}$	$\mathrm{Fix}(i) \times \mathrm{Fix}(\rho)$
12	$E_i \times E_\rho$	$\begin{pmatrix} i & 0 \\ 0 & -\rho \end{pmatrix}$	$\mathrm{Fix}(i) \times \{0\}$

Here E (resp. S) is an arbitrary elliptic curve (resp. abelian surface), $E_\tau = \boldsymbol{C}/(1, \tau)\boldsymbol{Z}$ for any τ in the upper half plane, ξ_d denotes a primitive d-th root of unity, and we abbreviate $i = \xi_4$, $\rho = \xi_3$. Moreover, S_5 denotes the abelian surface of CM-type $(\boldsymbol{Q}(\xi_5), (\xi_5, \xi_5^2))$, and

Π_1 and Π_2 the period matrices

$$\Pi_1 = \begin{pmatrix} 1 & 0 & x & y \\ 0 & 1 & -y & x+y \end{pmatrix} ; \ \Pi_2 = \begin{pmatrix} 1 & 0 & x & y \\ 0 & 1 & -y & x \end{pmatrix}$$

with $(x, y) \in \boldsymbol{C}^2 - \boldsymbol{R}^2$, defining abelian varieties.

Table 2 uses heavily Fujiki's list of automorphisms of abelian surfaces [F].

REMARK 3.9. (a) The quotient $\boldsymbol{C}^2/\Pi_i\boldsymbol{Z}^4$ is a complex torus for every $(x, y) \in \boldsymbol{C}^2 - \boldsymbol{R}^2$, but not always an abelian surface. In the cases where it is not, the quotient $X = B_1 \times B_2/\Gamma$ is a non algebraic hyperelliptic surface. Since the existence of non algebraic hyperelliptic varieties contradicts Theorem II of [J], we give an explicit example: The complex torus $B_2 = \boldsymbol{C}^2/\Pi\boldsymbol{Z}^4$ with

$$\Pi = \begin{pmatrix} 1 & i & 0 & \alpha \\ 0 & 0 & 1 & i \end{pmatrix}$$

fits into an exact sequence $0 \to E_i \to B_2 \to E_i \to 0$. If $\alpha \notin \boldsymbol{Q}(i)$, then E_i is the only nontrivial complex subtorus of S (see [BL] Section 1.6). In particular, B_2 is a non algebraic complex torus. Its automorphism group is isomorphic to $\boldsymbol{Z}/4\boldsymbol{Z}$ and generated by $g = \left(\begin{smallmatrix} i & -\alpha \\ 0 & -i \end{smallmatrix}\right)$ and $\mathrm{Fix}(g) = \{0, \overline{(x, y)}\}$ with $x = -(i/2)\alpha$, $y = (1-i)/2$. Choosing an elliptic curve B_1 and a 4-division point $x_1 \in B_1$, the group G acts on $B_1 \times B_2$ by $t_{x_1} \circ (1 \times g)$ and its quotient is a non algebraic hyperelliptic threefold.

(b) The automorphism of 3-dimensional abelian varieties were classified in [BGL]. Therefore one could also write a table of cyclic hyperelliptic varieties of order 3. This will be omitted, since the table would be too long.

4. Abelian hyperelliptic varieties. A hyperelliptic variety is called *abelian* if its associated group G is an abelian group. In this section we prove a theorem describing such varieties for abelian groups with two generators, which allows to construct abelian hyperelliptic varieties in any dimensions and will be applied in Section 6 to give a list of all abelian hyperelliptic threefolds.

Let A be an abelian variety of dimension $n(\geq 3)$ and G a group of automorphisms of A isomorphic to $\boldsymbol{Z}/d_1\boldsymbol{Z} \oplus \boldsymbol{Z}/d_2\boldsymbol{Z}$ with $d_1|d_2$. Suppose Γ is a group of biholomorphic maps of A, isomorphic to G and acting fixed point freely on A. Then

$$X := A/\Gamma$$

is an abelian hyperelliptic variety associated to the group G. We call it *of type* (d_1, d_2). Let g_1 and g_2 be automorphisms of A of order d_1 and d_2 generating G, and f_1, f_2 the corresponding generators of Γ

$$f_1 = t_x \circ g_1 \quad \text{and} \quad f_2 = t_{x'} \circ g_2 \,.$$

Define abelian subvarieties A_i by

$$A_1 = (\mathrm{Ker}(1-g_1) \cap \mathrm{Ker}(1-g_2))^0 \,, \quad A_2 = (\mathrm{Ker}(1-g_1) \cap \mathrm{Im}(1-g_2))^0 \,,$$
$$A_3 = (\mathrm{Im}(1-g_1) \cap \mathrm{Ker}(1-g_2))^0 \,, \quad A_4 = (\mathrm{Im}(1-g_1) \cap \mathrm{Im}(1-g_2))^0 \,.$$

Applying Lemma 3.2 twice, we conclude that the addition map induces an isogeny $A_1 \times A_2 \times A_3 \times A_4 \to A$. In this section we assume that $B_1 := A_1$ is positive dimensional. If B_2 denotes the image of the induced map $A_2 \times A_3 \times A_4 \to A$, we obtain that the addition map induces an isogeny

$$\mu : B_1 \times B_2 \to A\,.$$

The maps g_1 and g_2 restrict to the identity on B_1 and to automorphisms of B_2, which we also denote by g_1 and g_2.

LEMMA 4.1. $B_1 \cap B_2 \subset \mathrm{Fix}(g_1|B_2) \cap \mathrm{Fix}(g_2|B_2)$ *which is finite.*

PROOF. The only nontrivial assertion is that $\mathrm{Fix}(g_1|B_2) \cap \mathrm{Fix}(g_2|B_2)$ is a finite set. But

$$\mathrm{Fix}(g_1|B_2) \subset (\mathrm{Ker}(1-g_1) \cap \mathrm{Im}(1-g_1)) \cup (\mathrm{Ker}(1-g_1) \cap \mathrm{Im}(1-g_2))\,.$$

The first set is finite by Lemma 3.2(a) and the second is a union of finitely many translates of A_2. Similarly,

$$\mathrm{Fix}(g_2|B_2) \subset (\mathrm{Ker}(1-g_2) \cap \mathrm{Im}(1-g_2)) \cup (\mathrm{Ker}(1-g_2) \cap \mathrm{Im}(1-g_1))\,.$$

Again the first set is finite by Lemma 3.2(a) and the second is a union of finitely many translates of A_3. Hence the assertion follows from the fact that $A_2 \cap A_3$ is finite. □

Consider decompositions

$$x = x_1 + x_2\,, \qquad x' = x'_1 + x'_2$$

with $x_1, x'_1 \in B_1$ and $x_2, x'_2 \in B_2$.

LEMMA 4.2. (a) $d_1 x_1 = -\sum_{\nu=0}^{d_1-1} g_1^{\nu}(x_2) \in B_1 \cap B_2$.
(b) $d_2 x'_1 = -\sum_{\nu=0}^{d_2-1} g_2^{\nu}(x'_2) \in B_1 \cap B_2$.

PROOF. According to Equation (3.3), $\sum_{\nu=0}^{d_1-1} g_1^{\nu}(x) = 0$. This implies (a), since $g_1(x_1) = x_1$. The proof of (b) is the same. □

Define biholomorphic maps $\tilde{f}_1$ and $\tilde{f}_2 : B_1 \times B_2 \to B_1 \times B_2$ by

$$\tilde{f}_1 := t_{(x_1,x_2)} \circ (1 \times g_1) \quad \text{and} \quad \tilde{f}_2 := t_{(x'_1,x'_2)} \circ (1 \times g_2)\,.$$

For $i = 1$ and 2 the following diagram is commutative

$$\begin{array}{ccc} B_1 \times B_2 & \xrightarrow{\tilde{f}_i} & B_1 \times B_2 \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\mu} \\ A & \xrightarrow{f_i} & A \end{array}$$

LEMMA 4.3. (a) $\tilde{f}_1^{d_1} = t_{(d_1 x_1, \sum_{\nu=0}^{d_1-1} g_1^{\nu}(x_2))}, \ \tilde{f}_2^{d_2} = t_{(d_2 x'_1, \sum_{\nu=0}^{d_2-1} g_2^{\nu}(x'_2))}$.
(b) $(1-g_1)(x'_2) = (1-g_2)(x_2)$.

PROOF. (a) is a consequence of $f_i^{d_i} = 1$ for $i = 1, 2$. For (b) note that $f_1 f_2 = f_2 f_1$ implies $g_1(x') + x = g_2(x) + x'$. This gives the assertion since $g_1(x_1') = x_1'$ and $g_2(x_1) = x_1$. □

LEMMA 4.4. *Let T denote the group of translations $t_{(b,-b)}$ of $B_1 \times B_2$ with $b \in B_1 \cap B_2$. Then the subgroup $\tilde{\Gamma}$ of* Bihol$(B_1 \times B_2)$ *generated by $\tilde{f}_1, \tilde{f}_2$ and T is a finite abelian group with*

$$X \simeq B_1 \times B_2/\tilde{\Gamma}\,.$$

PROOF. It suffices to show that $\tilde{f}_1, \tilde{f}_2$ and T commute. But this is an immediate computation using Lemma 4.3 and $g_1(b) = g_2(b) = b$. □

Combining everything we have proved part of the following theorem.

THEOREM 4.5. *For a variety X of dimension $n (\geq 3)$ and positive integers d_1, d_2 with $d_1 | d_2$ the following statements are equivalent*:

(i) *X is an abelian hyperelliptic variety of type (d_1, d_2).*

(ii) *There are*

- *abelian varieties B_1 and B_2 with* $\dim B_1 + \dim B_2 = n$,
- *commuting automorphisms g_i of B_2 of order d_i for $i = 1, 2$ with* $\mathrm{Fix}(g_1) \cap \mathrm{Fix}(g_2)$ *finite,*
- *points (x_1, x_2) and $(x_1', x_2') \in B_1 \times B_2$ with $(1 - g_1)(x_2') = (1 - g_2)(x_2)$,*
- *a subgroup τ of* $\mathrm{Fix}(g_1) \cap \mathrm{Fix}(g_2)$ *containing $\sum_{\nu=0}^{d_1-1} g_1^\nu(x_2)$ and $\sum_{\nu=0}^{d_2-1} g_2^\nu(x_2')$ and an injective homomorphism $\iota : \tau \to B_1$ with $\iota\left(\sum_{\nu=0}^{d_1-1} g_1^\nu(x_2)\right) = -d_1 x_1$ and $\iota\left(\sum_{\nu=0}^{d_2-1} g_2^\nu(x_2')\right) = -d_2 x_1'$,*

such that

(a) *$X \simeq B_1 \times B_2/\Gamma$, where Γ is the subgroup of* Bihol$(B_1 \times B_2)$ *generated by $f_1 = t_{(x_1,x_2)} \circ (1 \times g_1)$, $f_2 = t_{(x_1',x_2')} \circ (1 \times g_2)$ and $T = \{t_{(\iota(y),y)} | y \in \tau\}$, and*

(b) *for all $i = 1, \dots, d_1 - 1$, $j = 1, \dots, d_2 - 1$ and $y \in \tau$ we have*

$$j x_1' + i x_1 + \iota(y) \neq 0 \quad \textit{or}$$

$$\sum_{\nu=0}^{i-1} g_1^\nu(x_2) + \sum_{\nu=0}^{j-1} g_1^i g_2^\nu(x_2') \notin \mathrm{Im}(g_1^i g_2^j)\,.$$

PROOF. Note first that Γ is a finite abelian subgroup of Bihol$(B_1 \times B_2)$. This follows from the assumptions with the same computations as in the proof of Lemma 4.4. It remains to show that Γ acts fixed point freely on $B_1 \times B_2$ if and only if Condition (b) is satisfied. But this follows immediately from the fact that the elements of Γ are just

$$t_{(\iota(y),y)} \circ f_1^i \circ f_2^j = t_{(ix_1 + jx_1' + \iota(y)), \sum_{\nu=0}^{i-1} g_1^\nu(x_2) + \sum_{\nu=0}^{j-1} g_1^i g_2^\nu(x_2') + y)} \circ (1 \times g_1^i g_2^j)$$

with $1 \leq i \leq d_1 - 1$, $1 \leq j \leq d_2 - 1$ and $y \in \tau$. □

The easiest way to construct abelian hyperelliptic varieties is given by the following corollary.

COROLLARY 4.6. *Suppose we are given abelian varieties B_1 of dimension $n_1 > 0$ and B_2 of dimension $n - n_1 > 0$, a finite subgroup $\Gamma = \langle x_1 \rangle \oplus \langle x_1' \rangle \oplus T$ of B_1 and a faithful representation $\rho : \Gamma \to \mathrm{Bihol}(B_2)$ such that*

(a) *$\rho(\langle x_1 \rangle \oplus \langle x_1' \rangle)$ is a group of automorphisms of B_2 isomorphic to $\mathbf{Z}/d_1\mathbf{Z} \oplus \mathbf{Z}/d_2\mathbf{Z}$ with $\mathrm{Fix}(\rho(x_1)) \cap \mathrm{Fix}(\rho(x_1'))$ finite.*

(b) *$\rho(T)$ is a group of translations of B_2 by elements of $\mathrm{Fix}(\rho(x_1)) \cap \mathrm{Fix}(\rho(x_1'))$. If G acts on $B_1 \times B_2$ by $(x, (b_1, b_2)) \mapsto (t_x(b_1), \rho(x)b_2)$, then*

$$X \simeq B_1 \times B_2/\Gamma$$

is an abelian hyperelliptic variety of dimension n of type (d_1, d_2).

PROOF. Choose $x_2 = x_2' = 0$. Let τ denote the subgroup $\tau = \{\rho(t)(0) | t \in T\}$ of B_2 and $\iota : \tau \to T$ the obvious isomorphism. Then all the conditions of (ii) of Theorem 4.5 are satisfied. □

REMARK 4.7. (a) One can also easily prove Corollary 4.6 directly without using Theorem 4.5. Moreover, it seems obvious how to generalize the corollary to construct abelian hyperelliptic varieties of arbitrary type $(d_1, \dots, d_\nu)$. We will omit this, since it will not be used in the sequel.

(b) Theorem 4.5 is trivially valid also for $\dim B_1 = 0$. In fact, in this case Condition (ii) reduces to the definition of a hyperelliptic variety associated to the group G.

(c) One might try to obtain a better description of abelian hyperelliptic varieties of type (d_1, d_2) using the isogeny $\mu : A_1 \times A_2 \times A_3 \times A_4 \to A$ of the beginning of this section. In fact, f_1 and f_2 lift to biholomorphic maps of $A_1 \times \cdots \times A_4$, and so this can be done. However, there are some difficulties to the effect that the result seems not easier to apply than Theorem 4.5: First of all, the kernel of μ seems complicated. Moreover, the liftings of f_1 and f_2 do not commute in general. These difficulties vanish in the special case $(d_1, d_2) = (2, 2)$ as we shall see in the next section.

5. Abelian hyperelliptic varieties of type (2,2). For a hyperelliptic variety X associated to a group $G \simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ one can use the isogeny $A_1 \times A_2 \times A_3 \times A_4 \to X$ of the last section to obtain a better description of X.

Let the notation be as at the beginning of the last section with $d_1 = d_2 = 2$. So, Γ is a group of biholomorphic maps, isomorphic to $\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$, generated by $f_1 = t_x \circ g_1$ and $f_2 = t_{x'} \circ g_2$ and acting fixed point freely on an abelian variety A of dimension $n (\geq 3)$ such that $X \simeq A/\Gamma$. Moreover

$$\mu : A_1 \times A_2 \times A_3 \times A_4 \to A$$

is an isogeny, where $A_1, \dots, A_4$ are abelian subvarieties of A defined as above. Here we have

$$g_1 | A_1 \times A_2 = 1\,, \qquad g_1 | A_3 \times A_4 = -1\,,$$
$$g_2 | A_1 \times A_3 = 1\,, \qquad g_2 | A_2 \times A_4 = -1\,,$$

and moreover $K = \mathrm{Ker}(\mu)$ consists of 2-division points. Consider decompositions

$$x = x_1 + x_2 + x_3 + x_4 \quad \text{and} \quad x' = x_1' + x_2' + x_3' + x_4'$$

with $x_i, x_i' \in A_i$.

LEMMA 5.1. (a) *f_1 and f_2 generate a group $\simeq \mathbf{Z}/2\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z}$ if and only if x_1, x_2, x_1', x_3' and $x_4 - x_4'$ are 2-division points.*

(b) *$\Gamma = \langle f_1, f_2\rangle$ acts fixed point freely on A if and only if*

(i) $x_1 + x_2 \notin A_3 + A_4$,

(ii) $x_1 + x_3 \notin A_2 + A_4$, *and*

(iii) $g_1(y) + x \notin A_2 + A_3$.

PROOF. Assertion (a) is obvious. For (b)(i) note that f_1 acts fixed point freely if and only if $x \notin \mathrm{Im}(1 - g_1) = A_3 + A_4$. (ii) and (iii) mean that f_2 and $f_1 f_2$ act fixed point freely. □

Define biholomorphic maps $\tilde{f}_1$ and $\tilde{f}_2$ on $A_1 \times A_2 \times A_3 \times A_4$ by

$$\tilde{f}_1 = t_{(x_1,x_2,x_3,x_4)} \circ (1 \times 1 \times (-1) \times (-1))\,,$$
$$\tilde{f}_2 = t_{(x_1',x_2',x_3',x_4')} \circ (1 \times (-1) \times 1 \times (-1))\,.$$

For $i = 1, 2$ the following diagram commutes

$$\begin{array}{ccc} A_1 \times A_2 \times A_3 \times A_4 & \xrightarrow{\tilde{f}_i} & A_1 \times A_2 \times A_3 \times A_4 \\ \mu\downarrow & & \downarrow\mu \\ A & \xrightarrow{f_i} & A \end{array}$$

Using Lemma 5.1 (a) one easily checks that $\tilde{f}_1$ and $\tilde{f}_2$ generate a group isomorphic to $\mathbf{Z}/2\mathbf{Z}\oplus\mathbf{Z}/2\mathbf{Z}$. Moreover, if T denotes the group of translations of $A_1 \times \cdots \times A_4$ by elements of $\mathrm{Ker}(\mu)$, then the sum $\langle f_1\rangle + \langle f_2\rangle + T$ is direct and

$$\tilde{\Gamma} = \langle \tilde{f}_1\rangle \oplus \langle \tilde{f}_2\rangle \oplus T$$

acts fixed point freely on $A_1 \times \cdots \times A_4$. Hence we have

LEMMA 5.2. $X \simeq A_1 \times A_2 \times A_3 \times A_4/\tilde{\Gamma}$.

LEMMA 5.3. *Choosing suitable zero points of A_2, A_3 and A_4, we may assume that*

$$\tilde{f}_1 = t_{(x_1,x_2,0,0)} \circ g_1 \quad \textit{and} \quad \tilde{f}_2 = t_{(x_1',0,x_3',x_4')} \circ g_2$$

with 2-division points $x_1, x_2, x_1', x_3', x_4'$, $g_1 = 1 \times 1 \times (-1) \times (-1)$ and $g_2 = 1 \times (-1) \times 1 \times (-1)$.

PROOF. Let $z_3 \in A_3$ with $2z_3 = x_3$. Then

$$\tilde{f}_1(z_3) = -z_3 + x_3 = z_3\,.$$

Choose z_3 as the new zero point of A_3 and let $\tilde{g}_i = t_{z_3} \circ g_i \circ t_{-z_3}$ for $i = 1, 2$ the corresponding automorphisms on A_3. Then for all $a_3 \in A_3$:

$$\tilde{g}_1(a_3) = -a_3 + 2z_3 = \tilde{f}_1(a_3)\,, \quad \text{and}$$
$$t_{x_3'} \circ \tilde{g}_2(a_3) = a_3 + x_3' = \tilde{f}_2(a_3)\,.$$

Hence we may assume that $x_3 = 0$. The proof that we may assume $x_2' = 0$ is the same. Finally, let $z_4 \in A_4$ with $2z_4 = x_4$. Then

$$\tilde{f}_1(z_4) = -z_4 + x_4 = z_4\,.$$

Choose z_4 as the new zero point of A_4 and let $\tilde{g}_i = t_{z_4} \circ g_i \circ t_{-z_4}$ for $i = 1, 2$ the corresponding automorphism on A_4. Then for all $a_4 \in A_4$:

$$\tilde{g}_1(a_4) = g_1(a_4) - g_1(z_4) + z_4 = -a_4 + 2z_4 = \tilde{f}_1(a_4)\,.$$

According to Lemma 5.1 (a), $x_4' = x_4 + p_4$ with a 2-division point p_4 of A_4. Thus

$$t_{p_4} \circ \tilde{g}_2(a_4) = g_2(a_4) - g_2(z_4) + z_4 + p_4 = -a_4 + x_4' = \tilde{f}_2(a_4)\,.$$

Hence we may assume that $x_4 = 0$ and $x_4' = p_4$ is a 2-division point. □

LEMMA 5.4. $\mathrm{Ker}(\mu) \cap \langle (x_1, x_2, 0, 0), (x_1', 0, x_3', x_4') \rangle = \{0\}$.

PROOF. Otherwise $\tilde{\Gamma}$ contains one of the automorphisms g_1, g_2 or $g_1 g_2$ and thus admits fixed points on $A_1 \times \cdots \times A_4$. □

Combining everything we have proved most of the following theorem, the remaining assertions being easy to check.

THEOREM 5.5. *For a variety X of dimension $n\,(\geq 3)$ the following statements are equivalent*:

(i) *X is an abelian hyperelliptic variety of type* $(2, 2)$.

(ii) *There are*

- *abelian varieties A_i of dimension $n_i \geq 0$ for $i = 1, \ldots, 4$ with $\sum_{i=1}^4 n_i = n$,*
- *2-division points $x = (x_1, x_2, 0, 0) \neq 0$ and $x' = (x_1', 0, x_3', x_4') \neq 0$ of $A_1 \times \cdots \times A_4$ with $(x_1 + x_1', x_4') \neq (0, 0)$,*
- *A subgroup τ of 2-division points of $A_1 \times \cdots \times A_4$ with $\tau \cap \langle x, x' \rangle = \{0\}$,*

such that if $f_1 = t_x \circ (1 \times 1 \times (-1) \times (-1))$ and $f_2 = t_{x'} \circ (1 \times (-1) \times 1 \times (-1))$ on $A_1 \times \cdots \times A_4$ and T denotes the group of translations by elements of τ, then

$$X \simeq A_1 \times \cdots \times A_4/\Gamma$$

with $\Gamma = \langle f_1 \rangle \oplus \langle f_2 \rangle \oplus\ T$.

Note that Theorem 5.5 can be easily applied to construct all abelian hyperelliptic varieties of type $(2, 2)$.

6. Hyperelliptic threefolds. The first aim of this section is the proof of the following theorem. Finally, we complete the classification of all hyperelliptic threefolds.

THEOREM 6.1. *Any hyperelliptic threefold is abelian.*

For the proof, assume that X is a hyperelliptic threefold associated to a non abelian group. According to [UY] this group is necessarily the dihedral group D_4 of order 8. Hence there is an abelian threefold A and a group $\Gamma \subset \mathrm{Bihol}(A)$ acting fixed point freely on A and isomorphic to D_4 such that $X = A/\Gamma$. Let Γ be generated by f_1 and f_2 with $f_1^4 = f_2^2 = 1$, $f_2 f_1 f_2 = f_1^{-1}$. Then

$$f_1 = t_x \circ g_1 \quad \text{and} \quad f_2 = t_y \circ g_2$$

with $g_1, g_2 \in \mathrm{Aut}(A)$ with $g_1^4 = g_2^2 = 1$, $g_2 g_1 g_2 = g_1^{-1}$ and $x, y \in A$. Comparing the relations for f_1, f_2 with the relations for g_1, g_2, we obtain

LEMMA 6.2. (a) $g_1^3(x) + g_1^2(x) + g_1(x) + x = 0$.
(b) $g_2(y) + y = 0$.
(c) $(1 + g_1 g_2)(x) = -(g_1 + g_2)(y)$.

LEMMA 6.3. *Let* $B_1 := \mathrm{Ker}(1 - g_1)^0$ *and* $B_2 := \mathrm{Im}(1 - g_1)$. *Then* $\dim B_1 = 1$, $\dim B_2 = 2$ *and the addition map* $\mu : B_1 \times B_2 \to A$ *is an isogeny. Moreover, the group* $G = \langle g_1, g_2 \rangle$ *acts on* B_1 *and* B_2.

PROOF. The group G acts on the tangent space $T_0 A = \boldsymbol{C}^3$. This representation of G must contain the two-dimensional representation of G, since otherwise the action of G on A would be commutative. The eigenvalues of g_1 of the two-dimensional representation are $\pm i$. Hence the one-dimensional representation has to be trivial on g_1. So B_1 is one-dimensional and B_2 is two-dimensional. Certainly, also g_2 acts on B_1 and B_2. □

B_2 is an abelian surface with automorphism group D_4. These surfaces have been classified by Fujiki. In fact, according to Table 8 of [F], we know that B_2 is isomorphic to $E \times E$ with an arbitrary elliptic curve E or the quotient of $E \times E$ by a group $H \simeq \mathbf{Z}/2\mathbf{Z}$ or $\simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ of diagonal 2-division points, and D_4 acts on $E \times E$ by

$$g_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad g_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and on the quotients by the corresponding quotient actions. In any case, we write the elements of B_2 as pairs (b, b'), $b, b' \in E$. In the case of the quotients $E \times E/H$ we have to identify pairs which differ by a diagonal 2-division point of H. According to Lemma 3.3 (e),

$$B_1 \cap B_2 \quad \subseteq \quad \Delta(2) \quad \simeq \quad \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z},$$

where Δ denotes the image of the diagonal of $E \times E$ in $E \times E/H$. Hence we may write the elements b of $A \simeq B_1 \times B_2/B_1 \cap B_2$ as

$$b = b_1 + (b_2, b_2')$$

with $b_1 \in B_1$ and $(b_2, b_2') \in B_2$. Again two such representations of $b \in A$ differ at most by 2-division points.

Only two of the four one-dimensional representations of D_4 can occur since $g_1|B_1$ has to be the identity. Hence there are two cases

$$\begin{array}{lll} \textit{Case } 1: & g_1|B_1 = 1\,, & g_2|B_1 = 1\,. \\ \textit{Case } 2: & g_1|B_1 = 1\,, & g_2|B_1 = -1\,. \end{array}$$

In both cases choose decompositions

$$x = x_1 + (x_2, x_2') \quad \text{and} \quad y = y_1 + (y_2, y_2')$$

for the translation points x and y of f_1 and f_2. Then the proof of Theorem 6.1 is completed if we show that in both cases the action of Γ on A is not fixed point free.

Case 1: According to Lemma 6.2, $g_2(y) = -y$. Hence y is contained in the eigenspace of -1 of g_2, which is the antidiagonal $\widetilde{\Delta} := \{(b, -b) \in B_2\}$ of $E \times E$ or its image $E \times E/H$. Hence we may assume $y = (y_2, -y_2)$. On the other hand, $\mathrm{Im}(1 - g_2)$ is just the antidiagonal $\widetilde{\Delta}$ of B_2, i.e., $y \in \mathrm{Im}(1 - g_2)$. According to Lemma 3.1 this implies the assertion.

Case 2: Again we have $g_2(y) = -y$, which in this case is equivalent to $y_2' = -y_2$. Hence $y = y_1 + (y_2, -y_2) \in B_1 + \widetilde{\Delta}$. Now $\mathrm{Im}(1 - g_2) = B_1 + \widetilde{\Delta}$, i.e., $y \in \mathrm{Im}(1 - g_2)$. Again Lemma 3.1 implies the assertion. □

Hence we may assume that X is a hyperelliptic threefold associated to an abelian group G acting on an abelian threefold A. If G is cyclic, then X is necessarily of the type already described in Section 3. Hence we may assume that G is not cyclic. Under these assumptions we have

LEMMA 6.4. *The group G is generated by two elements.*

PROOF. Suppose G is non cyclic and cannot be generated by two elements. Then G admits a subgroup isomorphic to $(\mathbf{Z}/p\mathbf{Z})^3$ with a prime p. Its generators, g_1, g_2 and g_3 say, cannot have a common eigenspace of the eigenvalue 1, since there is no such automorphism group of an abelian surface (see [F]). But then it is easy to see that a suitable product of g_1, g_2 and g_3 does not admit an eigenvalue 1. □

Hence we may assume that $G \simeq \mathbf{Z}/d_1\mathbf{Z} \times \mathbf{Z}/d_2\mathbf{Z}$ with $d_1|d_2$ and is generated by g_i of order d_i for $i = 1, 2$. Then we have

LEMMA 6.5. *There are two possibilities*: *Either*

(i) *The eigenspaces of* 1 *of g_1 and g_2 have a nontrivial intersection*, *or*

(ii) $G \simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$.

PROOF. Suppose the contrary, i.e., G is not of type (i) or (ii). We may choose the coordinates of $\boldsymbol{C}^3$ in such a way that $g_1 = \mathrm{diag}(1, \alpha_2, \alpha_3)$ and $g_2 = (\beta_1, \beta_2, 1)$ with $\alpha_3 \neq 1 \neq \beta_1$. The element $g_1 g_2$ has to admit an eigenvalue 1, implying $\alpha_2 = \beta_2^{-1}$. Hence $g_1 g_2^2 = \mathrm{diag}(\beta_1^2, \beta_2, \alpha_3)$. Since $\beta_2 \neq 1$ (otherwise $\alpha_2 = \beta_2 = 1$), this implies $\beta_1 = -1$. But then $g_1^2 g_2 = \mathrm{diag}(-1, \beta_2^{-1}, \alpha_3^2)$ gives $\alpha_3 = -1$. Thus we have $g_1 = \mathrm{diag}\ (1, \beta_2^{-1}, -1)$, $g_2 = \mathrm{diag}\ (-1, \beta_2, 1)$ with $\beta_2 \neq \pm 1$. But now $g_1 g_2^3 = \mathrm{diag}\ (-1, \beta_2^2, -1)$ admits no eigenvalue 1, a contradiction. □

Applying Theorem 5.5, it is easy to construct all hyperelliptic threefolds of type $(2, 2)$. So we are left with the case that X is of type (d_1, d_2) with $d_2 > 2$ such that g_1 and g_2 admit a common eigenspace of 1. Since an elliptic curve admits only cyclic automorphism groups, Theorem 4.5 gives us an elliptic curve B_1 and an abelian surface B_2 admitting commuting automorphisms g_1 and g_2 of order d_1 and d_2 with $\mathrm{Fix}(g_1) \cap \mathrm{Fix}(g_2)$ finite, and points $(x_1, x_2), (x_1', x_2') \in B_1 \times B_2$ and a group of translation T of $B_1 \times B_2$ with some additional properties such that

$$X = B_1 \times B_2 / \langle f_1 \rangle \oplus \langle f_2 \rangle \oplus T$$

with $f_1 = t_{(x_1,x_2)} \circ (1 \times g_1)$ and $f_2 = t_{(x_1',x_2')} \circ (1 \times g_2)$. Table 3 below gives all quadruples

TABLE 3.

(d_1, d_2)	B_2	g_1	g_2	$\mathrm{Fix}(g_1) \cap \mathrm{Fix}(g_2)$
(2, 4)	$E \times E_i$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$	$E(2) \times \mathrm{Fix}(i)$
(2, 4)	$E_i \times E_i$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$	$\mathrm{Fix}(i) \times \mathrm{Fix}(i)$
(2, 4)	$E_i \times E_i$	$\begin{pmatrix} 1 & 0 \\ 1+i & -1 \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$	$\mathrm{Fix}(i) \times \mathrm{Fix}(i)$
(2, 4)	$E_i \times E_i$	$\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$	$\{0\} \times \mathrm{Fix}(i)$
(2, 6)	$E \times E_\rho$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -\rho \end{pmatrix}$	$E(2) \times \{0\}$
(2, 6)	$E_\rho \times E_\rho$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\rho & 0 \\ 0 & -\rho \end{pmatrix}$	$\{0\}$
(2, 6)	$E_\rho \times E_\rho$	$\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} -\rho & 0 \\ 0 & -\rho \end{pmatrix}$	$\{0\}$
(2, 6)	$E_\rho \times E_\rho$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\rho & 0 \\ 0 & \rho^2 \end{pmatrix}$	$\{0\} \times \mathrm{Fix}(\rho)$
(2, 12)	$E_i \times E_\rho$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & -\rho \end{pmatrix}$	$\mathrm{Fix}(i) \times \{0\}$
(3, 3)	$E_\rho \times E_\rho$	$\begin{pmatrix} \rho & 0 \\ 0 & \rho^2 \end{pmatrix}$	$\begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}$	$\mathrm{Fix}(\rho) \times \mathrm{Fix}(\rho)$
(3, 3)	$E_\rho \times E_\rho$	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	$\begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}$	$\mathrm{Fix}(\rho \| \Delta)$
(3, 6)	$E_\rho \times E_\rho$	$\begin{pmatrix} \rho & 0 \\ 0 & \rho^2 \end{pmatrix}$	$\begin{pmatrix} -\rho & 0 \\ 0 & -\rho \end{pmatrix}$	$\{0\}$
(3, 6)	$E_\rho \times E_\rho$	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	$\begin{pmatrix} -\rho & 0 \\ 0 & -\rho \end{pmatrix}$	$\{0\}$
(3, 6)	$E_\rho \times E_\rho$	$\begin{pmatrix} \rho & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -\rho \end{pmatrix}$	$\mathrm{Fix}(\rho) \times \{0\}$
(4, 4)	$E_i \times E_i$	$\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$	$\mathrm{Fix}(i) \times \mathrm{Fix}(i)$
(4, 4)	$E_i \times E_i$	$\begin{pmatrix} i & 0 \\ i & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ -i & i \end{pmatrix}$	$\{0\} \times \mathrm{Fix}(i)$
(6, 6)	$E_\rho \times E_\rho$	$\begin{pmatrix} -\rho & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -\rho \end{pmatrix}$	$\{0\}$

$(B_2, g_1, g_2, \mathrm{Fix}(g_1) \cap \mathrm{Fix}\,(g_2))$ with $d_2 > 2$ which yield hyperelliptic threefolds in this way.

For the proof Fujiki's paper [F] is heavily used. According to [F], there are also some other abelian surfaces admitting a group of automorphisms of type (d_1, d_2), but these are quotients of B_2 of table 3. Hence in order to construct the corresponding hyperelliptic threefold we may start with B_2 out of table 3. Applying Corollary 4.6, it is now easy to construct many hyperelliptic threefolds for any of the triples (B_2, g_1, g_2) of the table: Choose a pair of (d_1, d_2)-division points (x_1, x_2) of an elliptic curve B_1, a subgroup T of B_1 and an embedding $\iota : T \hookrightarrow \mathrm{Fix}\,(g_1) \cap \mathrm{Fix}\,(g_2)$ such that $\Gamma = \langle x_1 \rangle + \langle x_2 \rangle + T$ is a direct sum. If Γ acts by $(x_1, (b_1, b_2)) \mapsto (t_{x_1}(b_1), g_1(b_2))$, $(x_2, (b_1, b_2)) \mapsto (t_{x_2}(b_1), g_2(b_2))$ and $(x, (b_1, b_2)) \to (t_x(b_1), t_{\iota(x)}(b_2))$ for any $x \in T$, then $X = B_1 \times B_2/\Gamma$ is a hyperelliptic threefold. One can apply Theorem 4.5 in order to construct all hyperelliptic threefolds of this type. This is a bit more complicated, but can be done separately in every case. However, there are too many cases, so this will be omitted.

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