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PERMANENCE OF AN SIR EPIDEMIC MODEL WITH DISTRIBUTED TIME DELAYS

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Abstract. We consider permanence of an SIR epidemic model with distributed time delays. Based on some known techniques on limit sets of differential dynamical systems, we show that, for any time delay, the SIR epidemic model is permanent if and only if an endemic equilibrium exits.

1. Introduction. In this paper, we shall consider the following SIR epidemic model with distributed time delays,

$$\begin{cases} \dot{S}(t) = -\beta S(t)\int_0^h I(t-s)d\eta(s) - \mu_1 S(t) + b\,, \\ \dot{I}(t) = \beta S(t)\int_0^h I(t-s)d\eta(s) - \mu_2 I(t) - \lambda I(t)\,, \\ \dot{R}(t) = \lambda I(t) - \mu_3 R(t)\,. \end{cases} \tag{1.1}$$

In model (1.1), $S(t)+I(t)+R(t) \equiv N(t)$ denotes the number of a population at time t; $S(t)$, $I(t)$ and $R(t)$ denote the numbers of the population susceptible to the disease, of infective members and of members who have been removed from the possibility of infection through full immunity, respectively. It is assumed that all newborns are susceptible. The positive constants μ_1, μ_2 and μ_3 represent the death rates of susceptibles, infectives and recovered, respectively. It is biologically natural to assume that

$$\mu_1 \le \min\{\mu_2,\ \mu_3\}\,.$$

The positive constants b and λ represent the birth rate of the population and the recovery rate of infectives, respectively. The positive constant β is the average number of contacts per infective per day. The nonnegative constant h is the time delay. The function $\eta(s) : [0, h] \to \mathcal{R} = (-\infty, +\infty)$ is nondecreasing and has bounded variation such that

$$\int_0^h d\eta(s) = \eta(h) - \eta(0) = 1\,.$$

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The term $\beta S(t)\int_0^h I(t-s)d\eta(s)$ can be considered as the force of infection at time t. For the detailed biological meanings, we refer to [1–4], [6] and [11].

The initial condition of (1.1) is given as

$$S(\theta)=\varphi_1(\theta)\,,\quad I(\theta)=\varphi_2(\theta)\,,\quad R(\theta)=\varphi_3(\theta)\,,\quad -h\le\theta\le 0\,, \tag{1.2}$$

where $\varphi=(\varphi_1,\varphi_2,\varphi_3)^T\in C$ such that $\varphi_i(\theta)=\varphi_i(0)\ge 0$ $(-h\le\theta\le 0,\ i=1,3)$, $\varphi_2(\theta)\ge 0$ $(-h\le\theta\le 0)$, and C denotes the Banach space $C([-h,0],\mathcal{R}^3)$ of continuous functions mapping the interval $[-h,0]$ into $\mathcal{R}^3$. By a biological meaning, we further assume that $\varphi_i(0)>0$ for $i=1,2,3$.

From Lemma 1 in the following section, the solution $(S(t),I(t),R(t))$ of (1.1) with the initial condition (1.2) exists for all $t\ge 0$ and is unique. Furthermore, $S(t)>0,\ I(t)>0$ and $R(t)>0$ for all $t\ge 0$. Note that there are no time delay in the state variables $S(t)$ and $R(t)$ of (1.1). In the phase space C, the solution $(S(t),I(t),R(t))$ can also be denoted in the form of (S_t,I_t,R_t) for $t\ge 0$. Here $S_t=S(t+\theta)$, $I_t=I(t+\theta)$ and $R_t=R(t+\theta)$ for $t\ge 0$ and $-h\le\theta\le 0$.

For any parameters $h,\ \beta,\ b,\ \lambda,$ and μ_i $(i=1,2,3)$, (1.1) always has a disease free equilibrium (i.e., boundary equilibrium) $E_0=(S_0,\ 0,\ 0)$, where $S_0=b/\mu_1$. Furthermore, if

$$\frac{b}{\mu_1}>S^*\equiv\frac{\mu_2+\lambda}{\beta}\,, \tag{1.3}$$

then (1.1) also has an endemic equilibrium (i.e., interior equilibrium) $E_+=(S^*,I^*,R^*)$, where

$$S^*=\frac{\mu_2+\lambda}{\beta}\,,\quad I^*=\frac{b-\mu_1S^*}{\beta S^*}\,,\quad R^*=\frac{\lambda(b-\mu_1S^*)}{\mu_3\beta S^*}\,.$$

The model (1.1) is a natural generalization of the following well-known SIR model without time delay, which was first proposed and studied in [1] and [11],

$$\begin{cases}\dot S(t)=-\beta S(t)I(t)-\mu S(t)+\mu\,,\\ \dot I(t)=\beta S(t)I(t)-\mu I(t)-\lambda I(t)\,,\\ \dot R(t)=\lambda I(t)-\mu R(t)\,,\end{cases} \tag{1.4}$$

where $\beta,\ \mu$ and λ are positive constants. In (1.4), it is assumed that the total number of population $N(t)$ is constant (i.e., $N(t)=1$ for all $t\ge 0$) and that the birth and the death rates of population are the same. It is shown in [1] and [11] that the condition

$$\delta\equiv\frac{\beta}{\lambda+\mu}>1$$

is the threshold of (1.4) for an epidemic to occur. δ was called the average number of contacts in [1] and [11]. In biology and mathematics, the results given in [1] and [11] say that, if $\delta\le 1$, the disease will eventually disappear and all population will become susceptibles (i.e., the disease free equilibrium $E_0=(1,0,0)$ of (1.4) is globally asymptotically stable), and if $\delta>1$, the disease always remains endemic and the numbers of the susceptibles, infectives and removed will eventually tend to some positive constants, respectively (i.e., the endemic equilibrium

$$E_+ = \left(\frac{1}{\delta}, \quad \frac{\mu(\delta-1)}{\beta}, \quad \frac{\lambda(\delta-1)}{\beta}\right)$$

of (1.4) is globally asymptotically stable).

Recently, in [3], [4] and [13], it is tried to show such threshold phenomenon as for (1.4) still remains true for the model (1.1) with time delay h, i.e., the following conjecture may be true.

CONJECTURE. *For any time delay* h, (1.3) *is the threshold of* (1.1) *for an epidemic to occur*.

It is shown in [4] that, if $b/\mu_1 < S^*$ (or $b/\mu_1 = S^*$), the disease free equilibrium E_0 is globally asymptotically stable (or globally attractive, respectively) for any time delay h. If $b/\mu_1 > S^*$ (i.e., (1.3) is valid), the disease free equilibrium E_0 becomes unstable and the endemic equilibrium E_+ is locally asymptotically stable for any time delay h. In [4], it is also shown that the endemic equilibrium E_+ is also globally asymptotically stable for some small time delay h. For a class of simpler model than (1.1), [13] studied the global asymptotic stability of the endemic equilibrium E_+ under some *stronger conditions than* (1.3). It is not difficult to see that the results given in [13] still remain true for the model (1.1).

The purpose of the present paper is to give a complete answer to the conjecture in a certain sense. Indeed, we shall show that, for any time delay h, (1.3) is necessary and sufficient for the permanence of (1.1). In biology, our result says that (1.3) is the threshold for an endemic to occur for any time delay h. To prove our result, some analytic techniques on limit sets of differential dynamical systems developed in [5], [7] and [9] have been used.

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2. Main result.

DEFINITION ([12]). (1.1) is said to be *permanent* if there are positive constants ν_i and M_i $(i = 1, 2, 3)$ such that

$$\nu_1 \le \liminf_{t\to+\infty} S(t) \le \limsup_{t\to+\infty} S(t) \le M_1,$$

$$\nu_2 \le \liminf_{t\to+\infty} I(t) \le \limsup_{t\to+\infty} I(t) \le M_2,$$

$$\nu_3 \le \liminf_{t\to+\infty} R(t) \le \limsup_{t\to+\infty} R(t) \le M_3$$

hold for any solution of (1.1) with the initial condition (1.2). Here ν_i and M_i $(i = 1, 2, 3)$ are independent of (1.2).

The following is our main result of the paper.

THEOREM. *For any time delay* h, (1.3) *is necessary and sufficient for the permanence of* (1.1).

Note that the disease free equilibrium E_0 of (1.1) is globally asymptotically stable or globally attractive if (1.3) is not valid. We only need to prove the sufficiency. Let us first show the following Lemmas 1–4.

LEMMA 1. *The solution* $(S(t), R(t), I(t))$ *of* (1.1) *with* (1.2) *exists and is positive for* $t \geq 0$. *Further*,

$$\limsup_{t\to+\infty} N(t) \leq \frac{b}{\mu_1}\,. \tag{2.1}$$

PROOF. Note that the right hand side of (1.1) is completely continuous and locally Lipschitzian on C. It follows from [9] and [12] that the solution $(S(t), I(t), R(t))$ of (1.1) exists and is unique on $[0, \alpha)$ for some $\alpha > 0$. It is easy to see that $S(t) > 0$ for all $t \in [0, \alpha)$. Indeed, this follows from that $\dot{S}(t) = b > 0$ for any $t \in [0, \alpha)$ when $S(t) = 0$. Let us show that $I(t) > 0$ for all $t \in [0, \alpha)$. In fact, assume that there exists some $t_1 \in (0, \alpha)$ such that $I(t_1) = 0$ and $I(t) > 0$ for $t \in [0, t_1)$. Integrating the second equation of (1.1) from 0 to t_1, we see that

$$I(t_1) = I(0)e^{-(\mu_2+\lambda)t_1} + \beta\int_0^{t_1}\left(S(u)\int_0^h I(u-s)d\eta(s)\right)e^{-(\mu_2+\lambda)(t_1-u)}du > 0\,,$$

which contradicts $I(t_1) = 0$. From (1.1), we also have that $R(t) > 0$ for all $t \in [0, \alpha)$. Thus, for $t \in [0, \alpha)$,

$$\dot{N}(t) \leq -\mu_1 N(t) + b\,, \tag{2.2}$$

which implies that $(S(t), I(t), R(t))$ is uniformly bounded on $[0, \alpha)$. It follows from [9] and [12] that $(S(t), I(t), R(t))$ exists and is unique and positive for $t \geq 0$. From (2.2), we also have (2.1). This completes the proof of Lemma 1.

REMARK 1. For any nonnegative initial function $\varphi \in C$, by a similar method as that used in Lemma 1, we can show that the following (i), (ii) and (iii) are true.

(i) The solution $(S(t), I(t), R(t))$ of (1.1) exists and $S(t) > 0$ $(t > 0)$, $I(t) \geq 0$ and $R(t) \geq 0$ $(t \geq 0)$.

(ii) If $\varphi_1(0) > 0$ and $\varphi_2(0) + \int_0^h \varphi_2(-s)d\eta(s) > 0$, then the solution $(S(t), I(t), R(t))$ of (1.1) exists and $S(t) > 0$ $(t \geq 0)$, $I(t) > 0$ and $R(t) > 0$ $(t > 0)$.

(iii) If $\varphi_2(\theta) = \varphi_3(0) = 0$ for any $\theta \in [-h, 0]$, then the solution $(S(t), I(t), R(t))$ of (1.1) exists and $S(t) > 0$ $(t > 0)$ and $I(t) = R(t) = 0$ $(t \geq 0)$.

In fact, let the solution $(S(t), I(t), R(t))$ exist and be unique on $[0, \alpha)$ for some $\alpha > 0$. It is easy to show that $S(t) > 0$ for $t \in (0, \alpha)$. From the proof of Lemma 1 and the continuity of the solution $(S(t), I(t), R(t))$ of (1.1) with respect to the initial function φ, we easily show that $I(t) \geq 0$ and $R(t) \geq 0$ for $t \in (0, \alpha)$. Thus, (i) of Remark 1 holds.

If $\varphi_2(0) + \int_0^h \varphi_2(-s)d\eta(s) > 0$, then from (1.1), we have that $\dot{I}(0) > 0$. This implies that $I(t) > 0$ for small $t > 0$, from which we can further show that $I(t) > 0$ for all $t \in (0, \alpha)$. Furthermore, from (1.1), we also have that $R(t) > 0$ for $t \in (0, \alpha)$. This shows that (ii) of Remark 1 holds.

If $\varphi_2(\theta) = \varphi_3(0) = 0$ for all $\theta \in [-h, 0]$, it is clear that

$$S(t) = \Big(\varphi_1(0) - \frac{b}{\mu_1}\Big)e^{-\mu_1 t} + \frac{b}{\mu_1} > 0\,,$$

and $I(t) = R(t) = 0$ for all $t \geq 0$. This shows that (iii) of Remark 1 holds.

LEMMA 2. *The solution* $(S(t), R(t), I(t))$ *of* (1.1) *with* (1.2) *satisfies*

$$\liminf_{t\to+\infty} S(t) \geq \frac{\mu_1 b}{b\beta + \mu_1^2} \equiv \nu_1\,. \tag{2.3}$$

PROOF. For any sufficiently small positive constant ε, it follows from Lemma 1 that there is some sufficiently large $t_1 > 0$ such that for $t \geq t_1$, $I(t) \leq b/\mu_1 + \varepsilon$. Thus, from (1.1) we have that for $t \geq t_1 + h$,

$$\dot{S}(t) \geq -\bigg[\beta\Big(\frac{b}{\mu_1} + \varepsilon\Big) + \mu_1\bigg]S(t) + b\,,$$

which implies that

$$\liminf_{t\to+\infty} S(t) \geq \frac{b\mu_1}{\beta(b + \mu_1\varepsilon) + \mu_1^2}\,.$$

Note that ε may be arbitrarily small so that (2.3) holds. This proves Lemma 2.

LEMMA 3. *The set* Q *is positively invariant for* (1.1) *and attracts all solutions of* (1.1). *Here* Q *is the set of* $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in C$ *satisfying*

$$\frac{b}{\mu_0} \leq \varphi_1(\theta) + \varphi_2(\theta) + \varphi_3(\theta) \leq \frac{b}{\mu_1}\,, \quad \nu_1 \leq \varphi_1(\theta) \leq \frac{b}{\mu_1}\,,$$

$$\varphi_2(\theta) \geq 0\,, \quad \varphi_3(\theta) \geq 0\,, \quad -h \leq \theta \leq 0\,,$$

and $\mu_0 = \max\{\mu_1,\ \mu_2,\ \mu_3\}$.

PROOF. By Lemmas 1 and 2 and the fact that $\dot{N}(t) \geq b - \mu_0 N(t)$, it is enough to show that Q is positively invariant for (1.1).

For any initial function $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in Q$, let $(S(t), I(t), R(t))$ be the solution of (1.1). From Remark 1, we have that $(S(t), I(t), R(t))$ is nonnegative for all $t \geq 0$.

Now, let us show that $S(t) \leq b/\mu_1$ for all $t \geq 0$. If not, there exists some $t_1 > 0$ such that $S(t_1) > b/\mu_1$ and $\dot{S}(t_1) > 0$ by the mean value theorem. Thus, it follows from (1.1) that

$$\dot{S}(t_1) = -\beta S(t_1)\int_0^h I(t_1 - s)d\eta(s) - \mu_1 S(t_1) + b < 0\,,$$

which is a contradiction. Moreover, note that for any $t \geq 0$,

$$-\mu_0 N(t) + b \leq \dot{N}(t) \leq -\mu_1 N(t) + b\,.$$

Hence we see that $b/\mu_0 \leq N(t) \leq b/\mu_1$ for any $t \geq 0$.

Let us show that $S(t) \geq \nu_1$ for all $t \geq 0$. If not, we can find some $t_2 \geq 0$ such that $S(t_2) = \nu_1$, $S(t) \geq \nu_1$ for all $-h \leq t \leq t_2$ and $\dot{S}(t_2) \leq 0$. On the other hand, it follows from

(1.1) that

$$\begin{aligned}\dot{S}(t_2) &= -\beta S(t_2)\int_0^h I(t_2-s)d\eta(s) - \mu_1 S(t_2) + b\\ &\geq -\beta\nu_1\left(\frac{b}{\mu_1}-\nu_1\right) - \mu_1\nu_1 + b\\ &= \beta\nu_1^2 > 0\,.\end{aligned}$$

Note that the first inequality of the above is true since we have $I(t) \leq b/\mu_1 - \nu_1$ for $t-h < t < t_2$ because of $N(t) \leq b/\mu_1$, $S(t) \geq \nu_1$ and $I(t) \leq N(t) - S(t)$ for $t-h < t < t_2$. Thus, we again have a contradiction. These shows that Q is positively invariant for (1.1). The proof of Lemma 3 is completed.

LEMMA 4. *If* (1.3) *holds, then the solution* $(S(t), I(t), R(t))$ *of* (1.1) *with* (1.2) *satisfies*

$$\liminf_{t\to+\infty} I(t) > 0\,.$$

PROOF. From Lemma 3, we see that it is enough to consider the solution $(S(t), I(t), R(t))$ $(t \geq 0)$ with the initial function $\varphi \in Q$. From Lemma 1, we see that the omega limit set $\omega(\varphi)$ of $(S(t), I(t), R(t))$ $(t \geq 0)$ is nonempty, compact, invariant and $\omega(\varphi) \subset Q$ ([8], [9] and [14]).

If $\liminf_{t\to+\infty} I(t) = 0$, we shall show that there is a contradiction.

Indeed, from $\liminf_{t\to+\infty} I(t) = 0$, we see that there exists a positive time sequence $\{t_n\}$: $t_n \to +\infty$ $(n \to +\infty)$ such that

$$\lim_{n\to+\infty} I(t_n) = 0\,, \quad \dot{I}(t_n) \leq 0\,, \quad I(t) \geq I(t_n) \quad (t_n - h \leq t \leq t_n)\,.$$

Note that the solution $(S(t), I(t), R(t))$ is bounded on $[0, +\infty)$ by Lemma 1. It follows from (1.1) that $(S(t), I(t), R(t))$ is uniformly continuous on $[0, +\infty)$. Hence, it follows from Ascoli's theorem that there is a subsequence of $\{t_n\}$, still denoted by $\{t_n\}$, such that

$$\lim_{n\to+\infty}(S(t+t_n), I(t+t_n), R(t+t_n)) = (\tilde{S}(t), \tilde{I}(t), \tilde{R}(t))$$

holds uniformly on $\mathcal{R}$ in the wider sense. From Lemma 3, we have that

$$(\tilde{S}_t, \tilde{I}_t, \tilde{R}_t) \in Q$$

for any $t \in \mathcal{R}$, and that for any $\tau \in \mathcal{R}$, the function $(\tilde{S}(t+\tau), \tilde{I}(t+\tau), \tilde{R}(t+\tau))$ of t is the solution of (1.1) with the initial function $(\tilde{S}_\tau, \tilde{I}_\tau, \tilde{R}_\tau)$. Here we note that $\tilde{I}(0) = 0$ and $\nu_1 \leq \tilde{S}(t) \leq b/\mu_1$ for any $t \in \mathcal{R}$.

We claim that $(\tilde{S}(t), \tilde{I}(t), \tilde{R}(t)) = (b/\mu_1, 0, 0)$ for any $t \in \mathcal{R}$.

From $\tilde{I}(0) = 0$ and (ii) of Remark 1, we have that $\int_0^h \tilde{I}(t-s)d\eta(s) + \tilde{I}(t) = 0$ for all $t < 0$, from which we further have that $\tilde{I}(t) = 0$ for any $t < 0$. Thus, it follows from (1.1) that $\tilde{I}(t) \equiv 0$ for any $t \in \mathcal{R}$, and that

$$\frac{d}{dt}\tilde{S}(t) = -\mu_1\tilde{S}(t) + b$$

for any $t \geq \tau$. Hence

$$\tilde{S}(t) = e^{-\mu_1 t}\tilde{S}(0) + \frac{b}{\mu_1}(1 - e^{-\mu_1 t})$$

for any $t \geq \tau$. Note that from the arbitrariness of τ, we have that

$$\tilde{S}(t) = \frac{b}{\mu_1} + \left(\tilde{S}(0) - \frac{b}{\mu_1}\right)e^{-\mu_1 t}$$

for any $t \in \mathcal{R}$. Since $\tilde{S}(t)$ is bounded for $t \in \mathcal{R}$, we must have that $\tilde{S}(0) = b/\mu_1$, which implies that $\tilde{S}(t) = b/\mu_1$ for any $t \in \mathcal{R}$. It follows from (1.1) and Lemma 1 that $(\tilde{S}(t), \tilde{I}(t), \tilde{R}(t)) = (b/\mu_1, 0, 0)$ for any $t \in \mathcal{R}$. This shows that the above claim holds.

Specially, we have that

$$\lim_{t\to+\infty} S(t_n) = \tilde{S}(0) = \frac{b}{\mu_1}\,.$$

For sufficiently large n, we have that $S(t_n) > (\mu_2 + \lambda)/\beta$ by (1.3). Hence

$$\begin{aligned}\dot{I}(t_n) &= \beta S(t_n)\int_0^h I(t_n - s)d\eta(s) - \mu_2 I(t_n) - \lambda I(t_n)\\ &\geq [\beta S(t_n) - (\mu_2 + \lambda)]I(t_n) > 0\,,\end{aligned}$$

which is a contradiction to $\dot{I}(t_n) \leq 0$. This completes the proof of Lemma 4.

PROOF OF THEOREM. We first show that the solution $(S(t), I(t), R(t))$ of (1.1) with (1.2) satisfies

$$\liminf_{t\to+\infty} I(t) \geq \nu_2\,. \tag{2.4}$$

Here ν_2 is some positive constant which does not depend on the initial function φ.

For any initial functions sequence $\{\varphi_n\} = \{(\varphi_1^{(n)}, \varphi_2^{(n)}, \varphi_3^{(n)})\} \subset Q$, let $(S^{(n)}(t), I^{(n)}(t), R^{(n)}(t))$ be the solution of (1.1) with the initial function φ_n. Let $\omega_n(\varphi_n)$ be the omega limit set of $(S^{(n)}(t), I^{(n)}(t), R^{(n)}(t))$. By a completely similar argument as that used in [5] and [10], we have that there exits some compact and invariant set $\omega^* \subset Q$ such that $\mathrm{dist}(\omega_n(\varphi_n), \omega^*) \to 0$ as $n \to +\infty$. Here, $\mathrm{dist}(\omega_n(\varphi_n), \omega^*)$ means Hausdorff distance.

If (2.4) does not hold, for some initial function sequence $\{\varphi_n\} = \{(\varphi_1^{(n)}, \varphi_2^{(n)}, \varphi_3^{(n)})\} \subset Q$ such that $\varphi_2^{(n)}(0) > 0$ and $\varphi_3^{(n)}(0) > 0$, we have that there is some $\bar{\varphi} = (\bar{\varphi}_1, \bar{\varphi}_2, \bar{\varphi}_3) \in \omega^*$ such that $\bar{\varphi}_2(\theta_0) = 0$ for some $-h \leq \theta_0 \leq 0$. Now, let $(\bar{S}(t), \bar{I}(t), \bar{R}(t))$ be the solution of (1.1) with the initial function $\bar{\varphi}$. Then, by the invariance of ω^*, we have that $(\bar{S}_t, \bar{I}_t, \bar{R}_t) \in \omega^*$ for all $t \in \mathcal{R}$. Note that Remark 1 and $\bar{\varphi}_2(\theta_0) = 0$, we easily have that $\int_0^h \bar{I}(t-s)d\eta(s) + \bar{I}(t) = 0$ for all $t \leq \theta_0$. Hence, it follows from (1.1) that $I(t) = 0$ for all $t \leq 0$. This imples that $\bar{\varphi}_2(\theta) = 0$ for all $-h \leq \theta \leq 0$. It follows from Remark 1 and (1.1) that $\bar{S}(t) = \bar{g}_1(t)$, $\bar{I}(t) = 0$ and $\bar{R}(t) = \bar{g}_2(t)$ for all $t \in \mathcal{R}$, where

$$\bar{g}_1(t) = \frac{b}{\mu_1} - \left(\frac{b}{\mu_1} - \bar{\varphi}_1(0)\right)e^{-\mu_1 t}\,, \quad \bar{g}_2(t) = \bar{\varphi}_3(0)e^{-\mu_3 t}\,.$$

If $\bar{\varphi}_1(0) < b/\mu_1$ or $\bar{\varphi}_3(0) > 0$, we see that the negative semi-orbit $(\bar{S}(t), \bar{I}(t), \bar{R}(t))$ $(t \leq 0)$ is unbounded. This is a contradiction.

If $\bar{\varphi}_1(0) = b/\mu_1$, we have that $\bar{\varphi}_2(0) = \bar{\varphi}_3(0) = 0$. Hence, $\bar{S}(t) = b/\mu_1$ and $\bar{I}(t) = \bar{R}(t) = 0$ for all $t \in \mathcal{R}$. This shows that $\bar{\varphi} = (b/\mu_1, 0, 0) = E_0 \in \omega^*$.

Let us show that E_0 is factually isolated (see [5] or [10]). That is, there exists some neighborhood U of E_0 in Q such that E_0 is the largest invariant set in U.

In fact, let us choose

$$U = \{\varphi \mid \varphi = (\varphi_1,\ \varphi_2,\ \varphi_3) \in \bar{Q},\ \|\varphi - E_0\| < \varepsilon\}$$

for some sufficiently small positive constant ε. We shall show that E_0 is the largest invariant set in U for some ε.

If not, for any sufficiently small ε, there exists some invariant set W ($W \subset U$) such that $W \backslash E_0$ is not empty. Let $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in W \backslash E_0$ and (S_t, I_t, R_t) be the solution of (1.1) with the initial function φ. Then, $(S_t, I_t, R_t) \in W$ for all $t \in \mathcal{R}$.

If $\varphi_2(0) + \int_0^h \varphi_2(-s)d\eta(s) = 0$, by the invariance of W and Remark 1, we also have the contradiction that $\varphi = E_0$ or that the negative semi-orbit (S_t, I_t, R_t) $(t < 0)$ of (1.1) through φ is unbounded.

If $\varphi_2(0) + \int_0^h \varphi_2(-s)d\eta(s) > 0$, from Remark 1 we see that $I(t) > 0$ for all $t \geq 0$. Now, let us consider the continuous function

$$P(t) = I(t) + \rho \int_0^h \int_{t-\tau}^t I(u)dud\eta(\tau)$$

for some constant $\rho > 0$. We see that for $t \geq 0$, the time derivative of $P(t)$ along the solution $(S(t), I(t), R(t))$ satisfies

$$\begin{aligned}
\dot{P}(t) &= \dot{I}(t) + \rho\left(I(t) - \int_0^h I(t-\tau)d\eta(\tau)\right) \\
&= [\rho - (\mu_2 + \lambda)]I(t) + [\beta S(t) - \rho] \int_0^h I(t-\tau)d\eta(\tau) \\
&\geq [\rho - (\mu_2 + \lambda)]I(t) + \left[\beta\left(\frac{b}{\mu_1} - \varepsilon\right) - \rho\right] \int_0^h I(t-\tau)d\eta(\tau) \\
&= \left[\frac{\beta b}{\mu_1} - (\mu_2 + \lambda) - \beta\varepsilon\right] I(t),
\end{aligned} \tag{2.5}$$

for $t \geq 0$. Here, we choose $\rho = \beta(b/\mu_1 - \varepsilon) > 0$ and used the inequality $S(t) \geq b/\mu_1 - \varepsilon$ for all $t \in \mathcal{R}$. From Lemma 4, we have that $I(t) \geq \eta > 0$ for some constant η and all large $t \geq t_1 > 0$. Hence, it follows from (2.5) and (1.3), that for some sufficiently small ε,

$$\dot{P}(t) \geq \eta\left[\frac{\beta b}{\mu_1} - (\mu_2 + \lambda) - \beta\varepsilon\right] > 0$$

for all $t \geq t_1$. Thus, $P(t) \to +\infty$ as $t \to +\infty$. This contradicts Lemma 1, and shows that E_0 is isolated.

We easily see that the semigroup defined by the solution of (1.1) satisfies the conditions of Lemma 4.3 in [10] with $M = E_0$. Thus, by Lemma 4.3 in [10], we have that there is some $\xi = (\xi_1, \xi_2, \xi_3)$ such that $\xi \in \omega^* \cap (W^s(E_0) \backslash E_0)$. Here, $W^s(E_0)$ denotes the stable set of E_0.

If $\xi_2(0) + \int_0^h \xi_2(-s)d\eta(s) = 0$, again by the invariance of W and Remark 1, we also have the contradiction that $\xi = E_0$ or that the negative semi-orbit $(\hat{S}_t, \hat{I}_t, \hat{R}_t)$ $(t < 0)$ of (1.1) through ξ is unbounded.

If $\xi_2(0) + \int_0^h \xi_2(-s)d\eta(s) > 0$, from Remark 1, we see that $\hat{S}(t) > 0$, $\hat{I}(t) > 0$ and $\hat{R}(t) > 0$ for all $t > 0$. It follows from $\xi \in \omega^* \cap (W^s(E_0)\backslash E_0)$ that

$$\lim_{t\to+\infty} \hat{S}(t) = \frac{b}{\mu_1}, \quad \lim_{t\to+\infty} \hat{I}(t) = \lim_{t\to+\infty} \hat{R}(t) = 0,$$

which contradicts Lemma 4. This shows that (2.4) holds. From (1.1) and (2.4) we easily have that

$$\liminf_{t\to+\infty} R(t) \geq \frac{\lambda\nu_2}{\mu_3} \equiv \nu_3 > 0.$$

Thus, (1.1) is permanent by Lemmas 1 and 2. This proves our theorem.

3. Conclusion. In this paper, we considered permanence of (1.1). In biology, our theorem together with results in [3, 4] and [13] show that, for any time delay h, the condition (1.3) is the threshold of (1.1) for an endemic to occur. On the other hand, the simulations for (1.1) given below suggest that the condition (1.3) maybe also necessary and sufficient for *the global asymptotic stability of the endemic equilibrium* E_+ *of* (1.1) *for any time delay* h. Unfortunately, we cannot give a complete proof to the problem. We can only show that the endemic equilibrium E_+ of (1.1) is globally asymptotically stable for *small time delay* h [4].

EXAMPLE. Note that the first two equations of (1.1) are independent of the state variable $R(t)$ and that the third equation of (1.1) is linear with respect to $I(t)$ and $R(t)$. We consider the following sub-systems (3.1) and (3.2) with discrete and distributed time delays, respectively.

$$\begin{cases} \dot{S}(t) = -0.1S(t)I(t-h) - 0.1S(t) + 0.5, \\ \dot{I}(t) = 0.1S(t)I(t-h) - \alpha I(t), \end{cases} \tag{3.1}$$

$$\begin{cases} \dot{S}(t) = -0.1S(t)\displaystyle\int_0^h \left(\frac{e^{-s}}{1-e^{-h}}\right) I(t-s)ds - 0.1S(t) + 0.5, \\ \dot{I}(t) = 0.1S(t)\displaystyle\int_0^h \left(\frac{e^{-s}}{1-e^{-h}}\right) I(t-s)ds - \alpha I(t), \end{cases} \tag{3.2}$$

where $\alpha > 0$ and $h > 0$. It is clear that the condition (1.3) is reduced to

$$0 < \alpha < \frac{1}{2}. \tag{3.3}$$

There exists the disease free equilibrium $E_0 = (5, 0)$ for (3.1) and (3.2). If (3.3) holds, there also exists the endemic equilibrium $E_+ = (10\alpha, (0.5-\alpha)/\alpha)$ for (3.1) and (3.2).

Figures 1 and 2 illustrate our theorem and further suggest that, *for large time delay* h, *the endemic equilibrium* E_+ *of* (1.1) *is also globally asymptotically stable if and only if* (1.3) *holds.*

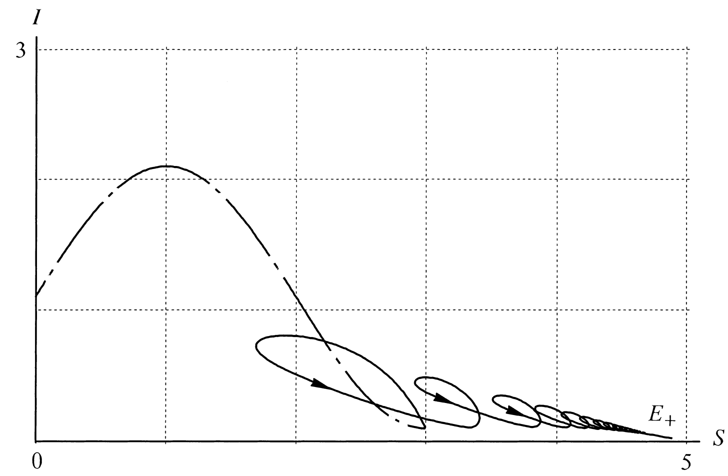

FIGURE 1. The graph of the trajectory of (3.1) with $\alpha = 0.49$, $h = 30$ and the initial function $\varphi_1(\theta) = 0.1\theta + 3$ and $\varphi_2(\theta) = 1.1 - \cos(0.05\pi\theta)$ for $\theta \in [-h, 0]$.

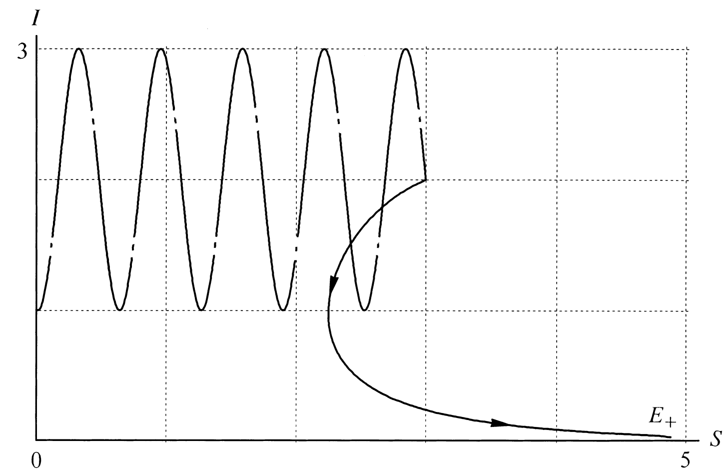

FIGURE 2. The graph of the trajectory of (3.2) with $\alpha = 0.49$, $h = 30$ and the initial function $\varphi_1(\theta) = 0.1\theta + 3$ and $\varphi_2(\theta) = 2 - \sin(\theta)$ for $\theta \in [-h, 0]$.

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