# Existence of solution for a coupled system of Urysohn-Stieltjes functional integral equations

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**Abstract**

We present an existence theorem for at least one continuous solution for a coupled system of nonlinear functional (delay) integral equations of Urysohn-Stieltjes type.

*2010 Mathematics Subject Classification.* **74H10**. 45G10, 47H30
*Keywords.* Coupled system, Urysohn-Stieltjes integral equations, continuous solution.

## 1 Introduction and preliminaries

Urysohn-Stieltjes integral operators and Urysohn-Stieltjes integral equations have been studied by some authors (see [1]-[8]). The coupled system of integral equations have been studied, recently, by some authors (see [10]-[12]). Our aim here is to study the existence of at least one solution for a coupled system of nonlinear functional (delay) integral equations of Urysohn-Stieltjes type in the space of continuous functions.

In what follows let $I = [0,1]$ be a fixed interval. Denote by $C(I) = C[0,1]$ the Banach space consisting of all continuous functions acting from the interval $I$ into $R$ with the standard norm

$$\| x \| = \sup_{t\in I} | x(t) | .$$

Consider the nonlinear Urysohn-Stieltjes integral equation

$$x(t) = p(t) + \int_0^1 f(t,s,x(s))\ d_s g(t,s),\ t \in I = [0,1] \tag{1}$$

where $g : I \times I \to R$ and the symbol $d_s$ indicates the integration with respect to $s$.
Equations of type (1) and some of their generalizations were considered in paper (see [3]), for the properties of the Urysohn-Stieltjes integral (see Banaś [1]).

In this paper, we generalize this result for the coupled system of Urysohn-Stieltjes functional (delay) integral equations

$$\begin{aligned} x(t) &= p_1(t) + \int_0^1 f_1(t,s,y(\psi_1(s)))\ d_s g_1(t,s),\ t \in I \\ y(t) &= p_2(t) + \int_0^1 f_2(t,s,x(\psi_2(s)))\ d_s g_2(t,s),\ t \in I \end{aligned} \tag{2}$$

**Tbilisi Mathematical Journal** 11(1) (2018), pp. 117–125.
Tbilisi Centre for Mathematical Sciences.
*Received by the editors:* 14 August 2017.
*Accepted for publication:* 03 January 2018.

in the Banach space $C(I)$.

## 2 Existence of solutions

In this section we discuss the existence of solutions for the coupled system of nonlinear Urysohn-Stieltjes integral equations in C(I). In our further considerations, we shall assume that the following conditions are satisfied:

(i) $p_i : I \to R$ are continuous functions on $I$, $p = \sup_t \mid p_i(t) \mid$, $i = 1, 2$.

(ii) $\psi_i : I \to I$ are continuous functions such that $\psi_i(t) \leq t$, $i = 1, 2$.

(iii) $f_i : I \times I \times R \to R$, $i = 1, 2$ are continuous functions such that there exist continuous functions $a_i : I \times I \to I$ and continuous and nondecreasing functions $\varphi_i : R_+ \to R_+$ such that

$$\mid f_i(t, s, x) \mid \leq a_i(t, s)\varphi_i(\mid x \mid)$$

for $t, s \in I$, $x \in R$, $i = 1, 2$. Moreover, we put $k = \max\{a_i(t, s) : t, s \in I, i = 1, 2\}$.

(iv) $g_i : I \times I \to R$, $i = 1, 2$ and for all $t_1$, $t_2 \in I$ with $t_1 < t_2$, and the function $s \to g_i(t_2, s) - g_i(t_1, s)$ is nondecreasing on $I$.

(v) $g_i(0, s) = 0$ for any $s \in I$.

(vi) The functions $t \to g_i(t, 1)$ and $t \to g_i(t, 0)$ are continuous on $I$. Put

$$\mu = \max\{\sup \mid g_i(t, 1) \mid + \sup \mid g_i(t, 0) \mid \ on\ I\}, \quad i = 1, 2.$$

(vii) There exists a positive number $r$ satisfying the inequality

$$p + (k\varphi_i(r))\mu \leq r.$$

**Remark 2.1.** Observe that Assumptions (iv) and (v) imply that the function $s \to g(t, s)$ is nondecreasing on the interval $I$, for any fixed $t \in I$ (Remark 1 in [4]). Indeed, putting $t_2 = t$, $t_1 = 0$ in (iv) and keeping in mind (v), we obtain the desired conclusion. From this observation, it follows immediately that, for every $t \in I$, the function $s \to g(t, s)$ is of bounded variation on $I$.

Now, let $X$ be the Banach space of all ordered pairs $(x, y)$, $x, y \in C(I)$ with the norm

$$\| (x, y) \|_X = \max\{||x||, ||y||\}$$

where

$$\| x \| = \sup_{t \in I} \mid x(t) \mid, \qquad \| y \| = \sup_{t \in I} \mid y(t) \mid.$$

It is clear that $(X, ||.||_X)$ is Banach space.

**Theorem 2.2.** Let the assumptions (i)-(vii) be satisfied, then the coupled system (2) has at least one solution in $X$.

**Proof.** Define the operator $T$ by

$$T(x,y)(t) = (T_1y(t), T_2x(t))$$

where

$$T_1y(t) = x(t) = p_1(t) + \int_0^1 f_1(t,s,y(\psi_1(s)))\ d_sg_1(t,s)$$

$$T_2x(t) = y(t) = p_2(t) + \int_0^1 f_2(t,s,x(\psi_2(s)))\ d_sg_2(t,s).$$

We prove a few results concerning the continuity and compactness of these operators in the space of continuous functions.
We define the set $U$ by

$$U = \{u = (x(t),y(t)) \mid (x(t),y(t)) \in X : \| (x,y) \|_X \leq r\}$$

Let $(x,y) \in U$ and define

$$\theta(\varepsilon) = \sup\{\mid f_1(t_2,s,y) - f_1(t_1,s,y) \mid, \mid f_2(t_2,s,x) - f_2(t_1,s,x) \mid : t_1,t_2 \in I,\ \ \mid t_2 - t_1 \mid \leq \varepsilon,\ x \in R\}.$$

Now, for $(x,y) \in U$, we have

$$\begin{aligned}
\mid T_1y(t) \mid \quad &\leq \quad \mid p_1(t) \mid + \mid \int_0^1 f_1(t,s,y(\psi_1(s)))\ d_sg_1(t,s) \mid \\
&\leq \quad \sup_t \mid p_1(t) \mid + \int_0^1 \mid f_1(t,s,y(\psi_1(s))) \mid\ \ d_s(\bigvee_{z=0}^{s} g_1(t,z)) \\
&\leq \quad p + \int_0^1 (a_1(t,s)\varphi_1(\mid y(\psi_1(s)) \mid))\ d_s(\bigvee_{z=0}^{s} g_1(t,z)) \\
&\leq \quad p + (k\varphi_1(\sup_s \mid y(\psi_1(s)) \mid)) \int_0^1 d_sg_1(t,s) \\
&\leq \quad p + (k\varphi_1(\| y \|))[g_1(t,1) - g_1(t,0)] \\
&\leq \quad p + (k\varphi_1(r))[\mid g_1(t,1) \mid + \mid g_1(t,0) \mid] \\
&\leq \quad p + (k\varphi_1(r))[\sup_t \mid g_1(t,1) \mid + \sup_t \mid g_1(t,0) \mid] \\
&\leq \quad p + (k\varphi_1(r))\mu
\end{aligned}$$

then

$$\| T_1y \| \ \ \leq\ \ p + (k\varphi_1(r))\mu.$$

By a similar way can deduce that

$$\| T_2x \| \ \ \leq\ \ p + (k\varphi_2(r))\mu.$$

Therefore,

$$\| Tu \|_X = \| T(x,y) \|_X = \| (T_1y, T_2x) \|_X = \max\{\| T_1y \|, \| T_2x \|\} \leq r.$$

Thus for every $u = (x, y) \in U$, we have $Tu \in U$ and hence $TU \subset U$, ( i.e $T : U \to U$).
This means that the functions of $TU$ are uniformly bounded on $I$, it is clear that the set $U$ is nonempty, bounded, closed and convex.

Now, we prove that the set $TU$ is relatively compact.
For $u = (x, y) \in U$, for all $\varepsilon > 0$, $\delta > 0$ and for each $t_1, t_2 \in I$, and $t_1 < t_2$ such that $| t_2 - t_1 | < \delta$, then

$$\begin{aligned}
| T_1y(t_2) - T_1y(t_1) | &\leq | p_1(t_2) - p_1(t_1) | \\
&+ | \int_0^1 f_1(t_2, s, y(\psi_1(s)))\ d_sg_1(t_2, s) - \int_0^1 f_1(t_1, s, y(\psi_1(s)))\ d_sg_1(t_1, s) | \\
&\leq | p_1(t_2) - p_1(t_1) | \\
&+ | \int_0^1 f_1(t_2, s, y(\psi_1(s)))\ d_sg_1(t_2, s) - \int_0^1 f_1(t_1, s, y(\psi_1(s)))\ d_sg_1(t_2, s) | \\
&+ | \int_0^1 f_1(t_1, s, y(\psi_1(s)))\ d_sg_1(t_2, s) - \int_0^1 f_1(t_1, s, y(\psi_1(s)))\ d_sg_1(t_1, s) | \\
&\leq | p_1(t_2) - p_1(t_1) | \\
&+ | \int_0^1 [f_1(t_2, s, y(\psi_1(s))) - f_1(t_1, s, y(\psi_1(s)))]\ d_sg_1(t_2, s) | \\
&+ | \int_0^1 f_1(t_1, s, y(\psi_1(s)))\ d_s(g_1(t_2, s) - g_1(t_1, s)) | \\
&\leq | p_1(t_2) - p_1(t_1) | \\
&+ \int_0^1 | f_1(t_2, s, y(\psi_1(s))) - f_1(t_1, s, y(\psi_1(s))) |\ d_s(\bigvee_{z=0}^{s} g_1(t_2, z)) \\
&+ \int_0^1 | f_1(t_1, s, y(\psi_1(s))) |\ d_s(\bigvee_{z=0}^{s} [g_1(t_2, z) - g_1(t_1, z)]) \\
&\leq | p_1(t_2) - p_1(t_1) | + \int_0^1 \theta(\varepsilon)\ d_s(\bigvee_{z=0}^{s} g_1(t_2, z)) \\
&+ \int_0^1 (a_1(t_1, s)\varphi_1(| y(\psi_1(s)) |))\ d_s(\bigvee_{z=0}^{s} [g_1(t_2, z) - g_1(t_1, z)]) \\
&\leq \| p_1(t_2) - p_1(t_1) \| + \theta(\varepsilon) \int_0^1\ d_s(g_1(t_2, s)) \\
&+ (k\varphi_1(\| y \|)) \int_0^1\ d_s[g_1(t_2, s) - g_1(t_1, s)]
\end{aligned}$$

$$\begin{aligned}
&\leq \ \| p_1(t_2) - p_1(t_1) \| + \theta(\varepsilon)[g_1(t_2,1) - g_1(t_2,0)] \\
&+ \ (k\varphi_1(r))\{[g_1(t_2,1) - g_1(t_1,1)] - [g_1(t_2,0) - g_1(t_1,0)]\} \\
&\leq \ \| p_1(t_2) - p_1(t_1) \| + \theta(\varepsilon)[g_1(1,1) - g_1(1,0)] \\
&+ \ (k\varphi_1(r))\{[| g_1(t_2,1) - g_1(t_1,1) | + | g_1(t_2,0) - g_1(t_1,0) |]\}
\end{aligned}$$

Hence

$$\begin{aligned}
\| T_1y(t_2) - T_1y(t_1) \| &\leq \| p_1(t_2) - p_1(t_1) \| + \theta(\varepsilon)[g_1(1,1) - g_1(1,0)] \\
&+ (k\varphi_1(r))[| g_1(t_2,1) - g_1(t_1,1) | + | g_1(t_2,0) - g_1(t_1,0) |].
\end{aligned}$$

As done above we can obtain

$$\begin{aligned}
\| T_2x(t_2) - T_2x(t_1) \| &\leq \| p_2(t_2) - p_2(t_1) \| + \theta(\varepsilon)[g_2(1,1) - g_2(1,0)] \\
&+ (k\varphi_2(r))[| g_2(t_2,1) - g_2(t_1,1) | + | g_2(t_2,0) - g_2(t_1,0) |].
\end{aligned}$$

Now, from the definition of the operator $T$ we get

$$\begin{aligned}
Tu(t_2) - Tu(t_1) &= T(x,y)(t_2) - T(x,y)(t_1) \\
&= (T_1y(t_2), T_2x(t_2)) - (T_1y(t_1), T_2x(t_1)) \\
&= (T_1y(t_2) - T_1y(t_1), T_2x(t_2) - T_2x(t_1))
\end{aligned}$$

Therefore,

$$\begin{aligned}
\| Tu(t_2) - Tu(t_1) \|_X &= \| (T_1y(t_2) - T_1y(t_1), T_2x(t_2) - T_2x(t_1)) \|_X \\
&= \max\{\| T_1y(t_2) - T_1y(t_1) \|, \| T_2x(t_2) - T_2x(t_1) \|\} \\
&\leq \max\{\| p_1(t_2) - p_1(t_1) \| + \theta(\varepsilon)[g_1(1,1) - g_1(1,0)] \\
&+ (k + rb)[| g_1(t_2,1) - g_1(t_1,1) | + | g_1(t_2,0) - g_1(t_1,0) |] \\
&, \quad \| p_2(t_2) - p_2(t_1) \| + \theta(\varepsilon)[g_2(1,1) - g_2(1,0)] \\
&+ (k + rb)[| g_2(t_2,1) - g_2(t_1,1) | + | g_2(t_2,0) - g_2(t_1,0) |]\}.
\end{aligned}$$

This means that the class of $\{Tu(t)\}$ is equi-continuous on $I$, then by Arzéla-Ascoil theorem $TU$ is relatively compact.

Now, we will show that the operator $T : U \to U$ is continuous.
Firstly, we prove that $T_1$ is continuous, for all $\varepsilon > 0$ and $\delta > 0$, let $y_1(t)$ and $y_2(t) \in C[0,1]$ and $| y_1(t) - y_2(t) | < \delta$, then

$$\begin{aligned}
| T_1y_1(t) - T_1y_2(t) | &\leq \ | \int_0^1 f_1(t,s,y_1(\psi_1(s)))\, d_sg_1(t,s) - \int_0^1 f_1(t,s,y_2(\psi_1(s)))\, d_sg_1(t,s) | \\
&\leq \int_0^1 | f_1(t,s,y_1(\psi_1(s))) - f_1(t,s,y_2(\psi_1(s))) | \ d_s(\bigvee_{z=0}^{s} g_1(t,z)) \\
&\leq \varepsilon^* \int_0^1 \ d_s(\bigvee_{z=0}^{s} g_1(t,z))
\end{aligned}$$

$$\begin{aligned} &\leq \varepsilon^* \int_0^1 d_s g_1(t,s) \\ &\leq \varepsilon^* \, [g_1(t,1) - g_1(t,0)] \\ &\leq \varepsilon^* \, [| \, g_1(t,1) \, | + | \, g_1(t,0) \, |] \\ &\leq \varepsilon^* \, \mu = \varepsilon \end{aligned}$$

Therefore,

$$| \, T_1 y_1(t) - T_1 y_2(t) \, | \leq \varepsilon.$$

This means that the operator $T_1$ is continuous.
By a similar way as done above we can prove that for any $x_1(t), \; x_2(t) \in C[0,1]$ and $| \, x_1(t) - x_2(t) \, | < \delta$, we have

$$| \, T_2 x_1(t) - T_2 x_2(t) \, | \leq \varepsilon.$$

Hence $T_2$ is continuous operator.
The operators $T_1$ and $T_2$ are continuous operators imply that $T$ is continuous operator.
Since all conditions of Schauder fixed point theorem are satisfied, then $T$ has at least one fixed point $u = (x, y) \in U$, which completes the proof. ■

**Corollary 2.3.** Under the assumptions of Theorem 2.2 (with $g_i(t,s) = g_i(s)$), the coupled system of Urysohn-Stieltjes integral equations

$$x(t) = p_1(t) + \int_0^1 f_1(t, s, y(\psi_1(s))) \; d_s g_1(s), \;\; t \in I$$

$$y(t) = p_2(t) + \int_0^1 f_2(t, s, x(\psi_2(s))) \; d_s g_2(s), \;\; t \in I$$

has a solution $u = (x, y) \in U$.

In what follows, we provide some examples illustrating the above obtained results.
**Example 1.** Consider the functions $g_i : I \times I \to R$ defined by the formula

$$\begin{aligned} g_1(t,s) &= \begin{cases} t \ln \frac{t+s}{t}, & \text{for } t \in [0,1], \quad s \in I, \\ 0, & \text{for } t = 0, \quad s \in I. \end{cases} \\ g_2(t,s) &= t(t+s-1), \quad t \in I. \end{aligned}$$

It can be easily seen that the functions $g_1(t,s)$ and $g_2(t,s)$ satisfies assumptions (iv)-(vi) given in Theorem 1. In this case, the coupled system of Urysohn-Stieltjes integral equations (2) has the form

$$\begin{aligned} x(t) &= p_1(t) + \int_0^1 \frac{t}{t+s} f_1(t, s, y(\psi_1(s))) \; ds, \quad t \in I \\ y(t) &= p_2(t) + \int_0^1 t f_2(t, s, x(\psi_2(s))) \; ds, \quad t \in I, \end{aligned} \tag{3}$$

Therefore, the coupled system (3) has at least one solution $x, y \in C[0,1]$.

**Example 2.** Assume that the functions $\varphi_i : R_+ \to R_+$ have the form $\varphi_i(r) = 1 + r$ and the functions $a_i \in C(I)$. Denote by $b_i = \| a_i \| = \max[| a_i(t,s) | : t, s \in I]$ . Then,

$$| f_i(t,s,x) | \leq a_i(t,s)(1+ | x |) \leq a_i(t,s) + b_i | x | .$$

Notice that this assumption is a special case of assumption (iii).

Consider now the assumptions $(iii)^*$ and $(vii)^*$ having the form
$(iii)^*$ $f_i : I \times R \to R$ are continuous and satisfy the Lipschitz condition

$$| f_i(t,s,x) - f_i(t,s,y) | \leq b_i | x - y |, \quad i = 1, 2.$$

From this assumption we can deduce that

$$| f_i(t,s,x) | - | f_i(t,s,0) | \leq | f_i(t,s,x) - f_i(t,s,0) | \leq b_i | x |$$

which implies that

$$| f_i(t,s,x) | \leq | f_i(t,s,0) | + b_i | x | = | a_i(t,s) | + b_i | x |$$

$(vii)^*$ $\mu b < 1$.

**Corollary 2.4.** Let the assumptions $(i)-(ii), (iii)^*, (iv)-(vi)$ and $(vii)^*$ be satisfied, then the coupled system (2) has an unique solution $(x,y) \in X$.

**Proof.** Let $u_1 = (x_1, y_1)$ and $u_2 = (x_2, y_2)$ be two solutions of the coupled system (2), we have

$$\begin{aligned} \| (x_1,y_1) - (x_2,y_2) \|_X &= \| (x_1 - x_2, y_1 - y_2) \|_X \\ &= \max\{\| x_1 - x_2 \|, \| y_1 - y_2 \|\} \end{aligned}$$

Now,

$$\begin{aligned} | x_1 - x_2 | &= | p_1(t) + \int_0^1 f_1(t,s,y_1(\psi_1(s)))\ d_s g_1(t,s) - p_1(t) + \int_0^1 f_1(t,s,y_2(\psi_1(s)))\ d_s g_1(t,s) | \\ &\leq \int_0^1 | f_1(t,s,y_1(\psi_1(s))) - f_1(t,s,y_2(\psi_1(s))) |\ d_s(\bigvee_{z=0}^{s} g_1(t,z)) \\ &\leq \int_0^1 b_1 | y_1(\psi_1(s)) - y_2(\psi_1(s)) |\ d_s g_1(t,s) \\ &\leq b \| y_1 - y_2 \| \int_0^1 d_s g_1(t,s) \\ &\leq b \| y_1 - y_2 \| [g_1(t,1) - g_1(t,0)] \\ &\leq b \| y_1 - y_2 \| [| g_1(t,1) | + | g_1(t,0) |] \\ &\leq \mu b \| y_1 - y_2 \| \end{aligned}$$

Therefore,

$$\| x_1 - x_2 \| \leq \mu b \| y_1 - y_2 \| .$$

Also

$$\begin{aligned}
| y_1 - y_2 | &= | p_2(t) + \int_0^1 f_2(s, x_1(\psi_2(s))) \; d_s g_2(t,s) - p_2(t) + \int_0^1 f_2(s, x_2(\psi_2(s))) \; d_s g_2(t,s) | \\
&\leq \int_0^1 | f_2(s, x_1(\psi_2(s))) - f_2(s, x_2(\psi_2(s))) | \; d_s(\bigvee_{z=0}^{s} g_2(t,z)) \\
&\leq \int_0^1 b_2 | x_1(\psi_2(s)) - x_2(\psi_2(s)) | \; d_s g_2(t,s) \\
&\leq b \| x_1 - x_2 \| \int_0^t d_s g_2(t,s) \\
&\leq b \| x_1 - x_2 \| [g_2(t,1) - g_2(t,0)] \\
&\leq b \| x_1 - x_2 \| [| g_2(t,1) | + | g_2(t,0) |] \\
&\leq \mu b \| x_1 - x_2 \| .
\end{aligned}$$

Hence

$$\| y_1 - y_2 \| \leq \mu b \| x_1 - x_2 \| .$$

Then

$$\begin{aligned}
\| (x_1, y_1) - (x_2, y_2) \|_X &= \max\{\| x_1 - x_2 \|, \| y_1 - y_2 \|\} \\
&\leq \max\{\mu b \| y_1 - y_2 \|, \mu b \| x_1 - x_2 \|\} \\
&\leq \mu b \max\{\| y_1 - y_2 \|, \| x_1 - x_2 \|\} \\
&= \mu b \| (x_1, y_1) - (x_2, y_2) \|_X
\end{aligned}$$

which implies that

$$(1 - \mu b) \| (x_1, y_1) - (x_2, y_2) \|_X \leq 0,$$

therefore,

$$\| (x_1, y_1) - (x_2, y_2) \|_X = 0$$

This means that

$$(x_1, y_1) = (x_2, y_2) \quad \Rightarrow \quad x_1 = x_2, \quad y_1 = y_2.$$

Thus, the solution of the coupled system (2) is unique.

**Example 3.** Similarly as above, take the functions $a_i(t,s) \in C(I)$ . Let us take the functions $\varphi_i : R_+ \to R_+$ having the form $\varphi_i(r) = 1 + r^{\alpha}$, where $\alpha > 0$ is a fixed number. Then

$$| f_i(t,s,x) | \leq a_i(t,s)(1 + | x |^{\alpha}) \leq a_i(t,s) + b_i | x |^{\alpha},$$

where $b_i = \| a_i \|$ .

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