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TWINING CHARACTERS AND KOSTANT'S HOMOLOGY FORMULA

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Abstract. Let $\mathfrak{g}$ be a symmetrizable generalized Kac-Moody algebra and $\mathfrak{n}_-$ the sum of all its negative root spaces. We obtain a formula for the twining characters of the Lie algebra homology modules of $\mathfrak{n}_-$ with coefficients in the irreducible highest weight $\mathfrak{g}$-module $L(\Lambda)$ of symmetric, dominant integral highest weight Λ. This formula gives a new (and convincing) proof of the formula for the twining character of $L(\Lambda)$ above.

Introduction. In [FSS] and [FRS], Fuchs, Schweigert, *et al.* introduced a new type of character-like quantities, called twining characters, corresponding to a Dynkin diagram automorphism for certain highest weight modules over a symmetrizable (generalized) Kac-Moody algebra $\mathfrak{g}$. Moreover, they gave a formula (see Theorem 2.2.1) for the twining character of an irreducible highest weight $\mathfrak{g}$-module $L(\Lambda)$ of symmetric, dominant integral highest weight Λ.

In this paper, we give a new proof of this result of theirs. In our proof, we use an extension of Kostant's homology formula to generalized Kac-Moody algebras in [N2] to obtain a formula for the twining characters of the Lie algebra homology modules $H_j(\mathfrak{n}_-, L(\Lambda))$, $j \geq 0$, of $\mathfrak{n}_-$ with coefficients in $L(\Lambda)$, where $\mathfrak{n}_-$ is the sum of all negative root spaces of $\mathfrak{g}$. Then, by an Euler-Poincaré principle, we get the twining character formula for $L(\Lambda)$ of symmetric, dominant integral highest weight Λ.

This new proof will give us a satisfactory explanation of why we need the subgroup $\tilde{W}$ of the Weyl group W consisting of elements which commute with the Dynkin diagram automorphism.

This paper is organized as follows. In Section 1, we recall the definition of a generalized Kac-Moody algebra and fix our notation. Furthermore, we review an extension of Kostant's homology formula to generalized Kac-Moody algebras in [N2]. In Section 2, following [FSS] and [FRS], we review the definition of a twining character and the twining character formula for $L(\Lambda)$.

Section 3 is the main part of this paper. There we show a formula for the twining characters of the Lie algebra homology modules $H_j(\mathfrak{n}_-, L(\Lambda))$, $j \geq 0$, and then give a new proof of the twining character formula for $L(\Lambda)$.

1. Preliminaries and notation.

1.1. Generalized Kac-Moody algebras. Let $I = \{1, 2, \dots, n\}$ be a finite index set, and let $A = (a_{ij})_{i,j \in I}$ be an $n \times n$ real matrix satisfying:

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(C1) either $a_{ii} = 2$ or $a_{ii} \le 0$ for all $i \in I$;
(C2) $a_{ij} \le 0$ if $i \ne j \in I$, and $a_{ij} \in \boldsymbol{Z}$ for $j \ne i$ if $a_{ii} = 2$;
(C3) $a_{ij} = 0$ if and only if $a_{ji} = 0$ for $i, j \in I$.

Such a matrix $A = (a_{ij})_{i,j \in I}$ is called a generalized GCM (GGCM). In this paper, following [K], we define a generalized Kac-Moody algebra (GKM algebra) $\mathfrak{g}$ over $\boldsymbol{C}$ to be the contragredient Lie algebra $\mathfrak{g}(A)$ associated to a GGCM $A = (a_{ij})_{i,j \in I}$. Let $\mathfrak{h}$ be the Cartan subalgebra and e_i, f_i for $i \in I$ the Chevalley generators. Let $\Delta_+ \subset \mathfrak{h}^* := \mathrm{Hom}_{\boldsymbol{C}}(\mathfrak{h}, \boldsymbol{C})$ be the set of positive roots, $\Delta_- = -\Delta_+$ the set of negative roots, and $\mathfrak{g}_\alpha$ the root space of $\mathfrak{g}$ corresponding to a root $\alpha \in \Delta = \Delta_- \sqcup \Delta_+$. We set

$$\mathfrak{n}_\pm := \bigoplus_{\alpha \in \Delta_\pm} \mathfrak{g}_\alpha\,, \quad \mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}_+\,. \tag{1.1.1}$$

We denote by $\Pi = \{\alpha_i \mid i \in I\}$ the set of simple roots, and by $\Pi^\vee = \{h_i \mid i \in I\}$ the set of simple coroots. We set $I^{\mathrm{re}} := \{i \in I \mid a_{ii} = 2\}$, $I^{\mathrm{im}} := \{i \in I \mid a_{ii} \le 0\}$, and call $\Pi^{\mathrm{re}} := \{\alpha_i \in \Pi \mid i \in I^{\mathrm{re}}\}$ the set of real simple roots, $\Pi^{\mathrm{im}} := \{\alpha_i \in \Pi \mid i \in I^{\mathrm{im}}\}$ the set of imaginary simple roots. Note that $\mathfrak{g}_{\alpha_i} = \boldsymbol{C}e_i$, $\mathfrak{g}_{-\alpha_i} = \boldsymbol{C}f_i$ for all $i \in I$.

The Weyl group W of the GKM algebra $\mathfrak{g}$ is defined by

$$W := \langle r_i \mid i \in I^{\mathrm{re}} \rangle \subset GL(\mathfrak{h}^*)\,, \tag{1.1.2}$$

where $r_i \in GL(\mathfrak{h}^*)$ for $i \in I^{\mathrm{re}}$ is the simple reflection of $\mathfrak{h}^*$. The length function of the Coxeter system $(W, \{r_i \mid i \in I^{\mathrm{re}}\})$ is denoted by

$$\ell : W \to \boldsymbol{Z}\,. \tag{1.1.3}$$

Throughout this paper, we assume that a GGCM $A = (a_{ij})_{i,j \in I}$ is symmetrizable, i.e., that there exist a diagonal matrix $D = \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_n)$ with $\varepsilon_i > 0$ for all $i \in I$ and a symmetric matrix $B = (b_{ij})_{i,j \in I}$ such that $A = DB$. Hence there exists a nondegenerate, symmetric, invariant bilinear form $(\cdot|\cdot)$ on $\mathfrak{g} = \mathfrak{g}(A)$. The restriction of this bilinear form $(\cdot|\cdot)$ to $\mathfrak{h}$ is again nondegenerate, so that it induces a nondegenerate, symmetric, W-invariant bilinear form on $\mathfrak{h}^*$, which is also denoted by $(\cdot|\cdot)$.

1.2. Kostant's homology formula. For $\lambda \in \mathfrak{h}^*$, let

$$M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \boldsymbol{C}(\lambda) \tag{1.2.1}$$

be the Verma module of highest weight λ over $\mathfrak{g}$, where $U(\mathfrak{a})$ denotes the universal enveloping algebra of a Lie algebra $\mathfrak{a}$ and $\boldsymbol{C}(\lambda)$ is the one-dimensional (irreducible) $\mathfrak{h}$-module of weight λ on which $\mathfrak{n}_+$ acts trivially. We then define a $\mathfrak{g}$-module $L(\lambda)$ to be the unique irreducible quotient of $M(\lambda)$, that is,

$$L(\lambda) := M(\lambda)/J(\lambda)\,, \tag{1.2.2}$$

where $J(\lambda)$ is the unique maximal proper submodule of $M(\lambda)$.

Let

$$P_+ := \{\Lambda \in \mathfrak{h}^* \mid \Lambda(h_i) \ge 0 \text{ for all } i \in I, \text{ and } \Lambda(h_i) \in \boldsymbol{Z} \text{ if } a_{ii} = 2\} \tag{1.2.3}$$

be the set of dominant integral weights. We denote by $H_j(\mathfrak{n}_-, L(\Lambda))$, $j \geq 0$, the Lie algebra homology modules of $\mathfrak{n}_-$ with coefficients in $L(\Lambda)$ for $\Lambda \in P_+$. Recall from [GL] that, for $j \geq 0$, the $H_j(\mathfrak{n}_-, L(\Lambda))$ is defined to be the j-th homology of the chain complex $\{C_j(\mathfrak{n}_-, L(\Lambda)), d_j\}_{j\geq 0}$ with $C_j(\mathfrak{n}_-, L(\Lambda)) := (\bigwedge^j \mathfrak{n}_-) \otimes_{\mathbf{C}} L(\Lambda)$, where $\bigwedge^j \mathfrak{n}_-$ denotes the j-th exterior power of $\mathfrak{n}_-$. Note that the boundary operator $d_j : C_j(\mathfrak{n}_-, L(\Lambda)) \to C_{j-1}(\mathfrak{n}_-, L(\Lambda))$ commutes with the action of $\mathfrak{h}$, and hence $H_j(\mathfrak{n}_-, L(\Lambda))$ is an $\mathfrak{h}$-module in the usual way.

In order to state an extension of Kostant's homology formula to GKM algebras, we introduce some notation. Let $i \neq j \in I^{\mathrm{im}}$ and $\Lambda \in P_+$. Two distinct imaginary simple roots α_i and α_j are said to be pairwise perpendicular if $a_{ij} = 0 = a_{ji}$, and an imaginary simple root α_i is said to be perpendicular to Λ if $\Lambda(h_i) = 0$. We denote by $\mathcal{S}(\Lambda)$ the set of sums of distinct, pairwise perpendicular, imaginary simple roots perpendicular to Λ. In addition, for an element $\beta = \sum_{i\in I^{\mathrm{im}}} k_i\alpha_i \in \mathcal{S}(\Lambda)$, we set $\mathrm{ht}(\beta) := \sum_{i\in I^{\mathrm{im}}} k_i \in \mathbf{Z}_{\geq 0}$ (note that $k_i = 0, 1$ for all $i \in I^{\mathrm{im}}$ by the definition of $\mathcal{S}(\Lambda)$). Now we take and fix an element $\rho \in \mathfrak{h}^*$ (called a Weyl vector) such that $\rho(h_i) = (1/2) \cdot a_{ii}$ for all $i \in I$. For $(w, \beta) \in W \times \mathcal{S}(\Lambda)$, we set

$$(w, \beta) \circ \Lambda := w(\Lambda + \rho - \beta) - \rho\,. \tag{1.2.4}$$

We know from [N2, Propositions 3.2, 3.3, and Theorem 5.3] the following theorem.

THEOREM 1.2.1. *Let $\Lambda \in P_+$ and $j \in \mathbf{Z}_{\geq 0}$.*

(1) *We have the following isomorphism of $\mathfrak{h}$-modules:*

$$H_j(\mathfrak{n}_-, L(\Lambda)) \cong \bigoplus_{\substack{(w,\beta)\in W\times\mathcal{S}(\Lambda)\\ \ell(w)+\mathrm{ht}(\beta)=j}} \mathbf{C}((w,\beta)\circ\Lambda)\,.$$

Here the sum above is a direct sum of inequivalent irreducible $\mathfrak{h}$-modules, i.e., the weights $(w, \beta) \circ \Lambda$ for $(w, \beta) \in W \times \mathcal{S}(\Lambda)$ with $\ell(w) + \mathrm{ht}(\beta) = j$ are all distinct.

(2) *If we set $\mu := (w, \beta) \circ \Lambda$ for $(w, \beta) \in W \times \mathcal{S}(\Lambda)$ with $\ell(w) + \mathrm{ht}(\beta) = j$, then the multiplicities of μ (= the dimensions of the μ-weight space) in the $\mathfrak{h}$-modules $(\bigwedge^* \mathfrak{n}_-) \otimes_{\mathbf{C}} L(\Lambda)$ and $(\bigwedge^j \mathfrak{n}_-) \otimes_{\mathbf{C}} L(\Lambda)$ are both equal to one.*

Here we recall from the proof of [N2, Proposition 3.3] the construction of a nonzero weight vector $v_{(w,\beta)} \in (\bigwedge^j \mathfrak{n}_-) \otimes_{\mathbf{C}} L(\Lambda)$ of weight $\mu = (w, \beta) \circ \Lambda$ in part (2) of Theorem 1.2.1. First, we note that $w(\rho) - \rho = -\sum_{\alpha\in\Delta_w} \alpha$ and that the number of elements of the set Δ_w equals $\ell(w)$, where $\Delta_w := \{\alpha \in \Delta_+ \mid w^{-1}(\alpha) \in \Delta_-\}$. Second, we write β in the form $\beta = \sum_{k=1}^m \alpha_{i_k}$, where $m = \mathrm{ht}(\beta)$, $\alpha_{i_k} \in \Pi^{\mathrm{im}}$, and $i_r \neq i_t$ for $1 \leq r \neq t \leq m$. Now we take nonzero root vectors $F_k \in \mathfrak{g}_{-w(\alpha_{i_k})}$ for $1 \leq k \leq m$, $F_\alpha \in \mathfrak{g}_{-\alpha}$ for $\alpha \in \Delta_w$, and a nonzero weight vector $v_{w(\Lambda)} \in L(\Lambda)_{w(\Lambda)}$ of weight $w(\Lambda)$. Then we set

$$v_{(w,\beta)} := (F_1 \wedge \cdots \wedge F_m) \wedge \left(\bigwedge_{\alpha\in\Delta_w} F_\alpha\right) \otimes v_{w(\Lambda)} \in \left(\bigwedge^j \mathfrak{n}_-\right) \otimes_{\mathbf{C}} L(\Lambda)\,. \tag{1.2.5}$$

We see that the vector $v_{(w,\beta)} \in (\bigwedge^j \mathfrak{n}_-) \otimes_{\boldsymbol{C}} L(\Lambda)$ is nonzero, and of weight $\mu = (w, \beta) \circ \Lambda$, since

$$\begin{aligned}\mu &= w(\Lambda + \rho - \beta) - \rho \\ &= -w(\beta) + (w(\rho) - \rho) + w(\Lambda) \\ &= \sum_{k=1}^{m} (-w(\alpha_{i_k})) + \sum_{\alpha \in \Delta_w} (-\alpha) + w(\Lambda)\,.\end{aligned}$$

We deduce from Theorem 1.2.1 that the image $\bar{v}_{(w,\beta)}$ of the vector $v_{(w,\beta)} \in (\bigwedge^j \mathfrak{n}_-) \otimes_{\boldsymbol{C}} L(\Lambda)$ of weight μ by the natural quotient map $\bar{\ } : (\bigwedge^j \mathfrak{n}_-) \otimes_{\boldsymbol{C}} L(\Lambda) \to H_j(\mathfrak{n}_-, L(\Lambda))$ is nonzero, and hence that the μ-weight space $(H_j(\mathfrak{n}_-, L(\Lambda)))_\mu$ of $H_j(\mathfrak{n}_-, L(\Lambda))$ is spanned by the vector $\bar{v}_{(w,\beta)}$:

(1.2.6) $$(H_j(\mathfrak{n}_-, L(\Lambda)))_\mu = \boldsymbol{C}\,\bar{v}_{(w,\beta)}\,.$$

2. Twining character formula for $L(\Lambda)$.

2.1. Twining characters. We recall the definition of the twining character of a certain highest weight module, following [FRS] and [FSS] (see also [N4]).

Let $A = (a_{ij})_{i,j \in I}$ be a symmetrizable GGCM indexed by a finite set I. A bijection $\omega : I \to I$ such that

(2.1.1) $$a_{\omega(i),\omega(j)} = a_{ij} \quad \text{for all } i, j \in I$$

is called a (Dynkin) diagram automorphism, since such an ω induces an automorphism of the Dynkin diagram of the GGCM $A = (a_{ij})_{i,j \in I}$ as a graph. Let N be the order of $\omega : I \to I$, and N_i the number of elements of the ω-orbit of $i \in I$ in I. We may (and will henceforth) assume that $\varepsilon_{\omega(i)} = \varepsilon_i$ for all $i \in I$ in the decomposition $A = DB$ with $D = \mathrm{diag}(\varepsilon_1, \dots, \varepsilon_n)$ (see [N4, §3.1]).

The diagram automorphism $\omega : I \to I$ can be extended (cf. [FSS, §3.2] and [K, §2.2]) to an automorphism ω of order N of the GKM algebra $\mathfrak{g} = \mathfrak{g}(A)$ associated to the GGCM $A = (a_{ij})_{i,j \in I}$ so that

(2.1.2) $$\begin{cases} \omega(e_i) := e_{\omega(i)} & \text{for } i \in I\,, \\ \omega(f_i) := f_{\omega(i)} & \text{for } i \in I\,, \\ \omega(h_i) := h_{\omega(i)} & \text{for } i \in I\,, \\ \omega(\mathfrak{h}) := \mathfrak{h}\,, & \\ (\omega(x)|\omega(y)) = (x|y) & \text{for } x, y \in \mathfrak{g}\,. \end{cases}$$

Notice that the $\omega : \mathfrak{g} \to \mathfrak{g}$ extends to a unique algebra automorphism $\omega : U(\mathfrak{g}) \to U(\mathfrak{g})$ by

$$\omega(x_1 \cdots x_k) = \omega(x_1) \cdots \omega(x_k) \quad \text{for } x_1, \dots, x_k \in \mathfrak{g}.$$

We call these two automorphisms ω also diagram automorphisms by abuse of notation.

The restriction of the diagram automorphism $\omega : \mathfrak{g} \to \mathfrak{g}$ to the Cartan subalgebra $\mathfrak{h}$ induces a transposed map $\omega^* : \mathfrak{h}^* \to \mathfrak{h}^*$ by

$$\omega^*(\lambda)(h) := \lambda(\omega(h)) \quad \text{for } \lambda \in \mathfrak{h}^*,\ h \in \mathfrak{h}\,. \tag{2.1.3}$$

We set

$$(\mathfrak{h}^*)^0 := \{\lambda \in \mathfrak{h}^* \mid \omega^*(\lambda) = \lambda\}\,, \tag{2.1.4}$$

and call an element of $(\mathfrak{h}^*)^0$ a symmetric weight. Note that we may (and will henceforth) assume that the Weyl vector ρ is a symmetric weight, i.e.,

$$\omega^*(\rho) = \rho\,. \tag{2.1.5}$$

Let $\lambda \in (\mathfrak{h}^*)^0$ be a symmetric weight, and let $V(\lambda)$ be either the Verma module $M(\lambda)$ or the irreducible highest weight module $L(\lambda)$ of highest weight λ. Then there exists a unique linear automorphism $\tau_\omega : V(\lambda) \to V(\lambda)$ such that

$$\tau_\omega(xv) = \omega^{-1}(x)\tau_\omega(v) \quad \text{for } x \in \mathfrak{g},\ v \in V(\lambda)\,, \tag{2.1.6}$$

and

$$\tau_\omega(v) = v \quad \text{for } v \in V(\lambda)_\lambda\,, \tag{2.1.7}$$

where $V(\lambda)_\lambda$ is the (one-dimensional) highest weight space of $V(\lambda)$.

REMARK 2.1.1. Because $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \boldsymbol{C}(\lambda)$ by definition, we can take the linear automorphism $\omega^{-1} \otimes \mathrm{id} : U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \boldsymbol{C}(\lambda) \to U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \boldsymbol{C}(\lambda)$ for $\tau_\omega : M(\lambda) \to M(\lambda)$ above. Moreover, since this map $\omega^{-1} \otimes \mathrm{id} : M(\lambda) \to M(\lambda)$ stabilizes the unique maximal proper submodule $J(\lambda)$ of $M(\lambda)$, we can take for $\tau_\omega : L(\lambda) \to L(\lambda)$ above the linear map $M(\lambda)/J(\lambda) \to M(\lambda)/J(\lambda)$ induced from $\omega^{-1} \otimes \mathrm{id} : M(\lambda) \to M(\lambda)$.

REMARK 2.1.2. Let V be an $\mathfrak{h}$-module admitting a weight space decomposition

$$V = \bigoplus_{\chi \in \mathfrak{h}^*} V_\chi$$

with finite-dimensional weight spaces V_χ, and let $f : V \to V$ be a linear map such that $f(hv) = \omega^{-1}(h) f(v)$ for $h \in \mathfrak{h},\ v \in V$. Then it follows that

$$f(V_\chi) \subset V_{\omega^*(\chi)}$$

for all $\chi \in \mathfrak{h}^*$. Thus, we define a formal sum:

$$\mathrm{Tr}_V\, f \exp := \sum_{\chi \in (\mathfrak{h}^*)^0} \mathrm{Tr}(f|_{V_\chi})\, e(\chi)\,,$$

where $\mathrm{Tr}(f|_{V_\chi})$ for $\chi \in (\mathfrak{h}^*)^0$ denotes the trace of the restriction of f to the χ-weight space V_χ of V.

Let $\lambda \in (\mathfrak{h}^*)^0$ be a symmetric weight. The twining character $\mathrm{ch}^{\omega}(V(\lambda))$ of $V(\lambda)$ $(= M(\lambda), L(\lambda))$ is then defined to be the formal sum

$$\mathrm{ch}^{\omega}(V(\lambda)) := \mathrm{Tr}_{V(\lambda)}\, \tau_\omega \, \exp = \sum_{\chi \in (\mathfrak{h}^*)^0} \mathrm{Tr}(\tau_\omega|_{V(\lambda)_\chi})\, e(\chi)\,. \tag{2.1.8}$$

2.2. Twining character formula for $L(\Lambda)$. We review the twining character formula for $L(\Lambda)$ of symmetric, dominant integral highest weight Λ, which is the main result of [FSS] and [FRS].

We choose a set of representatives $\hat{I}$ of the ω-orbits in I, and then introduce the following subset of $\hat{I}$:

$$\check{I} := \left\{ i \in \hat{I} \;\middle|\; \sum_{k=0}^{N_i-1} a_{i,\omega^k(i)} = 1,\ 2 \right\}. \tag{2.2.1}$$

We define the following subgroup of the Weyl group W:

$$\tilde{W} := \{ w \in W \mid \omega^* w = w \omega^* \}\,. \tag{2.2.2}$$

We know from [FRS, Proposition 3.3] that the group $\tilde{W}$ is a Coxeter group with the canonical generator system $\{w_i \mid i \in \check{I}\}$, where for $i \in \check{I}$,

$$w_i := \begin{cases} \displaystyle\prod_{k=0}^{N_i/2-1} (r_{\omega^k(i)}\, r_{\omega^{k+N_i/2}(i)}\, r_{\omega^k(i)}) & \text{if} \quad \displaystyle\sum_{k=0}^{N_i-1} a_{i,\omega^k(i)} = 1\,, \\ \displaystyle\prod_{k=0}^{N_i-1} r_{\omega^k(i)} & \text{if} \quad \displaystyle\sum_{k=0}^{N_i-1} a_{i,\omega^k(i)} = 2\,. \end{cases} \tag{2.2.3}$$

Here we note that if $\sum_{k=0}^{N_i-1} a_{i,\omega^k(i)} = 1$, then N_i is an even integer. We denote the length function of the Coxeter system $(\tilde{W}, \{w_i \mid i \in \check{I}\})$ by

$$\hat{\ell} : \tilde{W} \to \mathbf{Z}\,. \tag{2.2.4}$$

We also recall from [FRS, Equation (1) on page 529] that for a symmetric weight $\lambda \in (\mathfrak{h}^*)^0$ and $i \in \check{I}$,

$$w_i(\lambda) = \lambda - \frac{2s_i(\lambda|\alpha_i)}{(\alpha_i|\alpha_i)} \sum_{k=0}^{N_i-1} \alpha_{\omega^k(i)}\,, \tag{2.2.5}$$

where $s_i := 2/\sum_{k=0}^{N_i-1} a_{i,\omega^k(i)}$.

If $\Lambda \in P_+ \cap (\mathfrak{h}^*)^0$ is a symmetric, dominant integral weight, then each element $\beta \in \mathcal{S}(\Lambda) \cap (\mathfrak{h}^*)^0$ can be written in the form $\beta = \sum_{i \in \hat{I}} k_i \beta_i$, where $\beta_i := \sum_{k=0}^{N_i-1} \alpha_{\omega^k(i)} \in (\mathfrak{h}^*)^0$ and $k_i = 0,\ 1$ for $i \in \hat{I}$. For such a $\beta \in \mathcal{S}(\Lambda) \cap (\mathfrak{h}^*)^0$, we set

$$\widehat{\mathrm{ht}}(\beta) := \sum_{i \in \hat{I}} k_i\,. \tag{2.2.6}$$

Let $\Lambda \in P_+ \cap (\mathfrak{h}^*)^0$ be a symmetric, dominant integral weight. We know from [FRS, Proposition 3.5] that for every $w \in \tilde{W}$,

$$w(\mathrm{ch}^{\omega}(L(\Lambda))) = \mathrm{ch}^{\omega}(L(\Lambda)) \,, \tag{2.2.7}$$

since the $\mathfrak{g}$-module $L(\Lambda)$ is integrable. Moreover, we have

THEOREM 2.2.1 ([FRS, Theorem 3.1]). *Let $\Lambda \in P_+ \cap (\mathfrak{h}^*)^0$ be a symmetric, dominant integral weight. Then*

$$\mathrm{ch}^{\omega}(L(\Lambda)) = \frac{\displaystyle\sum_{\substack{w\in\tilde{W} \\ \beta\in\mathcal{S}(\Lambda)\cap(\mathfrak{h}^*)^0}} (-1)^{\hat{\ell}(w)+\widehat{\mathrm{ht}}(\beta)}\, e((w,\beta)\circ\Lambda)}{\displaystyle\sum_{\substack{w\in\tilde{W} \\ \beta\in\mathcal{S}(0)\cap(\mathfrak{h}^*)^0}} (-1)^{\hat{\ell}(w)+\widehat{\mathrm{ht}}(\beta)}\, e((w,\beta)\circ 0)} \,.$$

3. Twining character formula for $H_j(\mathfrak{n}_-, L(\Lambda))$.

3.1. Some lemmas. Since the inverse $\omega^{-1} : \mathfrak{g} \to \mathfrak{g}$ of the diagram automorphism $\omega : \mathfrak{g} \to \mathfrak{g}$ stabilizes $\mathfrak{n}_-$, i.e., $\omega^{-1}(\mathfrak{n}_-) = \mathfrak{n}_-$, it induces an algebra automorphism

$$\bigwedge\nolimits^* \omega^{-1} : \bigwedge\nolimits^* \mathfrak{n}_- \to \bigwedge\nolimits^* \mathfrak{n}_-$$

of the exterior algebra $\bigwedge^* \mathfrak{n}_-$ of $\mathfrak{n}_-$. The restriction of the $\bigwedge^* \omega^{-1} : \bigwedge^* \mathfrak{n}_- \to \bigwedge^* \mathfrak{n}_-$ to each homogeneous subspace $\bigwedge^j \mathfrak{n}_-$ for $j \geq 0$ is denoted by

$$\bigwedge\nolimits^j \omega^{-1} : \bigwedge\nolimits^j \mathfrak{n}_- \to \bigwedge\nolimits^j \mathfrak{n}_- \,.$$

Let $\Lambda \in P_+ \cap (\mathfrak{h}^*)^0$ be a symmetric, dominant integral weight, and let $\tau_\omega : L(\Lambda) \to L(\Lambda)$ be the linear automorphism in Section 2.1. We define a linear automorphism

$$\Phi := \left(\bigwedge\nolimits^* \omega^{-1}\right) \otimes \tau_\omega : \left(\bigwedge\nolimits^* \mathfrak{n}_-\right) \otimes_{\boldsymbol{C}} L(\Lambda) \to \left(\bigwedge\nolimits^* \mathfrak{n}_-\right) \otimes_{\boldsymbol{C}} L(\Lambda) \,, \tag{3.1.1}$$

and for $j \geq 0$, we define a linear automorphism

$$\Phi_j := \left(\bigwedge\nolimits^j \omega^{-1}\right) \otimes \tau_\omega : \left(\bigwedge\nolimits^j \mathfrak{n}_-\right) \otimes_{\boldsymbol{C}} L(\Lambda) \to \left(\bigwedge\nolimits^j \mathfrak{n}_-\right) \otimes_{\boldsymbol{C}} L(\Lambda) \,. \tag{3.1.2}$$

REMARK 3.1.1. Let $j \geq 0$. It is easily seen that

$$\Phi_j(hv) = \omega^{-1}(h)\Phi_j(v)$$

for $h \in \mathfrak{h}$ and $v \in (\bigwedge^j \mathfrak{n}_-) \otimes_{\boldsymbol{C}} L(\Lambda)$. It also follows that for $h \in \mathfrak{h}$ and $v \in (\bigwedge^* \mathfrak{n}_-) \otimes_{\boldsymbol{C}} L(\Lambda)$,

$$\Phi(hv) = \omega^{-1}(h)\Phi(v) \,.$$

The follwing lemma immediately follows from the definitions of d_j and Φ_j, $j \geq 0$.

LEMMA 3.1.2. *We have the following commutative diagram for each* $j \geq 0$:

$$\begin{array}{ccc} (\bigwedge^j \mathfrak{n}_-) \otimes_{\boldsymbol{C}} L(\Lambda) & \xrightarrow{\Phi_j} & (\bigwedge^j \mathfrak{n}_-) \otimes_{\boldsymbol{C}} L(\Lambda) \\ {\scriptstyle d_j}\downarrow & & \downarrow{\scriptstyle d_j} \\ (\bigwedge^{j-1} \mathfrak{n}_-) \otimes_{\boldsymbol{C}} L(\Lambda) & \xrightarrow[\Phi_{j-1}]{} & (\bigwedge^{j-1} \mathfrak{n}_-) \otimes_{\boldsymbol{C}} L(\Lambda)\,, \end{array}$$

where $d_j : (\bigwedge^j \mathfrak{n}_-) \otimes_{\boldsymbol{C}} L(\Lambda) \to (\bigwedge^{j-1} \mathfrak{n}_-) \otimes_{\boldsymbol{C}} L(\Lambda)$ *is the boundary operator in Section* 1.2.

By Lemma 3.1.2, the linear automorphism $\Phi_j : (\bigwedge^j \mathfrak{n}_-) \otimes_{\boldsymbol{C}} L(\Lambda) \to (\bigwedge^j \mathfrak{n}_-) \otimes_{\boldsymbol{C}} L(\Lambda)$ induces in the usual way a linear automorphism

$$\bar{\Phi}_j : H_j(\mathfrak{n}_-, L(\Lambda)) \to H_j(\mathfrak{n}_-, L(\Lambda)) \tag{3.1.3}$$

for $j \geq 0$. Notice that for $j \geq 0$ and $h \in \mathfrak{h}$, $v \in H_j(\mathfrak{n}_-, L(\Lambda))$,

$$\bar{\Phi}_j(hv) = \omega^{-1}(h)\bar{\Phi}_j(v) \tag{3.1.4}$$

by Remark 3.1.1.

Now we state an easy but useful lemma, which will be often used later. Let V and V' be $\mathfrak{h}$-modules which decompose into a direct sum of finite-dimensional weight spaces:

$$V = \bigoplus_{\chi \in \mathfrak{h}^*} V_\chi \quad \text{and} \quad V' = \bigoplus_{\chi \in \mathfrak{h}^*} V'_\chi .$$

We further assume that there exist linear automorphisms $\tau_\omega : V \to V$ and $\tau'_\omega : V' \to V'$ such that for $h \in \mathfrak{h}$,

$$\tau_\omega(hv) = \omega^{-1}(h)\tau_\omega(v) \ \text{ for } v \in V, \ \text{ and } \quad \tau'_\omega(hv) = \omega^{-1}(h)\tau'_\omega(v) \ \text{ for } v \in V' .$$

Then the linear automorphism

$$\tau_\omega \otimes \tau'_\omega : V \otimes_{\boldsymbol{C}} V' \to V \otimes_{\boldsymbol{C}} V'$$

obviously satisfies

$$(\tau_\omega \otimes \tau'_\omega)(hv) = \omega^{-1}(h)(\tau_\omega \otimes \tau'_\omega)(v)$$

for $h \in \mathfrak{h}$ and $v \in V \otimes_{\boldsymbol{C}} V'$. We can easily show

LEMMA 3.1.3. *In the notation above*, *we have*

$$\mathrm{Tr}_{V \otimes V'} (\tau_\omega \otimes \tau'_\omega) \exp = (\mathrm{Tr}_V \, \tau_\omega \exp) \cdot (\mathrm{Tr}_{V'} \, \tau'_\omega \exp) .$$

3.2. Main result. We define the twining character $\mathrm{ch}^\omega(H_j(\mathfrak{n}_-, L(\Lambda)))$ of the Lie algebra homology module $H_j(\mathfrak{n}_-, L(\Lambda))$ for each $j \geq 0$ by

$$\mathrm{ch}^\omega(H_j(\mathfrak{n}_-, L(\Lambda))) := \mathrm{Tr}_{H_j(\mathfrak{n}_-, L(\Lambda))} \, \bar{\Phi}_j \exp , \tag{3.2.1}$$

where $\bar{\Phi}_j : H_j(\mathfrak{n}_-, L(\Lambda)) \to H_j(\mathfrak{n}_-, L(\Lambda))$ is as in Section 3.1.

PROPOSITION 3.2.1. *Let $\Lambda \in P_+ \cap (\mathfrak{h}^*)^0$ be a symmetric, dominant integral weight, and let $j \geq 0$. Then*

$$\mathrm{ch}^{\omega}(H_j(\mathfrak{n}_-, L(\Lambda))) = \sum_{\substack{w \in \tilde{W} \\ \beta \in \mathcal{S}(\Lambda) \cap (\mathfrak{h}^*)^0 \\ \ell(w)+\mathrm{ht}(\beta)=j}} c_{(w,\beta)}\, e((w,\beta)\circ \Lambda)\,,$$

where the scalar $c_{(w,\beta)} \in \boldsymbol{C}$ is defined by

$$c_{(w,\beta)} := \mathrm{Tr}(\bar{\Phi}_j|_{(H_j(\mathfrak{n}_-, L(\Lambda)))_{(w,\beta)\circ\Lambda}})\,.$$

PROOF. Let $w \in W$ and $\beta \in \mathcal{S}(\Lambda)$ with $\ell(w) + \mathrm{ht}(\beta) = j$. Set $\mu := (w,\beta)\circ\Lambda$. Let us show that $\omega^*(\mu) = \mu$ if and only if $w \in \tilde{W}$ and $\beta \in (\mathfrak{h}^*)^0$. Because $\omega^* r_i (\omega^*)^{-1} = r_{\omega^{-1}(i)}$ for $i \in I^{\mathrm{re}}$, we see that $\omega^* w (\omega^*)^{-1} \in W$. If we set $w' := \omega^* w (\omega^*)^{-1} \in W$, then we have

$$\begin{aligned}
\omega^*(\mu) &= \omega^*(w(\Lambda+\rho-\beta)-\rho) \\
&= \omega^*(w(\Lambda+\rho-\beta)) - \omega^*(\rho) \\
&= \omega^* w(\Lambda+\rho-\beta) - \rho \\
&= w'\omega^*(\Lambda+\rho-\beta) - \rho \\
&= w'(\omega^*(\Lambda+\rho-\beta)) - \rho \\
&= w'(\Lambda+\rho-\omega^*(\beta)) - \rho\,,
\end{aligned}$$

since $\omega^*(\Lambda) = \Lambda$ and $\omega^*(\rho) = \rho$. Now assume that $\omega^*(\mu) = \mu$, i.e., that $w'(\Lambda+\rho-\omega^*(\beta)) = w(\Lambda+\rho-\beta)$. Then, since $(\Lambda+\rho-\beta)(h_i) \geq 1$ and $(\Lambda+\rho-\omega^*(\beta))(h_i) \geq 1$ for all $i \in I^{\mathrm{re}}$, we deduce that $\Lambda+\rho-\omega^*(\beta) = \Lambda+\rho-\beta$ (i.e., $\omega^*(\beta) = \beta$) and that $w' = w$ (i.e., $\omega^* w(\omega^*)^{-1} = w$) by the proof of [K, Proposition 3.12 a) and b)]. Conversely, $\omega^* w (\omega^*)^{-1} = w$ and $\omega^*(\beta) = \beta$ immediately imply that $\omega^*(\mu) = \mu$. Therefore, the proposition follows directly from Theorem 1.2.1 and the definition (3.2.1) of $\mathrm{ch}^{\omega}(H_j(\mathfrak{n}_-, L(\Lambda)))$. □

We see from the comments just below Theorem 1.2.1 that for $w \in \tilde{W}$ and $\beta \in \mathcal{S}(\Lambda) \cap (\mathfrak{h}^*)^0$ with $\ell(w) + \mathrm{ht}(\beta) = j$,

$$\begin{aligned}
c_{(w,\beta)} &= \mathrm{Tr}(\bar{\Phi}_j|_{(H_j(\mathfrak{n}_-, L(\Lambda)))_{(w,\beta)\circ\Lambda}}) \\
&= \mathrm{Tr}(\bar{\Phi}_j|_{\boldsymbol{C}\,\bar{v}_{(w,\beta)}}) \\
&= \mathrm{Tr}(\Phi_j|_{\boldsymbol{C}\,v_{(w,\beta)}}) \\
&= \mathrm{Tr}\big(\Phi_j|_{((\bigwedge^j \mathfrak{n}_-)\otimes_{\boldsymbol{C}} L(\Lambda))_{(w,\beta)\circ\Lambda}}\big)\,.
\end{aligned} \tag{3.2.2}$$

To determine the scalar $c_{(w,\beta)} \in \boldsymbol{C}$, we define the twining character $\mathrm{ch}^{\omega}((\bigwedge^* \mathfrak{n}_-) \otimes_{\boldsymbol{C}} L(\Lambda))$ of $(\bigwedge^* \mathfrak{n}_-) \otimes_{\boldsymbol{C}} L(\Lambda)$ by

$$\mathrm{ch}^{\omega}\left(\left(\bigwedge^* \mathfrak{n}_-\right) \otimes_{\boldsymbol{C}} L(\Lambda)\right) := \mathrm{Tr}_{(\bigwedge^* \mathfrak{n}_-)\otimes_{\boldsymbol{C}} L(\Lambda)}\, \Phi\, \exp\,, \tag{3.2.3}$$

where $\Phi = (\bigwedge^* \omega^{-1}) \otimes \tau_{\omega}$ is as in Section 3.1. The following is our key proposition.

PROPOSITION 3.2.2. *Let $\Lambda \in P_+ \cap (\mathfrak{h}^*)^0$ be a symmetric, dominant integral weight. For every $w \in \tilde{W}$, we have*

$$w\left(e(\rho)\cdot \mathrm{ch}^{\omega}\left(\left(\bigwedge^* \mathfrak{n}_-\right)\otimes_{\boldsymbol{C}} L(\Lambda)\right)\right) = (-1)^{\ell(w)-\hat{\ell}(w)}\left(e(\rho)\cdot \mathrm{ch}^{\omega}\left(\left(\bigwedge^* \mathfrak{n}_-\right)\otimes_{\boldsymbol{C}} L(\Lambda)\right)\right).$$

PROOF. It follows from Lemma 3.1.3 that

$$\mathrm{ch}^{\omega}\left(\left(\bigwedge^* \mathfrak{n}_-\right)\otimes_{\boldsymbol{C}} L(\Lambda)\right) = \mathrm{ch}^{\omega}\left(\bigwedge^* \mathfrak{n}_-\right)\cdot \mathrm{ch}^{\omega}(L(\Lambda)), \tag{3.2.4}$$

where

$$\mathrm{ch}^{\omega}\left(\bigwedge^* \mathfrak{n}_-\right) := \mathrm{Tr}_{\bigwedge^* \mathfrak{n}_-}\, \omega^{-1} \exp. \tag{3.2.5}$$

Furthermore, we know from (2.2.7) that for $w \in \tilde{W}$,

$$w(\mathrm{ch}^{\omega}(L(\Lambda))) = \mathrm{ch}^{\omega}(L(\Lambda)).$$

Hence we may assume that $\Lambda = 0$.

It is well-known that $(-1)^{\ell(ww')} = (-1)^{\ell(w)}(-1)^{\ell(w')}$ for $w, w' \in W$, and $(-1)^{\hat{\ell}(ww')} = (-1)^{\hat{\ell}(w)}(-1)^{\hat{\ell}(w')}$ for $w, w' \in \tilde{W}$. Thus, we may assume that $\hat{\ell}(w) = 1$, i.e., $w = w_i$ for some $i \in \breve{I}$.

We set

$$\Delta_i := \Delta_+ \cap \left(\sum_{k=0}^{N_i-1} \boldsymbol{Z}\,\alpha_{\omega^k(i)}\right), \quad \Delta(i) := \Delta_+ \setminus \Delta_i, \tag{3.2.6}$$

and correspondingly

$$\mathfrak{n}_i := \bigoplus_{\alpha\in\Delta_i} \mathfrak{g}_{-\alpha}, \quad \mathfrak{n}(i) := \bigoplus_{\alpha\in\Delta(i)} \mathfrak{g}_{-\alpha}. \tag{3.2.7}$$

When $\sum_{k=0}^{N_i-1} a_{i,\omega^k(i)} = 2$, we have

$$\Delta_i = \{\alpha_{\omega^k(i)}\}_{k=0}^{N_i-1}.$$

We call this case "Case (a)". When $\sum_{k=0}^{N_i-1} a_{i,\omega^k(i)} = 1$, we have

$$\Delta_i = \{\alpha_{\omega^k(i)}\}_{k=0}^{N_i-1} \sqcup \{\alpha_{\omega^k(i)} + \alpha_{\omega^{(N_i/2)+k}(i)}\}_{k=0}^{N_i/2-1}.$$

We call this case "Case (b)". Since $\mathfrak{n}_- = \mathfrak{n}_i \oplus \mathfrak{n}(i)$, we have an isomorphism of $\mathfrak{h}$-modules:

$$\bigwedge^* \mathfrak{n}_- \cong \left(\bigwedge^* \mathfrak{n}_i\right)\otimes_{\boldsymbol{C}}\left(\bigwedge^* \mathfrak{n}(i)\right).$$

Here we note that $\omega^*(\Delta_i) = \Delta_i$ and $\omega^*(\Delta(i)) = \Delta(i)$. Then, since $\omega^{-1}(\mathfrak{g}_\alpha) = \mathfrak{g}_{\omega^*(\alpha)}$ for $\alpha \in \Delta$, we get that

$$\left(\bigwedge^* \omega^{-1}\right)\left(\bigwedge^* \mathfrak{n}_i\right) = \bigwedge^* \mathfrak{n}_i,$$

$$\Big(\bigwedge^* \omega^{-1}\Big)\Big(\bigwedge^* \mathfrak{n}(i)\Big) = \bigwedge^* \mathfrak{n}(i)\,.$$

Therefore, we can apply Lemma 3.1.3 to the $\mathfrak{h}$-module $\bigwedge^* \mathfrak{n}_- \cong (\bigwedge^* \mathfrak{n}_i) \otimes_{\boldsymbol{C}} (\bigwedge^* \mathfrak{n}(i))$ to obtain that

$$\mathrm{ch}^\omega\Big(\bigwedge^* \mathfrak{n}_-\Big) = \mathrm{ch}^\omega\Big(\bigwedge^* \mathfrak{n}_i\Big) \cdot \mathrm{ch}^\omega\Big(\bigwedge^* \mathfrak{n}(i)\Big)\,, \tag{3.2.8}$$

where $\mathrm{ch}^\omega(\bigwedge^* \mathfrak{n}_i)$ and $\mathrm{ch}^\omega(\bigwedge^* \mathfrak{n}(i))$ are defined by

$$\mathrm{ch}^\omega\Big(\bigwedge^* \mathfrak{n}_i\Big) := \mathrm{Tr}_{\bigwedge^* \mathfrak{n}_i} \bigwedge^* \omega^{-1} \exp\,, \tag{3.2.9}$$

$$\mathrm{ch}^\omega\Big(\bigwedge^* \mathfrak{n}(i)\Big) := \mathrm{Tr}_{\bigwedge^* \mathfrak{n}(i)} \bigwedge^* \omega^{-1} \exp\,. \tag{3.2.10}$$

STEP 1. First, we show that

$$w_i\Big(e(\rho) \cdot \mathrm{ch}^\omega\Big(\bigwedge^* \mathfrak{n}_i\Big)\Big) = (-1)^{\ell(w_i)-\hat{\ell}(w_i)}\Big(e(\rho) \cdot \mathrm{ch}^\omega\Big(\bigwedge^* \mathfrak{n}_i\Big)\Big)\,. \tag{3.2.11}$$

To show this, as an ordered basis of $\mathfrak{n}_i$ we take $\{f_{\omega^k(i)}\}_{k=0}^{N_i-1}$ in Case (a), and $\{f_{\omega^k(i)}\}_{k=0}^{N_i-1} \sqcup \{\dot{f}_{\omega^k(i)}\}_{k=0}^{N_i/2-1}$ with $\dot{f}_{\omega^k(i)} := [f_{\omega^k(i)}, f_{\omega^{(N_i/2)+k}(i)}]$ for $0 \le k \le (N_i/2) - 1$ in Case (b).

Case (a): We can easily deduce that the only basis vectors of $\bigwedge^* \mathfrak{n}_i$ which make a contribution to the trace of $\bigwedge^* \omega^{-1}$ are the following two vectors:

$$1 \in \bigwedge^0 \mathfrak{n}_i = \boldsymbol{C} \quad \text{and} \quad f_i \wedge f_{\omega(i)} \wedge \cdots \wedge f_{\omega^{N_i-1}(i)}\,.$$

Hence we immediately obtain that

$$\mathrm{ch}^\omega\Big(\bigwedge^* \mathfrak{n}_i\Big) = 1 + (-1)^{N_i-1} e(-\beta_i)\,, \tag{3.2.12}$$

where $\beta_i = \sum_{k=0}^{N_i-1} \alpha_{\omega^k(i)}$. Now we use Equation (2.2.5) to see that $w_i(\rho) = \rho - \beta_i$. So we have

$$\begin{aligned} w_i\Big(e(\rho) \cdot \mathrm{ch}^\omega\Big(\bigwedge^* \mathfrak{n}_i\Big)\Big) &= w_i(e(\rho) \cdot (1 + (-1)^{N_i-1} e(-\beta_i))) \\ &= e(\rho - \beta_i) \cdot (1 + (-1)^{N_i-1} e(\beta_i)) \\ &= (-1)^{N_i-1} e(\rho) \cdot (1 + (-1)^{N_i-1} e(-\beta_i)) \\ &= (-1)^{N_i-1} e(\rho) \cdot \mathrm{ch}^\omega\Big(\bigwedge^* \mathfrak{n}_i\Big) \\ &= (-1)^{\ell(w_i)-\hat{\ell}(w_i)}\Big(e(\rho) \cdot \mathrm{ch}^\omega\Big(\bigwedge^* \mathfrak{n}_i\Big)\Big)\,, \end{aligned}$$

since it is seen from the form (2.2.3) of w_i that $w_i(\beta_i) = -\beta_i$ and $\ell(w_i) = N_i$.

Case (b): We can easily deduce that the only basis vectors of $\bigwedge^* \mathfrak{n}_i$ which make a contribution to the trace of $\bigwedge^* \omega^{-1}$ are the following four vectors:

$$1 \in \bigwedge^0 \mathfrak{n}_i = \boldsymbol{C},\ f_i \wedge f_{\omega(i)} \wedge \cdots \wedge f_{\omega^{N_i-1}(i)},\ \dot{f}_i \wedge \dot{f}_{\omega(i)} \wedge \cdots \wedge \dot{f}_{\omega^{(N_i/2)-1}(i)},$$

and

$$f_i \wedge \cdots \wedge f_{\omega^{N_i-1}(i)} \wedge \dot{f}_i \wedge \cdots \wedge \dot{f}_{\omega^{(N_i/2)-1}(i)}.$$

Hence we immediately obtain that

$$\begin{aligned} \operatorname{ch}^{\omega}\left(\bigwedge^* \mathfrak{n}_i\right) &= (1+(-1)^{N_i-1}e(-\beta_i))(1+(-1)(-1)^{(N_i/2)-1}e(-\beta_i)) \\ &= (1+(-1)^{N_i-1}e(-\beta_i))(1+(-1)^{N_i/2}e(-\beta_i)), \end{aligned} \tag{3.2.13}$$

where $\beta_i = \sum_{k=0}^{N_i-1} \alpha_{\omega^k(i)}$. Now we use Equation (2.2.5) to see that $w_i(\rho) = \rho - 2\beta_i$. So we have

$$\begin{aligned} & w_i\left(e(\rho)\cdot \operatorname{ch}^{\omega}\left(\bigwedge^* \mathfrak{n}_i\right)\right) \\ &\quad = w_i(e(\rho)\cdot(1+(-1)^{N_i-1}e(-\beta_i))(1+(-1)^{N_i/2}e(-\beta_i))) \\ &\quad = e(\rho-2\beta_i)\cdot(1+(-1)^{N_i-1}e(\beta_i))(1+(-1)^{N_i/2}e(\beta_i)) \\ &\quad = e(\rho)\cdot(-1)^{(3/2)N_i-1}(1+(-1)^{N_i-1}e(-\beta_i))(1+(-1)^{N_i/2}e(-\beta_i)) \\ &\quad = (-1)^{(3/2)N_i-1}e(\rho)\cdot \operatorname{ch}^{\omega}\left(\bigwedge^* \mathfrak{n}_i\right) \\ &\quad = (-1)^{\ell(w_i)-\hat{\ell}(w_i)}\left(e(\rho)\cdot \operatorname{ch}^{\omega}\left(\bigwedge^* \mathfrak{n}_i\right)\right), \end{aligned}$$

since it is seen from the form (2.2.3) of w_i that $w_i(\beta_i) = -\beta_i$ and $\ell(w_i) = 3\cdot(N_i/2) = (3/2)N_i$.

STEP 2. Second, we show that

$$w_i\left(\operatorname{ch}^{\omega}\left(\bigwedge^* \mathfrak{n}(i)\right)\right) = \operatorname{ch}^{\omega}\left(\bigwedge^* \mathfrak{n}(i)\right), \tag{3.2.14}$$

which completes the proof of the proposition. Notice that $a_{\omega^k(i),\omega^k(i)} = 2$ for all $0 \le k \le N_i - 1$, since $i \in \check{I}$ implies $\sum_{k=0}^{N_i-1} a_{i,\omega^k(i)} > 0$ by definition. Hence, the operators $\operatorname{ad} e_{\omega^k(i)}$ and $\operatorname{ad} f_{\omega^k(i)}$ are locally nilpotent on $\mathfrak{g}$ for all $0 \le k \le N_i - 1$. We define linear automorphisms of $\mathfrak{g}$ by

$$x_k^{\mathrm{ad}} := (\exp(\operatorname{ad} f_{\omega^k(i)}))(\exp(-\operatorname{ad} e_{\omega^k(i)}))(\exp(\operatorname{ad} f_{\omega^k(i)})) \tag{3.2.15}$$

for $0 \leq k \leq N_i - 1$, and then by

$$X_i^{\mathrm{ad}} := \begin{cases} \prod_{k=0}^{N_i-1} x_k^{\mathrm{ad}} & \text{in Case (a)}, \\ \prod_{k=0}^{N_i/2-1} (x_k^{\mathrm{ad}}\, x_{(N_i/2)+k}^{\mathrm{ad}}\, x_k^{\mathrm{ad}}) & \text{in Case (b)}. \end{cases} \tag{3.2.16}$$

By [K, Lemma 3.8] we see that $X_i^{\mathrm{ad}}(\mathfrak{g}_\alpha) = \mathfrak{g}_{w_i(\alpha)}$ for $\alpha \in \Delta$. Moreover, we immediately see that

$$\omega^{-1}(X_i^{\mathrm{ad}}(v)) = (\omega^{-1}X_i^{\mathrm{ad}})(\omega^{-1}(v)) \tag{3.2.17}$$

for $v \in \mathfrak{g}$, where the linear automorphism $\omega^{-1}X_i^{\mathrm{ad}}$ of $\mathfrak{g}$ is defined by

$$\omega^{-1}X_i^{\mathrm{ad}} := \begin{cases} \prod_{k=0}^{N_i-1} x_{k-1}^{\mathrm{ad}} & \text{in Case (a)}, \\ \prod_{k=0}^{N_i/2-1} (x_{k-1}^{\mathrm{ad}}\, x_{(N_i/2)+k-1}^{\mathrm{ad}}\, x_{k-1}^{\mathrm{ad}}) & \text{in Case (b)}, \end{cases} \tag{3.2.18}$$

with $x_{-1}^{\mathrm{ad}} := x_{N_i-1}^{\mathrm{ad}}$. We see easily that $\omega^{-1}X_i^{\mathrm{ad}} = X_i^{\mathrm{ad}}$, since, in Case (b),

$$\begin{aligned} x_{-1}^{\mathrm{ad}}\, x_{(N_i/2)-1}^{\mathrm{ad}}\, x_{-1}^{\mathrm{ad}} &= x_{N_i-1}^{\mathrm{ad}}\, x_{(N_i/2)-1}^{\mathrm{ad}}\, x_{N_i-1}^{\mathrm{ad}} \\ &= x_{(N_i/2)-1}^{\mathrm{ad}}\, x_{N_i-1}^{\mathrm{ad}}\, x_{(N_i/2)-1}^{\mathrm{ad}} \end{aligned}$$

(see, for example, [KP]). Thus we get the commutative diagram for $\alpha \in \Delta_+$:

$$\begin{array}{ccc} \mathfrak{g}_{-\alpha} & \xrightarrow{X_i^{\mathrm{ad}}} & \mathfrak{g}_{-w_i(\alpha)} \\ {\scriptstyle \omega^{-1}}\downarrow & & \downarrow{\scriptstyle \omega^{-1}} \\ \mathfrak{g}_{-\omega^*(\alpha)} & \xrightarrow[X_i^{\mathrm{ad}}]{} & \mathfrak{g}_{-\omega^* w_i(\alpha)}\,, \end{array} \tag{3.2.19}$$

where $\omega^* w_i(\alpha) = w_i \omega^*(\alpha)$.

Because we see from Equation (2.2.5) that $w_i(\Delta(i)) = \Delta(i)$, we deduce that

$$X_i^{\mathrm{ad}}(\mathfrak{n}(i)) = \mathfrak{n}(i)\,. \tag{3.2.20}$$

By extending the linear automorphism $X_i^{\mathrm{ad}} : \mathfrak{g} \to \mathfrak{g}$ to the linear automorphism

$$\bigwedge^* X_i^{\mathrm{ad}} : \bigwedge^* \mathfrak{g} \to \bigwedge^* \mathfrak{g}$$

in the usual way, we finally obtain the following commutative diagram from (3.2.19) and (3.2.20):

$$\begin{array}{ccc} \bigwedge^* \mathfrak{n}(i) & \xrightarrow{\bigwedge^* X_i^{\mathrm{ad}}} & \bigwedge^* \mathfrak{n}(i) \\ {\scriptstyle \bigwedge^* \omega^{-1}}\downarrow & & \downarrow{\scriptstyle \bigwedge^* \omega^{-1}} \\ \bigwedge^* \mathfrak{n}(i) & \xrightarrow[\bigwedge^* X_i^{\mathrm{ad}}]{} & \bigwedge^* \mathfrak{n}(i)\,. \end{array} \tag{3.2.21}$$

Let $\chi \in (\mathfrak{h}^*)^0$ be a symmetric weight. Since $\omega^* w_i(\chi) = w_i \omega^*(\chi) = w_i(\chi)$, the following diagram commutes:

$$\begin{array}{ccc} \big(\bigwedge^* \mathfrak{n}(i)\big)_\chi & \xrightarrow{\bigwedge^* X_i^{\mathrm{ad}}} & \big(\bigwedge^* \mathfrak{n}(i)\big)_{w_i(\chi)} \\ {\scriptstyle \bigwedge^* \omega^{-1}}\downarrow & & \downarrow{\scriptstyle \bigwedge^* \omega^{-1}} \\ \big(\bigwedge^* \mathfrak{n}(i)\big)_\chi & \xrightarrow[\bigwedge^* X_i^{\mathrm{ad}}]{} & \big(\bigwedge^* \mathfrak{n}(i)\big)_{w_i(\chi)}\,. \end{array} \tag{3.2.22}$$

It follows from this commutative diagram that for $\chi \in (\mathfrak{h}^*)^0$,

$$\mathrm{Tr}\bigg(\Big(\bigwedge\nolimits^* \omega^{-1}\Big)\Big|_{(\bigwedge^* \mathfrak{n}(i))_\chi}\bigg) = \mathrm{Tr}\bigg(\Big(\bigwedge\nolimits^* \omega^{-1}\Big)\Big|_{(\bigwedge^* \mathfrak{n}(i))_{w_i(\chi)}}\bigg)\,. \tag{3.2.23}$$

Therefore, we conclude that

$$w_i\bigg(\mathrm{ch}^\omega\Big(\bigwedge\nolimits^* \mathfrak{n}(i)\Big)\bigg) = \mathrm{ch}^\omega\Big(\bigwedge\nolimits^* \mathfrak{n}(i)\Big)\,.$$

This proves the proposition. □

COROLLARY 3.2.3. *Let $\Lambda \in P_+ \cap (\mathfrak{h}^*)^0$ be a symmetric, dominant integral weight, and let $w \in \tilde{W}$ and $\beta \in \mathcal{S}(\Lambda) \cap (\mathfrak{h}^*)^0$. Then*

$$c_{(w,\beta)} = (-1)^{\ell(w)-\hat{\ell}(w)} \cdot (-1)^{\mathrm{ht}(\beta)-\widehat{\mathrm{ht}}(\beta)}\,.$$

PROOF. First, we show that

$$\mathrm{Tr}(\Phi|_{((\bigwedge^* \mathfrak{n}_-)\otimes_{\boldsymbol{C}} L(\Lambda))_{\Lambda-\beta}}) = (-1)^{\mathrm{ht}(\beta)-\widehat{\mathrm{ht}}(\beta)}\,. \tag{3.2.24}$$

The element $\beta \in \mathcal{S}(\Lambda) \cap (\mathfrak{h}^*)^0$ can be written in the form $\beta = \sum_{k=1}^{l} \beta_{i_k}$, where $\beta_{i_k} = \sum_{r=0}^{N_{i_k}-1} \alpha_{\omega^r(i_k)}$ for $1 \le k \le l$. Then obviously $\widehat{\mathrm{ht}}(\beta) = l$ and $\mathrm{ht}(\beta) = \sum_{k=1}^{l} N_{i_k}$. Recall from (1.2.5) that we have

$$\bigg(\Big(\bigwedge\nolimits^* \mathfrak{n}_-\Big) \otimes_{\boldsymbol{C}} L(\Lambda)\bigg)_{\Lambda-\beta} = \boldsymbol{C}\, v_{(1,\beta)}\,, \tag{3.2.25}$$

where $v_{(1,\beta)} = \hat{f}_{i_1} \wedge \hat{f}_{i_2} \wedge \cdots \wedge \hat{f}_{i_l} \otimes v_\Lambda$ with $\hat{f}_{i_k} := f_{i_k} \wedge f_{\omega(i_k)} \wedge \cdots \wedge f_{\omega^{N_{i_k}-1}(i_k)}$ for $1 \le k \le l$. Then it follows that

$$\begin{aligned} \Phi(v_{(1,\beta)}) &= \Big(\bigwedge^* \omega^{-1}\Big)(\hat{f}_{i_1} \wedge \cdots \wedge \hat{f}_{i_l}) \otimes v_\Lambda \\ &= \Big(\bigwedge^* \omega^{-1}\Big)(\hat{f}_{i_1}) \wedge \cdots \wedge \Big(\bigwedge^* \omega^{-1}\Big)(\hat{f}_{i_l}) \otimes v_\Lambda . \end{aligned} \tag{3.2.26}$$

Because for each $1 \le k \le l$,

$$\Big(\bigwedge^* \omega^{-1}\Big)(\hat{f}_{i_k}) = (-1)^{N_{i_k}-1} \hat{f}_{i_k} ,$$

we deduce that

$$\Phi(v_{(1,\beta)}) = (-1)^{(\sum_{k=1}^{l} N_{i_k})-l}\, v_{(1,\beta)} = (-1)^{\mathrm{ht}(\beta)-\widehat{\mathrm{ht}}(\beta)}\, v_{(1,\beta)} .$$

Thus we have shown that

$$\mathrm{Tr}(\Phi|_{((\bigwedge^* \mathfrak{n}_-)\otimes_{\mathbf{C}} L(\Lambda))_{\Lambda-\beta}}) = (-1)^{\mathrm{ht}(\beta)-\widehat{\mathrm{ht}}(\beta)}.$$

Now we set $j := \ell(w) + \mathrm{ht}(\beta)$. Then the scalar

$$\begin{aligned} c_{(w,\beta)} &= \mathrm{Tr}(\Phi_j|_{((\bigwedge^j \mathfrak{n}_-)\otimes_{\mathbf{C}} L(\Lambda))_{(w,\beta)\circ\Lambda}}) \\ &= \mathrm{Tr}(\Phi|_{((\bigwedge^* \mathfrak{n}_-)\otimes_{\mathbf{C}} L(\Lambda))_{(w,\beta)\circ\Lambda}}) \end{aligned} \tag{3.2.27}$$

is the coefficient of $e(w(\Lambda+\rho-\beta))$ in $e(\rho)\cdot \mathrm{ch}^\omega((\bigwedge^* \mathfrak{n}_-) \otimes_{\mathbf{C}} L(\Lambda))$, which is, by Proposition 3.2.2, equal to the coefficient of $e(\Lambda-\beta)$ in $\mathrm{ch}^\omega((\bigwedge^* \mathfrak{n}_-) \otimes_{\mathbf{C}} L(\Lambda))$ multiplied by $(-1)^{\ell(w)-\hat{\ell}(w)}$. But the coefficient of $e(\Lambda-\beta)$ in $\mathrm{ch}^\omega((\bigwedge^* \mathfrak{n}_-) \otimes_{\mathbf{C}} L(\Lambda))$ is by definition $\mathrm{Tr}(\Phi|_{((\bigwedge^* \mathfrak{n}_-)\otimes_{\mathbf{C}} L(\Lambda))_{\Lambda-\beta}})$. Thus, we obtain from (3.2.24) just proved that

$$c_{(w,\beta)} = (-1)^{\ell(w)-\hat{\ell}(w)} \cdot (-1)^{\mathrm{ht}(\beta)-\widehat{\mathrm{ht}}(\beta)} ,$$

as desired. This completes the proof. □

Combining Proposition 3.2.1 with Corollary 3.2.3, we obtain our main result.

THEOREM 3.2.4. *Let $\Lambda \in P_+ \cap (\mathfrak{h}^*)^0$ be a symmetric, dominant integral weight, and let $j \ge 0$. Then*

$$\begin{aligned} \mathrm{ch}^\omega(H_j(\mathfrak{n}_-, L(\Lambda))) &= \sum_{\substack{w\in\tilde{W} \\ \beta\in\mathcal{S}(\Lambda)\cap(\mathfrak{h}^*)^0 \\ \ell(w)+\mathrm{ht}(\beta)=j}} (-1)^{-(\ell(w)+\mathrm{ht}(\beta))} \cdot (-1)^{\hat{\ell}(w)+\widehat{\mathrm{ht}}(\beta)}\, e((w,\beta)\circ\Lambda) \\ &= \sum_{\substack{w\in\tilde{W} \\ \beta\in\mathcal{S}(\Lambda)\cap(\mathfrak{h}^*)^0 \\ \ell(w)+\mathrm{ht}(\beta)=j}} (-1)^{\hat{\ell}(w)+\widehat{\mathrm{ht}}(\beta)-j}\, e((w,\beta)\circ\Lambda) . \end{aligned}$$

3.3. Application. As an application of Theorem 3.2.4, we give a new proof of the twining character formula (Theorem 2.2.1) for $L(\Lambda)$ of symmetric, dominant integral highest weight Λ.

By virtue of Lemma 3.1.2, we can apply an Euler-Poincaré principle to deduce that

$$\sum_{j\geq 0}(-1)^j \operatorname{ch}^{\omega}(H_j(\mathfrak{n}_-, L(\Lambda))) = \sum_{j\geq 0}(-1)^j \operatorname{ch}^{\omega}\biggl(\Bigl(\bigwedge^j \mathfrak{n}_-\Bigr)\otimes_{\boldsymbol{C}} L(\Lambda)\biggr). \tag{3.3.1}$$

Also, by Lemma 3.1.2, we have for each $j \geq 0$,

$$\operatorname{ch}^{\omega}\biggl(\Bigl(\bigwedge^j \mathfrak{n}_-\Bigr)\otimes_{\boldsymbol{C}} L(\Lambda)\biggr) = \operatorname{ch}^{\omega}\Bigl(\bigwedge^j \mathfrak{n}_-\Bigr)\cdot \operatorname{ch}^{\omega}(L(\Lambda)). \tag{3.3.2}$$

Hence we get

$$\begin{aligned}
\sum_{j\geq 0}(-1)^j \operatorname{ch}^{\omega}\biggl(\Bigl(\bigwedge^j \mathfrak{n}_-\Bigr)\otimes_{\boldsymbol{C}} L(\Lambda)\biggr)
&= \sum_{j\geq 0}(-1)^j \operatorname{ch}^{\omega}\Bigl(\bigwedge^j \mathfrak{n}_-\Bigr)\cdot \operatorname{ch}^{\omega}(L(\Lambda)) \\
&= \operatorname{ch}^{\omega}(L(\Lambda))\cdot\Bigl(\sum_{j\geq 0}(-1)^j \operatorname{ch}^{\omega}\Bigl(\bigwedge^j \mathfrak{n}_-\Bigr)\Bigr) \\
&= \operatorname{ch}^{\omega}(L(\Lambda))\cdot\biggl(\sum_{j\geq 0}(-1)^j \operatorname{ch}^{\omega}\biggl(\Bigl(\bigwedge^j \mathfrak{n}_-\Bigr)\otimes_{\boldsymbol{C}} L(0)\biggr)\biggr) \\
&= \operatorname{ch}^{\omega}(L(\Lambda))\cdot\Bigl(\sum_{j\geq 0}(-1)^j \operatorname{ch}^{\omega}(H_j(\mathfrak{n}_-, L(0)))\Bigr),
\end{aligned}$$

since $L(0) = \boldsymbol{C}$ and $\operatorname{ch}^{\omega}(L(0)) = e(0) = 1$. Here, in the last equality above, we have used an Euler-Poincaré principle again.

On the other hand, we obtain from Theorem 3.2.4 that

$$\sum_{j\geq 0}(-1)^j \operatorname{ch}^{\omega}(H_j(\mathfrak{n}_-, L(\Lambda))) = \sum_{\substack{w\in\tilde{W}\\ \beta\in\mathcal{S}(\Lambda)\cap(\mathfrak{h}^*)^0}} (-1)^{\hat{\ell}(w)+\widehat{\mathrm{ht}}(\beta)}\, e((w,\beta)\circ\Lambda). \tag{3.3.3}$$

Putting all the above together, we conclude that

$$\begin{aligned}
&\sum_{\substack{w\in\tilde{W}\\ \beta\in\mathcal{S}(\Lambda)\cap(\mathfrak{h}^*)^0}} (-1)^{\hat{\ell}(w)+\widehat{\mathrm{ht}}(\beta)}\, e((w,\beta)\circ\Lambda) \\
&\qquad = \operatorname{ch}^{\omega}(L(\Lambda))\cdot\Biggl(\sum_{\substack{w\in\tilde{W}\\ \beta\in\mathcal{S}(0)\cap(\mathfrak{h}^*)^0}} (-1)^{\hat{\ell}(w)+\widehat{\mathrm{ht}}(\beta)}\, e((w,\beta)\circ 0)\Biggr).
\end{aligned}$$

Thus, we have given a new proof of Theorem 2.2.1.

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