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TORIC VARIETIES WHOSE BLOW-UP AT A POINT IS FANO

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Abstract. We classify smooth toric Fano varieties of dimension $n \geq 3$ containing a toric divisor isomorphic to the $(n-1)$-dimensional projective space. As a consequence of this classification, we show that any smooth complete toric variety X of dimension $n \geq 3$ with a fixed point $x \in X$ such that the blow-up $B_x(X)$ of X at x is Fano is isomorphic either to the n-dimensional projective space or to the blow-up of the n-dimensional projective space along an invariant linear codimension two subspace. As expected, such results are proved using toric Mori theory due to Reid.

Introduction. Smooth blow-ups and blow-downs between toric smooth Fano varieties have been intensively studied; see [Bat82], [Bat99], [Oda88], [WWa82] and more recently [Sat00] or [Cas01]. In this Note, we prove the following result using toric Mori theory (see also the uncorrect exercise V.3.7.10 mentionned in [Kol99]): As usual, T denotes the big torus acting on a toric variety; if Y is a smooth subvariety of a smooth variety X, $B_Y(X)$ denotes the blow-up of X along Y and a variety X is called Fano if and only if $-K_X$ is ample.

THEOREM 1. *Let X be a smooth and complete toric variety of dimension $n \geq 3$. Suppose there exists a T-fixed point x in X such that $B_x(X)$ is Fano. Then either $X \simeq \boldsymbol{P}^n$ and x can be chosen arbitrary or $X \simeq B_{\boldsymbol{P}^{n-2}}(\boldsymbol{P}^n)$ and x must be chosen outside the exceptional divisor.*

Let us say that when X is a toric surface with a T-fixed point n in X such that $B_n(X)$ is Fano, then X is isomorphic to $\boldsymbol{P}^2$ blown-up at m T-fixed points with $m = 0, 1, 2$ or 3 or to $\boldsymbol{P}^1 \times \boldsymbol{P}^1$. Recall also that smooth Fano toric varieties are classified in dimension less or equal to 4 ([Bat82], [Bat99], [Oda88], [WWa82] and [Sat00]) together with smooth blow-ups and blow-downs between them; in particular, Theorem 1 could be proved in dimension 3 and 4 just by looking at the classification.

In fact, we will obtain Theorem 1 as a consequence of the following result (which is inspired by a private communication of J. Wiśniewski):

THEOREM 2. *Let X be a smooth toric Fano variety of dimension $n \geq 3$. Then, there exists a toric divisor D of X isomorphic to $\boldsymbol{P}^{n-1}$ with $\mathcal{O}_{\boldsymbol{P}^{n-1}}(d)$ as normal bundle in X if and only if one of the following situations occurs*:

(i) $X \simeq \boldsymbol{P}^n$, $d = 1$ *and D is a linear codimension one subspace of X*,

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(ii) $X \simeq \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}(1)^{\oplus n-1}) \simeq B_{\boldsymbol{P}^{n-2}}(\boldsymbol{P}^n)$, $d = 0$ *and* D *is a fiber of the projection on* $\boldsymbol{P}^1$,

(iii) *there exists an integer* ν *satisfying* $0 \leq \nu \leq n-1$ *such that* X *is isomorphic to* $\boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^{n-1}} \oplus \mathcal{O}_{\boldsymbol{P}^{n-1}}(\nu))$ *and* D *is either the divisor* $\boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^{n-1}})$ (*and* $d = \nu$) *or the divisor* $\boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^{n-1}}(\nu))$ (*and* $d = -\nu$),

(iv) *there exists an integer* ν *satisfying* $0 \leq \nu \leq n-2$ *such that* X *is isomorphic to the blow-up of* $\boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^{n-1}} \oplus \mathcal{O}_{\boldsymbol{P}^{n-1}}(\nu+1))$ *along a linear* $\boldsymbol{P}^{n-2}$ *contained in the divisor* $\boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^{n-1}})$ *and* D *is either the strict transform of the divisor* $\boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^{n-1}})$ (*and* $d = \nu$) *or the strict transform of the divisor* $\boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^{n-1}}(\nu+1))$ (*and* $d = -\nu-1$).

Remark that the adjunction formula implies that $d \geq 1-n$. As an immediate consequence of Theorem 2, *there are exactly* $2n+1$ *distinct smooth toric Fano varieties of dimension* $n \geq 3$ *containing a toric divisor isomorphic to* $\boldsymbol{P}^{n-1}$.

1. Notation. We briefly review notation and very basic facts of toric geometry (see [Ful93] or [Oda88] for details).

A toric variety X is defined by a fan Δ in a lattice N (the elements of N are the one parameter subgroups of the big torus T). If X is smooth, any cone of Δ is simplicial, generated by a family of lattice vectors which is part of a basis of N. Any such cone $\langle e_1, \ldots, e_r\rangle$ defines a smooth T-invariant subvariety of codimension r which is the closure of a T-orbit. Recall that on a toric variety X, a T-invariant Cartier divisor is ample if and only if its intersection with any toric curve of X is strictly positive [Oda88].

The cone of effective curves modulo numerical equivalence (usually denoted by $\mathrm{NE}(X)$) of a smooth projective toric variety is polyhedral generated by the T-invariant curves of X [Rei83]. A T-invariant extremal curve C of X is called Mori extremal if moreover $-K_X \cdot C > 0$. Finally, if C is a T-invariant extremal curve with normal bundle $N_{C/X} = \bigoplus_{i=1}^{n-1} \mathcal{O}_{\boldsymbol{P}^1}(a_i)$ generating an extremal ray R of $\mathrm{NE}(X)$, let

$$\alpha = \mathrm{card}\{i \in [1, \ldots, n-1] \mid a_i < 0\} \quad \text{and} \quad \beta = \mathrm{card}\{i \in [1, \ldots, n-1] \mid a_i \leq 0\}.$$

Then, toric Mori theory, due to Reid [Rei83], says that the contraction of R defines a map $\varphi_R : X \to Y$, which is birational if and only if $\alpha \neq 0$. In that case, its exceptional locus $A(R)$ in X is $(n-\alpha)$-dimensional, $B(R) = \varphi_R(A(R))$ is $(\beta-\alpha)$-dimensional and the restriction of φ_R to $A(R)$ is a flat morphism, with fibers isomorphic to weighted projective spaces. If $\alpha = 0$, $\varphi_R : X \to Y$ is a smooth $\boldsymbol{P}^{n-\beta}$-fibration and Y is smooth and projective.

2. Fano varieties with a divisor isomorphic to a projective space. In this section, we prove Theorem 2.

2.1. Mori contraction on X. In this subsection, X is a smooth toric Fano variety of dimension $n \geq 3$ containing a toric divisor D isomorphic to $\boldsymbol{P}^{n-1}$ and $N_{D/X} = \mathcal{O}_{\boldsymbol{P}^{n-1}}(d)$. Let $[l_D] \in \mathrm{NE}(X)$ be the class in $\mathrm{NE}(X)$ of a line l_D contained in D (this class does not depend on the choice of the line).

PROPOSITION 1. *Suppose there exists a Mori extremal curve ω transverse to D such that $[\omega] \in \mathrm{NE}(X)$ does not belong to the ray generated by $[l_D]$. Denote by $\varphi_{[\omega]}$ the Mori contraction defined by ω. Then*

(i) *either $\nu := |d|$ satisfies $0 \le \nu \le n-1$, $X \simeq \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^{n-1}} \oplus \mathcal{O}_{\boldsymbol{P}^{n-1}}(\nu))$ and $\varphi_{[\omega]} : X \to \boldsymbol{P}^{n-1}$ is the natural fibration, or*

(ii) *there exists a smooth toric Fano variety X' with a T-invariant smooth divisor D' such that*

$$(D', N_{D'/X'}) \simeq (\boldsymbol{P}^{n-1}, \mathcal{O}_{\boldsymbol{P}^{n-1}}(d+1)),$$

$\varphi_{[\omega]} : X \to X'$ is the blow-up of X' along a toric subvariety $Y \simeq \boldsymbol{P}^{n-2}$ contained in D' and D is the strict transform of D'.

In Case (ii), we get a new smooth toric Fano variety X' containing a toric divisor D' isomorphic to $\boldsymbol{P}^{n-1}$. This motivates the following definition.

DEFINITION. When the situation (ii) of Proposition 1 occurs, we say that the pair (X, D) *can be simplified*.

PROOF OF PROPOSITION 1. Let

$$N_{\omega/X} = \bigoplus_{i=1}^{n-1} \mathcal{O}_{\boldsymbol{P}^1}(a_i)$$

be the normal bundle of ω in X and as in the previous section:

$$\alpha = \mathrm{card}\{i \in [1, \dots, n-1] \,|\, a_i < 0\} \quad \text{and} \quad \beta = \mathrm{card}\{i \in [1, \dots, n-1] \,|\, a_i \le 0\}.$$

Since $[\omega] \in \mathrm{NE}(X)$ does not belong to the ray generated by $[l_D]$, each a_i is less or equal to zero. Therefore, since $-K_X \cdot \omega = 2 + \sum_{i=1}^{n-1} a_i > 0$, there are only two possibilities:

(i) every $a_i = 0$; therefore $\alpha = 0$, $\beta = n-1$ and the Mori contraction $\varphi_{[\omega]} : X \to Z$ is a $\boldsymbol{P}^1$-fibration on Z. Since $D \simeq \boldsymbol{P}^{n-1}$ is a section of this fibration (by the transversality assumption, $D \cdot \omega = 1$), we get $Z \simeq \boldsymbol{P}^{n-1}$ and, if $\nu := |d|$, X is isomorphic to $\boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^{n-1}} \oplus \mathcal{O}_{\boldsymbol{P}^{n-1}}(\nu))$ which is Fano if and only if $0 \le \nu \le n-1$, or

(ii) there is exacly one of the a_i's equal to -1 and each other equal to 0. Therefore $\alpha = 1$, $\beta = n-1$ and $\varphi_{[\omega]} : X \to X'$ is a smooth blow-down on a smooth codimension two center. Denote by $E \subset X$ the exceptional divisor of $\varphi_{[\omega]}$. Since $D \cdot \omega = 1$, the center of the blow-up is isomorphic to $E \cap D$, i.e., isomorphic to $\boldsymbol{P}^{n-2}$. Therefore, since $N_{D/X} = \mathcal{O}_{\boldsymbol{P}^{n-1}}(d)$, the center of the blow-up $\varphi_{[\omega]}$ in X' is isomorphic to $\boldsymbol{P}^{n-2}$ with normal bundle $\mathcal{O}_{\boldsymbol{P}^{n-2}}(d+1) \oplus \mathcal{O}_{\boldsymbol{P}^{n-2}}(1)$. Therefore X' is Fano by Lemma 1 below (recall that $d \ge 1-n$). Moreover, $D' := \varphi_{[\omega]}(D)$ is a T-invariant smooth divisor containing the center of the blow-up $\varphi_{[\omega]}$ and satisfying $(D', N_{D'/X'}) \simeq (\boldsymbol{P}^{n-1}, \mathcal{O}_{\boldsymbol{P}^{n-1}}(d+1))$. □

LEMMA 1. *Let X be a smooth toric variety of dimension n. Suppose there exists a T-invariant subvariety Y isomorphic to $\boldsymbol{P}^{n-2}$ with normal bundle $\mathcal{O}_{\boldsymbol{P}^{n-2}}(a) \oplus \mathcal{O}_{\boldsymbol{P}^{n-2}}(b)$ such that $B_Y(X)$ is Fano. Then X is Fano if and only if $n-1+a+b > 0$.*

PROOF. Since $B_Y(X)$ is Fano, $-K_X$ has strictly positive intersection with any curve not contained in Y, and if C is a line contained in Y, then $-K_X \cdot C = n - 1 + a + b$. □

Let us end this part by the following lemma, which says that Case (ii) in Proposition 1 can not occur twice consecutively:

LEMMA 2. *With the previous notation, assume that the pair* (X, D) *can be simplified, and let* $\varphi_{[\omega]} : X \to X'$ *be the corresponding codimension two smooth blow-down as in Propositio* 1 (ii). *Then, the pair* $(X', \varphi_{[\omega]}(D))$ *can not be simplified.*

PROOF. By contradiction, suppose $(X', \varphi_{[\omega]}(D))$ can be simplified and denote by $\varphi_{[\omega']} : X' \to X''$ the corresponding smooth codimension two blow-down. The exceptional divisor $E' \subset X'$ of $\varphi_{[\omega']}$ intersects $D' := \varphi_{[\omega]}(D)$ along a $\boldsymbol{P}^{n-2}$ which itself meets the center Z of $\varphi_{[\omega]}$ (since two $\boldsymbol{P}^{n-2}$ contained in a $\boldsymbol{P}^{n-1}$ must intersect). Let C be the fiber of $\varphi_{[\omega']}$ containing a given point of $E' \cap D' \cap Z$. Then the strict transform of C in X is a curve with normal bundle in X equals to $\mathcal{O}_{\boldsymbol{P}^1}^{\oplus n-2} \oplus \mathcal{O}_{\boldsymbol{P}^1}(-2)$, and hence with zero intersection on $-K_X$, a contradiction, since X is Fano. □

2.2. Proof of Theorem 2. As before, X is a smooth toric Fano variety of dimension $n \geq 3$ containing a toric divisor D isomorphic to $\boldsymbol{P}^{n-1}$ and $N_{D/X} = \mathcal{O}_{\boldsymbol{P}^{n-1}}(d)$. Let $[l_D] \in \mathrm{NE}(X)$ be the class in $\mathrm{NE}(X)$ of a line contained in D.

PROPOSITION 2. *Suppose that* $d \geq 0$ *and* $[l_D]$ *spans an extremal ray of* $\mathrm{NE}(X)$. *Then*

(i) *either* $d = 0$ *and* $X \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^{n-1}$ *or* $X \simeq \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}(1)^{\oplus n-1})$, *or*

(ii) $d = 1$ *and* $X \simeq \boldsymbol{P}^n$.

PROOF. If l_D is a line contained in D, then

$$N_{l_D/X} = \mathcal{O}_{\boldsymbol{P}^1}(1)^{\oplus n-2} \oplus \mathcal{O}_{\boldsymbol{P}^1}(d)\,.$$

If $d = 0$, then the Mori contraction $\varphi_{[l_D]}$ is a smooth $\boldsymbol{P}^{n-1}$-fibration on $\boldsymbol{P}^1$, therefore X is isomorphic to $\boldsymbol{P}(\bigoplus_{i=1}^n \mathcal{O}_{\boldsymbol{P}^1}(a_i))$, which is Fano if and only if $X \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^{n-1}$ or $X \simeq \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}(1)^{\oplus n-1})$. If $d > 0$, the Mori contraction $\varphi_{[l_D]}$ maps X to a point, therefore $X \simeq \boldsymbol{P}^n$ and $d = 1$. □

Now, we are ready to prove Theorem 2: Let X be a smooth toric Fano variety of dimension $n \geq 3$. Suppose there exists a toric divisor D of X isomorphic to $\boldsymbol{P}^{n-1}$, and let $\mathcal{O}_{\boldsymbol{P}^{n-1}}(d)$ be its normal bundle in X. Let also $[l_D] \in \mathrm{NE}(X)$ be the class in $\mathrm{NE}(X)$ of a line contained in D.

• First case: Suppose that either $d < 0$ or $d \geq 0$ and $[l_D]$ does not span an extremal ray in $\mathrm{NE}(X)$. Since D is effective, there exists a Mori extremal curve ω transverse to D such that $[\omega] \in \mathrm{NE}(X)$ does not belong to the ray generated by $[l_D]$. Therefore Proposition 1 applies: $X \simeq \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^{n-1}} \oplus \mathcal{O}_{\boldsymbol{P}^{n-1}}(|d|))$ (and $0 < |d| \leq n - 1$) or the pair (X, D) can be simplified.

• Second case: $d \geq 0$ and $[l_D]$ spans an extremal ray of $\mathrm{NE}(X)$. Then apply Proposition 2.

As a result, either X satisfies one of the conclusions (i), (ii) or (iii) of Theorem 2, or the pair (X, D) can be simplified. In the latter case, let $\varphi_{[\omega]} : X \to X'$ be the corresponding codimension two smooth blow-down as in Proposition 1 (ii). Then, since the pair (X', D') can not be simplified by Lemma 2, applying the same process to the Fano variety X' with $D' = \varphi_{[\omega]}(D)$ and $d' = d + 1$, X' itself must satisfy one of the conclusions (i), (ii) or (iii) of Theorem 2. In case X' is isomorphic to $\boldsymbol{P}^n$, we get that $X \simeq \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}(1)^{\oplus n-1})$. Moreover, X' can not be isomorphic to $\boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}(1)^{\oplus n-1})$, because assuming the contrary, (X, D) could be simplified twice, a contradiction with Lemma 2. Finally, suppose $X' \simeq \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^{n-1}} \oplus \mathcal{O}_{\boldsymbol{P}^{n-1}}(|d+1|))$. Since X' is Fano, we get $0 \leq |d+1| \leq n-1$, which together with the inequality $d \geq 1-n$ shows that X satisfies conclusion (iv) of Theorem 2. $\square$

3. Proof of Theorem 1. Let X be a smooth toric complete variety of dimension $n \geq 3$. Suppose in the sequel that there exists a T-fixed point x in X such that $B_x(X)$ is Fano (it is well-known that X is therefore also Fano). Hence $B_x(X)$ is a Fano variety containing a toric divisor (the exceptional divisor of the blow-up $\pi : B_x(X) \to X$) isomorphic to $\boldsymbol{P}^{n-1}$ with normal bundle $\mathcal{O}_{\boldsymbol{P}^{n-1}}(-1)$. Applying Theorem 2 to $B_x(X)$ with $d = -1$ gives that either

- $B_x(X) \simeq \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^{n-1}} \oplus \mathcal{O}_{\boldsymbol{P}^{n-1}}(-1))$ therefore $X \simeq \boldsymbol{P}^n$, or
- $B_x(X)$ is isomorphic to the blow-up of $\boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^{n-1}} \oplus \mathcal{O}_{\boldsymbol{P}^{n-1}}) = \boldsymbol{P}^1 \times \boldsymbol{P}^{n-1}$ along a $\boldsymbol{P}^{n-2}$ contained in a fiber of the projection $\boldsymbol{P}^1 \times \boldsymbol{P}^{n-1} \to \boldsymbol{P}^1$. Therefore, $X \simeq \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}(1)^{\oplus n-1}) \simeq B_{\boldsymbol{P}^{n-2}}(\boldsymbol{P}^n)$ and x is outside the exceptional divisor of the blow-up $B_{\boldsymbol{P}^{n-2}}(\boldsymbol{P}^n) \to \boldsymbol{P}^n$. $\square$

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