A new extended generalized Burr-III family of distributions

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Abstract

In this paper, we introduce a new extended generalized Burr III family of distributions in the so-called T-BurrIII{Y} family by using the quantile functions of a few popular distributions. We derive the general mathematical properties of this extended family including explicit expressions for the quantile function, Shannon entropy, moments and mean deviations. Three new Burr III sub-families are then investigated, and four new extended Burr III models are discussed. The density of Burr III extended distributions can be symmetric, left-skewed, right-skewed or reversed-J shaped, and the hazard rate shapes can be increasing, decreasing, bathtub and upside-down bathtub. The potentiality of the newly generated distributions is demonstrated through applications to censored and complete data sets.

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1 Introduction

Statistical distributions are often used to describe real-world phenomenon and to get improved inferences. In many real-life situations, when a simple distribution cannot provide a good fit or is inadequate for practical purposes; there is need to propose its extended form to provide flexible model. In the past two decades, many new families of distributions have been introduced for applied statisticians. The main objective of generalizing distributions is to induct shape parameter(s) to the parent (or baseline) distributions.

Eugene et al. (2002) first introduced *Beta-G family* through which two additional shape parameters are added to the baseline distribution, and also studied the properties of *Beta-normal* distribution. Cordeiro and de-Castro (2011) introduced *Kumaraswamy-G family* by which two parameters are added in the baseline distributions. Alexander et al. (2012) proposed *McDonald-G family* of distributions as a generalization of Beta-G family, which allows adding three shape parameters to baseline distribution. For more detail on generalized families, the reader is referred to Tahir et al. (2015), and Tahir and Cordeiro (2016).

Alzaatreh et al. (2013) introduced a general idea for parameter induction by defining the *transformed-transformer (T-X) family* of distributions. Let $r(t)$ be the probability density function (PDF) of a random variable $T \in [a, b]$ for $-\infty \leq a < b < \infty$, and let $F(x)$ be the cumulative distribution function (CDF) of a random variable X. Consequently, the transformation $W(\cdot)$: $[0, 1] \longrightarrow [a, b]$ satisfies the following conditions: (i) $W(\cdot)$ is differentiable and monotonically non-decreasing, and (ii) $W(0) \to a$ and $W(1) \to b$. Alzaatreh et al. (2013) defined the CDF of the *T-X*

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family of distributions as

$$G(x) = \int_a^{W[F(x)]} r(t)\, dt. \tag{1}$$

If $T \in (0, \infty)$, then X is a continuous random variable; if $W[F(x)] = -\log[1 - F(x)]$, then, the PDF corresponding to Eq. (1) is given by

$$g(x) = \frac{f(x)}{1 - F(x)}\, r\Big(- \log\big[1 - F(x)\big]\Big) = h_f(x)\, r\Big(H_f(x)\Big), \tag{2}$$

where $h_f(x) = \frac{f(x)}{1-F(x)}$ and $H_f(x) = -\log[1 - F(x)]$ are the hazard rate function (HRF) and cumulative HRF corresponding to any baseline PDF $f(x)$, respectively.

Aljarrah et al. (2014) considered the function $W(F(x))$ to be the quantile function of a random variable Y and defined the *T-R*$\{Y\}$ *family*. Alzaatreh et al. (2014), Alzaatreh et al. (2016) and Almheidat et al. (2015) proposed and studied the T-normal{Y}, T-Gamma{Y} and T-Weibull{Y} families of distributions, respectively. These methods result into enhance flexibility and also improve goodness-of-fit of the proposed model.

In this paper, our aim to propose an extended generalized *T-BurrIII family* of distributions by using the *T-R*$\{Y\}$ *approach* pioneered by Aljarrah et al. (2014). If X is random variable that follows two-parameter Burr III distribution, the its CDF and PDF are, respectively, given by

$$B_{c,k}(x) = \left(1 + x^{-c}\right)^{-k} \tag{3}$$

and

$$b_{c,k}(x) = c\, k\, x^{-c-1} \left(1 + x^{-c}\right)^{-k-1}, \quad x > 0, \tag{4}$$

where $c > 0$ and $k > 0$ are both shape parameters. Henceforth, a random variable having PDF (4) is denoted by $X \sim \text{BurrIII}(c, k)$. Due to closed form CDF, several properties of the Burr III distribution can easily be explored.

Let T, R and Y be continuous random variables having CDFs $F_T(x) = \mathbb{P}(T \le x), F_R(x) = \mathbb{P}(R \le x)$ and $F_Y(x) = \mathbb{P}(Y \le x)$, respectively. Then, the quantile functions (QFs) of these CDFs are denoted by $Q_T(u), Q_Y(u)$ and $Q_R(u)$. The QF is the inverse mapping of the CDF. The densities of T, R and Y are denoted by $f_T(x)$, $f_Y(x)$ and $f_R(x)$, respectively. Let the random variables be $T, Y \in (a, b)$ for $-\infty \le a < b \le \infty$. Aljarrah et al. (2014) defined the CDF of the random variable X as

$$F_X(x) = \int_a^{Q_Y\left(F_R(x)\right)} f_T(t)\, dt = F_T\left(Q_y\big(F_R(x)\big)\right). \tag{5}$$

The PDF and HRF corresponding to Eq. (5) are, respectively, given by

$$f_X(x) \quad = \quad f_t\left(Q_Y(F_R(x))\right) Q'_Y\big(F_R(x)\big)\, f_R(x) = f_R(x)\, \frac{f_t\Big(Q_Y\big(F_R(x)\big)\Big)}{f_Y\Big(Q_y\big(F_R(x)\big)\Big)} \tag{6}$$

and

$$h_X(x) = h_R(x)\, \frac{h_T\Big(Q_Y(F_R(x))\Big)}{h_Y\Big(Q_Y(F_R(x))\Big)}. \tag{7}$$

The rest of the paper is organized as follows. In Section 2, we define the *extended generalized Burr III family* of distributions and discuss three Burr-III sub-families, namely, T-BurrIII{Lomax}, T-BurrIII{logistic} and T-BurrIII{Weibull}. In Section 3, some general properties of extended gemerlaized T-BurrIII $\{Y\}$ family of distributions are obtained including expressions for the modes, moments, Shannon entropy and mean deviations. In Section 4, four special sub-models, namely, log-logistic-BurrIII{Weibull}, Gumbel-BurrIII{logistic}, normal-BurrIII{logistic} and exponential-BurrIII{Lomax} are defined and their density and hazard rate plots are displayed. Later, the detail properties of exponential-Burr{Lomax} (EBIII{Lx}) model are obtained. In Section 5, a small simulation study is carried out to assess the performance of maximum likelihood method of EBIII{Lx} distribution. In Section 6, the applicability of EBIII{Lx} model is shown to three real-life data sets. The final remarks are presented in Section 7.

2 The new extended generalized Burr-III family

Let the random variable R follows the Burr-III density given in Eq. (3). Then from Eq. (5) and Eq. (3), the CDF of new family is defined by

$$F_X(x) = \int_a^{Q_Y\left(\left(1+x^{-c}\right)^{-k}\right)} r(t)\,dt = F_T\Big(Q_Y\left((1+x^{-c})^{-k}\right)\Big). \tag{8}$$

The PDF corresponding to Eq. (8) will be

$$\begin{aligned} f_X(x) &= c\,k\,x^{-c-1}\left(1+x^{-c}\right)^{-k-1} \frac{\left(f_t\left[Q_Y\left(\left(1+x^{-c}\right)^{-k}\right)\right]\right)}{f_y\left(Q_Y\left(\left(1+x^{-c}\right)^{-k}\right)\right)} \\ &= b_{c,k}(x)\, f_T\Big(Q_Y\left[B_{c,k}(x)\right]\Big) Q'_Y\Big(B_{c,k}(x)\Big), \end{aligned} \tag{9}$$

where $b_{c,k}(x)$ and $B_{c,k}(x)$ are the PDF and CDF of the Burr III random variables, respectively. Henceforth, the new extended family of distributions defined in Eq. (9) is denoted by *T-BurrIII*$\{Y\}$ *family.*

Table 1 contains QFs of three popular distributions. We can generate different Burr III sub-families from T-BurrIII$\{Y\}$ family using these QFs.

TABLE 1. QFs of different Y distributions.

S.No	Y	$Q_Y(u)$
1.	Weibull	$\left[-\alpha^{-1}\ \ln(1-u)\right]^{1/\beta}$
2.	Logistic	$-\lambda^{-1}\ \ln\ \left(u^{-1}-1\right)$
3.	Lomax	$\beta\left[\left(1-u\right)^{-1/\alpha}-1\right]$

Remark 2.1. If X follows the T-BurrIII$\{Y\}$ family of distributions given in Eq. (8), then the following relations hold:

(i) $X = \left(F_Y(T)^{-\frac{1}{k}} - 1\right)^{-1/c}$.

(ii) $Q_X(u) = \left(F_Y(Q_T(u))^{-\frac{1}{k}} - 1\right)^{-1/c}$, $u \in (0,1)$.

(iii) If $Y = BurrIII(c,k)$, then $X = T$.

2.1 T-BurrIII{Weibull} family of distributions

Let Q_Y be the QF of the Weibull distribution (given in Table 1) and $\alpha = 1$, then $F_X(x)$ given in Eq. (8) becomes

$$F_X(x) = F_T\left(\left[-\ln\left(1 - B_{c,k}(x)\right)\right]^{1/\beta}\right), \tag{10}$$

where $T \in (0,\infty)$.

The PDF corresponding to Eq. (10) is

$$f_X(x) = \frac{b_{c,k}(x)}{\beta\left[1 - B_{c,k}(x)\right]}\left[-\ln\left(1 - B_{c,k}(x)\right)\right]^{1/\beta - 1} f_T\left(\left[-\ln\left(1 - B_{c,k}(x)\right)\right]^{1/\beta}\right). \tag{11}$$

When $\beta = 1$, then the sub-family given in Eq. (10) reduces to T-BurrIII{exponential}, and when $\beta = 2$, then the sub-family given in Eq. (10) reduces to T-BurrIII{Rayleigh}.

2.2 T-BurrIII{logistic} family of distributions

Let Q_Y be the QF of logistic distribution (given in Table 1), then $F_X(x)$ given in Eq. (8) becomes

$$F_X(x) = F_T\left(-\lambda^{-1}\ln\left(B_{c,-k}(x) - 1\right)\right), \tag{12}$$

where the random variable $T \in (-\infty,\infty)$.

The PDF corresponding to Eq. (12) is

$$f_X(x) = \frac{b_{c,-k}(x)}{\lambda\left[B_{c,-k}(x) - 1\right]} f_T\left(-\lambda^{-1}\ln\left(B_{c,-k}(x) - 1\right)\right). \tag{13}$$

2.3 T-BurrIII{Lomax} family of distributions

Let Q_Y be the QF of Lomax distribution (given in Table 1) and $\beta = 1$, then $F_X(x)$ given in Eq. (8) becomes

$$F_x(x) = F_T\left(\left(1 - B_{c,k}(x)\right)^{-1/\alpha} - 1\right), \tag{14}$$

where the random variable $T \in (0,\infty)$.

The PDF corresponding to Eq. (14) is

$$f_x(x) = \frac{b_{c,k}(x)}{\alpha}\left(1 - B_{c,k}(x)\right)^{-1/\alpha - 1} f_T\left(\left(1 - B_{c,k}(x)\right)^{-1/\alpha} - 1\right). \tag{15}$$

3 General properties of the T-BurrIII{Y} family

In this section, some general properties of T-BurrIII{Y} family of distributions are described.

Lemma 3.1. (Transformation) If T is a random variable with CDF $F_T(x)$, then the random variable:

(i) $X = \left[\left\{1 - \mathrm{e}^{-\alpha T^{\beta}}\right\}^{-1/k} - 1\right]^{-1/c}$ follows the T-BurrIII{Weibull} distribution.

(ii) $X = \left[\left(1 + \mathrm{e}^{-\lambda T}\right)^{1/k} - 1\right]^{-1/c}$ follows the T-BurrIII{logistic} distribution.

(iii) $X = \left(\left[1 - \left(1 + \frac{T}{\beta}\right)^{-\alpha}\right]^{-1/k} - 1\right)^{-1/c}$ follows the T-BurrIII{Lomax} distribution.

Proof. The proof of this Lemma can easily be followed from Remark 1(i).

Lemma 3.2. The QFs for the T-BurrIII{Weibull}, T-BurrIII{logistic} and T-BurrIII{Lomax} sub-families are given by:

(i) $Q_X(u) = \left[\left(1 - \mathrm{e}^{-\alpha\, Q_T(u)^{\beta}}\right)^{-1/k} - 1\right]^{-1/c}$, $u \in (0,1)$.

(ii) $Q_X(u) = \left[\left(1 + \mathrm{e}^{-\lambda\, Q_T(u)}\right)^{1/k} - 1\right]^{-1/c}$, $u \in (0,1)$.

(iii) $Q_X(u) = \left(\left[1 - \left(1 + \frac{Q_T(u)}{\beta}\right)^{-\alpha}\right]^{-1/k} - 1\right)^{-1/c}$, $u \in (0,1)$.

Proof. The proof of this Lemma can easily be followed from Remark 1(ii).

The skewness and kurtosis are very useful measures to check the properties the distribution. Skewness measures the degree of the longer tail toward the left-side (negative skewness) or longer tail toward the right-side (positive skewness). Kurtosis is a measure of the degree of heavy-tail distribution. The distribution can be symmetric ($S = 0$), right-skewed ($S > 0$) or left-skewed ($S < 0$). When K increases, the tail of the distribution becomes heavier. The skewness S and kurtosis K measures can be computed using the QF as

$$S = \frac{Q_X(3/4) + Q_X(1/4) - 2Q_X(1/2)}{Q_X(3/4) - Q_X(1/4)}$$

and

$$K = \frac{Q_X(7/8) - Q_X(5/8) + Q_X(3/8) - Q_X(1/8)}{Q_X(6/8) - Q_X(2/8)},$$

where $Q(.)$ represents the QF of the T-BurrIII{Y} random variable.

3.1 Mode of T-BurrIII{Y} family

The mode(s) of T-BurrIII{Y} family can be obtained from the solution of the following equation:

$$\Psi\Big(Q_Y'\big(B_{c,k}(x)\big)\Big) + \Psi\Big(f_T\Big(Q_Y'\big(B_{c,k}(x)\big)\Big)\Big) + \frac{c(k+1)\,x^{-c-1}}{(1+x^{-c})} - \frac{c+1}{x} = 0, \tag{16}$$

where $\Psi(f) = f'/f$. The result in Eq. (16) can be obtained by equating the first derivative of the PDF given in Eq. (9) to zero.

3.2 Moments

From Remark 1(i), we can obtain the rth moment expression of T-BurrIII{Y} family as

$$\mathbb{E}(X^r) = \mathbb{E}\Big(F_Y(T)^{-1/k} - 1\Big)^{-r/c}.$$

Using negative binomial series expansion $(x+y)^{-v} = \sum_{j=0}^{\infty} \begin{pmatrix} v+j-1 \\ j \end{pmatrix} x^j\, y^{-v-j}$ where $(|x| < y)$, we have

$$\mathbb{E}(X^r) = \sum_{j=0}^{\infty} \begin{pmatrix} \frac{r}{c}+j-1 \\ j \end{pmatrix} \mathbb{E}\Big(F_Y(T)\Big)^{\frac{r}{c\,k}+\frac{j}{k}}. \tag{17}$$

Now, using binomial series $(1+x)^v = \sum_{j=0}^{\infty} \begin{pmatrix} v \\ j \end{pmatrix} x^j$, $|x| < 1$, on $[F_Y(T)]^r$, the Eq. (17) yields the rth moment of T-BurrIII{Weibull}, T-BurrIII{logistic} and T-BurrIII{Lomax} sub-families as

$$\mathbb{E}(X^r) = \sum_{j,i=0}^{\infty} \begin{pmatrix} \frac{r}{c}+j-1 \\ j \end{pmatrix} \begin{pmatrix} \frac{r}{c\,k}+\frac{j}{k} \\ i \end{pmatrix} (-1)^i\, \mathbb{E}\Big(\mathrm{e}^{-i\,\alpha\, T^{\beta}}\Big), \tag{18}$$

$$\mathbb{E}(X^r) = \sum_{j=0}^{\infty} \begin{pmatrix} \frac{r}{c}+j-1 \\ j \end{pmatrix} \mathbb{E}\Big(1+\mathrm{e}^{-\lambda\, T}\Big)^{-\left(\frac{r}{c\,k}+\frac{j}{k}\right)} \tag{19}$$

and

$$\mathbb{E}(X^r) = \sum_{j,i=0}^{\infty} \begin{pmatrix} \frac{r}{c}+j-1 \\ j \end{pmatrix} \begin{pmatrix} \frac{r}{c\,k}+\frac{j}{k} \\ i \end{pmatrix} (-1)^i\, \mathbb{E}\Big(1+\frac{T}{\beta}\Big)^{-\alpha\, i}. \tag{20}$$

3.3 Shannon entropy

Entropy has been used in various situations in science as a measure of variation of the uncertainty. Numerous measures of entropy have been studied and compared in the literature. The Shannon entropy (Shannon, 1948) is used as a measure of uncertainty and plays an important role in many fields such as engineering and information theory. Shannons entropy of a random variable X with pdf $f(x)$ is defined as $\eta_X = -\mathbb{E}[\log(f(X))]$. Using Theorem 2 of Aljarrah et al. (2014), the Shannon entropy of T-BurrIII{Y} is defined as

$$\eta_X = \eta_T + \mathbb{E}\left(\log f_Y(T)\right) + \mathbb{E}\left\{\log Q'_R\left[F_Y(T)\right]\right\}. \tag{21}$$

Using Eq. (21), the Shannon entropies for T-BurrIII{Weibull}, T-BurrIII{logistic} and T-BurrIII{Lomax} sub-families are, respectively, given by

$$\begin{aligned} \eta_X &= \log\left(\frac{\alpha\,\beta}{c\,k}\right) + (1+c)\,\mathbb{E}\left(\log X\right) - \left(\frac{k+1}{k}\right)\mathbb{E}\left[\log\left(1-\mathrm{e}^{-\alpha\, T^{\beta}}\right)\right] \\ &\quad +(\beta+1)\,\mathbb{E}(\log T) - \alpha\,\mathbb{E}(T^{\beta}) + \eta_T, \end{aligned} \tag{22}$$

$$\eta_X = \log\left(\frac{\lambda}{c\,k}\right) + (1+c)\,\mathbb{E}\left(\log X\right) + \left(\frac{1-k}{k}\right)\mathbb{E}\left[\log\left(1+\mathrm{e}^{-\lambda\, T}\right)\right] - \lambda\,\mathbb{E}(T) + \eta_T$$

and

$$\eta_X = \log\left(\frac{\alpha}{c\,k\,\beta}\right) + (1+c)\,\mathbb{E}\,(\log X) - \left(\frac{1+k}{k}\right)\,\mathbb{E}\left\{\log\left[1-\left(1+\frac{T}{\beta}\right)^{-\alpha}\right]\right\}$$
$$-(\alpha+1)\,\mathbb{E}\left[\log\left(1+\frac{T}{\beta}\right)\right] + \eta_T, \tag{23}$$

where $\mathbb{E}\big(\log X\big) = \lim_{n\to\,0}\frac{d}{dr}\mathbb{E}(X^r)$, and η_T is the Shannon entropy of the random variable T.

3.4 Mean deviation

The mean deviations from mean and median for the T-BurrIII{Y} family are, respectively, given by

$$\delta_1 = 2\mu\,F(\mu) - 2\,I_c\,(\mu) \quad \text{and} \quad \delta_2 = \mu - 2\,I_c\,(\mu), \tag{24}$$

where F_X is given in Eq. (8). The mean μ can be obtained from Eq. (18) with $r = 1$. The median can be obtained from Remark 1(ii) after replacing the value of u with 0.5. The first incomplete moment $I_c(s)$ is obtained as

$$I_c(s) = \int_0^s x\,f_X(x)dx = \int_0^{Q_Y(F_R(s))} Q_R(F_Y(t))\,f_T(t)\,dt. \tag{25}$$

Using generalized binomial theorem in Eq. (25), the incomplete moment expression for T-BurrIII {Weibull}, T-BurrIII{logistic} and T-BurrIII{Lomax} distributions can be obtained as

$$I_c(s) = \sum_{j,i=0}^{\infty}(-1)^i\binom{\frac{1}{c}+j-1}{j}\binom{\frac{1}{c\,k}+\frac{j}{k}}{i}\int_0^{\left\{-\frac{1}{\alpha}\,\ln[1-B_{c,k}(s)]\right\}^{\frac{1}{\beta}}} \mathrm{e}^{-i\,\alpha\,t^{\beta}}\,f_T(t)\,dt,$$

$$I_c(s) = \sum_{j=0}^{\infty}\binom{\frac{1}{c}+j-1}{j}\int_{-\infty}^{-\frac{1}{\lambda}\ln[B_{c,-k}(s)-1]}\left[1+\mathrm{e}^{-\lambda\,t}\right]^{-\left[\frac{1}{c\,k}+\frac{j}{k}\right]}\,f_T(t)\,dt$$

and

$$I_c(s) = \sum_{j=0}^{\infty}\binom{\frac{1}{c}+j-1}{j}\binom{\frac{1}{c\,k}+\frac{j}{k}}{i}(-1)^i\int_0^{\beta\left[(1-B_{c,k}(s))^{-\frac{1}{\alpha}}-1\right]}\left(1+\frac{t}{\beta}\right)^{-i\,\alpha}\,f_T(t)\,dt.$$

4 Special sub models

In this section, we select different distributions for T random variable to generate special models. We consider four special sub-models, namely, log logistic-BurrIII{Weibull}, Gumbel-BurrIII{logistic}, Normal-Burr{logistic}, and exponential-Burr {Lomax}. We also discuss the properties of the exponential-BurrIII{Lomax} distribution in detail.

4.1 Log-logistic-BurrIII{Weibull} distribution

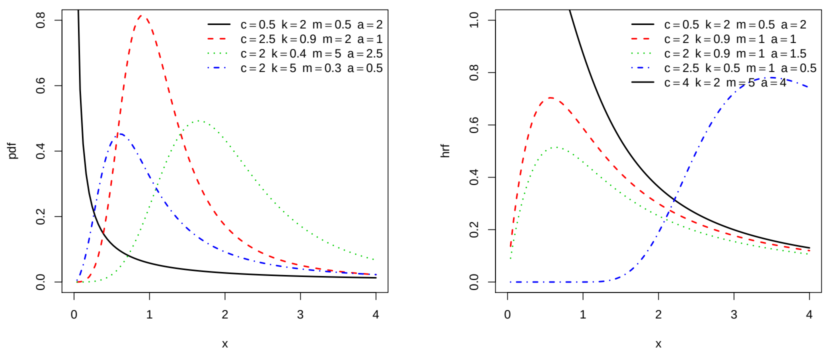
If $T \sim$ log-logistic(a,b) having CDF $F_T(t) = 1-[1+(x/a)^b]^{-1}$ and PDF $f_T(t) = \frac{b}{a^b}\,x^{b-1}\,[1+(x/a)^b]^{-1}$, then by using Eq. (10) and Eq. (11), the CDF and PDF of log-logistic-BurrIII{Weibull} distributions are, respectively, given by

$$F_X(x) = 1 - \left[1 + \left(\frac{-\log\left[1-(1+x^{-c})^{-k}\right]}{a}\right)^m\right]^{-1}$$

and

$$f_X(x) = \frac{m\,c\,k}{a^m}\,\frac{x^{-c-1}(1+x^{-c})^{-k-1}}{a^m(1-(1+x^{-c})^{-k})}\,\frac{\left\{-\log\left[1-(1+x^{-c})^{-k}\right]\right\}^{m-1}}{\left[1+\left(\frac{-\log[1-(1+x^{-c})^{-k}]}{a}\right)^m\right]^2},$$

where $b/\beta = m$. Henceforth, the random variable X is denoted by LLBIII{W}(m, c, k, a). Figure 1 shows some plots of density and HRF of LLBIII{W} distribution. The density shapes of LLBIII{W} distribution can be right-skewed and reversed-J and the HRF shapes can be increasing, decreasing and upside-down bathtub.



(a) (b)

FIGURE 1. Plots of (a) PDF and (b) HRF of LLBIII{W} distribution.

4.2 Gumbel-BurrIII{logistic} distribution

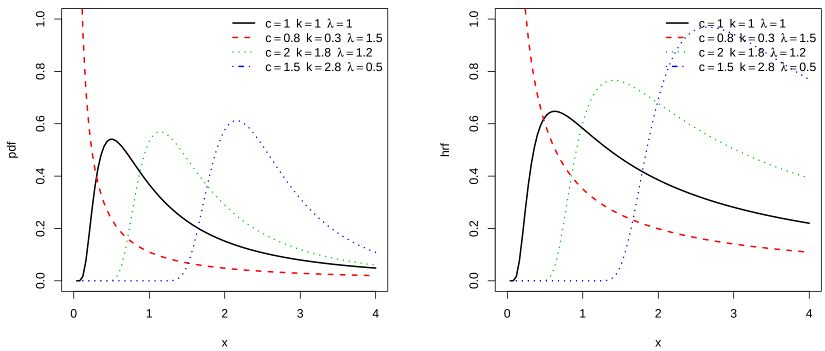
Let $T \sim$ Gumbel$(0, 1)$ having CDF $F_T(t) = \exp[-\exp(-x)]$ and PDF $f_T(t) = \exp[-x - \exp(-x)]$, then by using Eq. (12) and Eq. (13), the CDF and PDF of GBIII{L} distribution are, respectively, given by

$$F_X(x) = \exp\left[-\left[(1+x^{-c})^k - 1\right]^{1/\lambda}\right]$$

and

$$f_X(x) = \frac{c}{\lambda}\,k\,x^{-c-1}\,(1+x^{-c})^{k-1}\,\left[(1+x^{-c})^k - 1\right]^{1/\lambda-1}\exp\left[-\left[(1+x^{-c})^k - 1\right]^{1/\lambda}\right].$$

Henceforth, the random variable X is denoted by $X \sim$ GBIII{L}(c, k, λ). Figure 2 shows some plots of density and HRF of GBIII{L} distribution. The density shapes of GBIII{L} distribution can be right-skewed and reversed-J, and the HRF can have decreasing and upside-down bathtub shapes.



(a) (b)

FIGURE 2. Plots of (a) PDF and (b) HRF of GBIII{L} distribution.

4.3 Normal-BurrIII{logistic} distribution

If $T \sim$ Normal(μ, σ) having CDF $F_T(t) = \Phi\left(\frac{x-\mu}{\sigma}\right)$ and PDF $f_T(t) = \sigma^{-1}\,\varphi\left(\frac{x-\mu}{\sigma}\right)$, then by using Eq. (12) and Eq. (13), the CDF and PDF of Normal-BurrIII{logistic} distribution are, respectively, given by

$$F_X(x) = \Phi\Big(- \lambda^{-1}\,\ln\Big[(1+x^{-c})^k - 1\Big]\Big)$$

and

$$f_X(x) = \frac{c\,k\,x^{-c-1}(1+x^{-c})^{k-1}}{\sigma\,\lambda\Big[(1+x^{-c})^k - 1\Big]}\,\varphi\Big(- \lambda^{-1}\,\ln\Big[(1+x^{-c})^k - 1\Big]\Big).$$

Henceforth, the random variable X is denoted by $X \sim$ NBIII{L}$(\mu, \sigma, c, k, \lambda)$. Figure 3 shows some plots of density and HRF for NBIII{L} distribution. The shapes of the density of NBIII{L} can be right-skewed, symmetrical and reversed-J, and the HRF shapes can be increasing, decreasing, bathtub and upside-down bathtub.

4.4 Exponential-BurrIII{Lomax distribution}

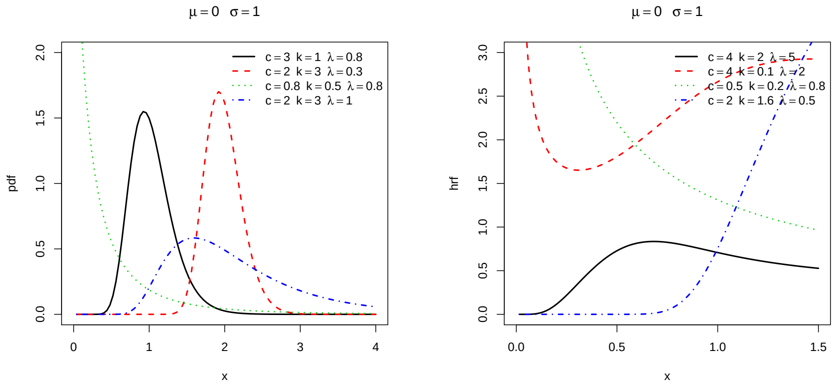
If T follow the exponential distribution with parameter a having CDF $F_T(t) = 1 - \mathrm{e}^{-a\,x}$, $(t > 0)$ and PDF $f(t) = a\,\mathrm{e}^{-a\,x}$, then by using Eq. (14) and Eq. (15), the CDF and PDF of exponential-BurrIII{Lomax} distribution are, respectively, given by

$$F_X(x) = 1 - \exp\left[-a\left(\left(1 - (1+x^{-c})^{-k}\right)^{-1/\alpha} - 1\right)\right] \tag{26}$$

and

$$f_X(x) = \frac{a\,c\,k\,\left[1-(1+x^{-c})^{-k}\right]^{-(\alpha+1)/\alpha}}{\alpha\,x^{c+1}\,(1+x^{-c})^{k+1}}\,\exp\left[-a\left(\left(1 - (1+x^{-c})^{-k}\right)^{-1/\alpha} - 1\right)\right]. \tag{27}$$

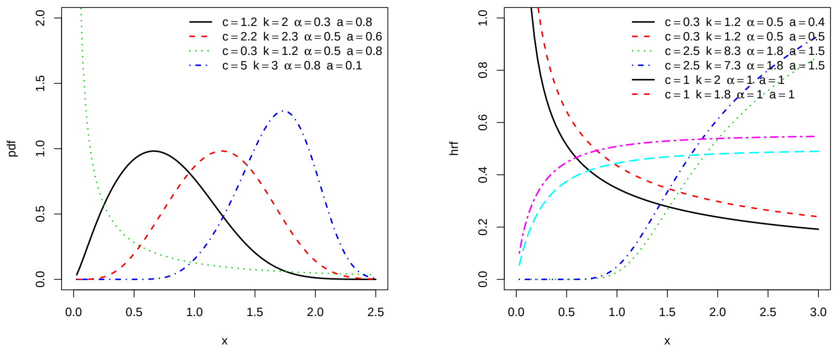
Henceforth, the random variable X is denoted by $X \sim$ EBIII{Lx}(α, c, k, a). Figure 4 shows some plots of density and HRF of EBIII{Lx} distribution. The density plots of EBIII{Lx} can exhibit



(a) (b)

FIGURE 3. Plots of (a) PDF and (b) HRF of NBIII{L} distribution.

symmetrical, right-skewed, left-skewed and reversed-J shapes, and the HRF can have increasing or decreasing shapes.



(a) (b)

FIGURE 4. Plots of (a) PDF and (b) HRF of EBIII{Lx} distribution.

4.5 Properties of EBIII{Lx} distribution

In this subsection, the properties of EBIII{Lx} distribution are obtained.

QF of EBIII{Lx} distribution. Using the Remark 1(ii), the QF of EBIII{Lx} distribution will be

$$Q_X(u) = \left(\left[1-\left(1-\frac{\ln(1-u)}{a}\right)^{-\alpha}\right]^{-1/k}\right)^{-1/c}.$$

Mode of EBIII{Lx} distribution. Using Eq. (16), the modes of EBIII{Lx} can be obtained from the solutions of the following equation:

$$-\frac{c+1}{x} - \frac{c\,(k+1)}{x^{c+1}\,(1+x^{-c})} - c\,k\left(\frac{\alpha+1}{\alpha}\right)\frac{(1+x^{-c})^{-k-1}}{1-(1+x^{-c})^{-k}}x^{-c-1} - a\,c\,k\frac{\left[1-(1+x^{-c})^{-k}\right]^{\frac{\alpha+1}{\alpha}}}{x^{c+1}\,(1+x^{-c})^{k+1}} = 0.$$

Moments of EBIII{Lx} distribution. The following theorem gives rth moment expression for the EBIII{Lx} distribution.

Theorem 4.1. If $T \sim \mathrm{Exp}(a)$, where $x, a > 0$, then the rth moment expression for EBIII{Lx} distribution is given by

$$\mathbb{E}(X^r) = \sum_{j,i=0}^{\infty}\binom{\frac{r}{c}+j-1}{j}\binom{\frac{r}{c\,k}+\frac{j}{k}}{i}(-1)^i\,\beta\,\mathrm{e}^{a\,\beta}\,\mathbb{E}_{a\,i}(a\,\beta), \tag{28}$$

where $\mathbb{E}_p(z) = \int_1^\infty t^p\,\mathrm{e}^{-z\,t}\,dt$ is the usual exponential integral function (Milgram, 1985).

Proof. From Eq. (18), we have the following result

$$\mathbb{E}\left(1+\frac{T}{\beta}\right)^{-\alpha\,i} = \int_0^\infty \left(1+\frac{t}{\beta}\right)^{-\alpha\,i}\mathrm{e}^{-a\,t}\,dt. \tag{29}$$

The above integral can easily be solved by using the transformation $u = 1+\frac{t}{\beta}$, which completes the proof.

Shannon entropy of EBIII{Lx} distribution. The following theorem provides the expression for the Shannon entropy of EBIII{Lx} distribution.

Theorem 4.2. The expression for the Shannon entropy of EBIII{Lx} distribution is given by

$$\eta_X = \log\left(\frac{\alpha\,\beta}{c\,k}\right) + (1+c)\,\mathbb{E}\,(\log X) - \left(\frac{k+1}{k}\right)\left[-\sum_{n=0}^{\infty}\frac{\beta\,e^{a\,\beta}}{n}\,E_{a\,n}(a\,\beta)\right] - \alpha\,\mathbb{E}_0^1(a\,\beta) + \eta_T,$$

where η_T is the Shannon entropy of the Lomax distribution.

Proof. Using Eq. (22), we obtain

$$\mathbb{E}\left[\log\left(1+\frac{T}{\beta}\right)\right] = \int_0^\infty \log\left(1+\frac{t}{\beta}\right)\mathrm{e}^{-a\,t}\,dt. \tag{30}$$

Let $u = 1+\frac{t}{\beta}$, then

$$\mathbb{E}\left[\log\left(1+\frac{T}{\beta}\right)\right] = \int_1^\infty \log u\,\mathrm{e}^{-a\,\beta\,(u-1)}\,du = \beta\,\mathrm{e}^{a\,\beta}\int_1^\infty \log u\,\mathrm{e}^{-a\,\beta\,u}\,du = \beta\,\mathrm{e}^{a\,\beta}\,\mathbb{E}_0^1(a\,\beta), \tag{31}$$

where $\mathbb{E}_0^1(a\,\beta)=\int_1^\infty \log u\,\mathrm{e}^{-z\,u}\,du$ is the generalized integer-exponential integral.

Using Eq. (22), we obtain

$$\mathbb{E}\left\{\log\left[1-\left(1+\frac{t}{\beta}\right)^{-\alpha}\right]\right\} \;=\; \int_0^\infty \log\left[1-\left(1+\frac{t}{\beta}\right)^{-\alpha}\right]\mathrm{e}^{-a\,t}\,dt.$$

Using infinite series $\log(1+u)=-\sum_{n=1}^\infty(-1)^{n+1}\frac{u^n}{n}$, $|u|<1$ (Polyanina and Manzhiror, 2008) to the quantity $\log\left[1-\left(1+\frac{t}{\beta}\right)^{-\alpha}\right]$ and then interchanging the order of integration and summation, we obtain

$$\mathbb{E}\left\{\log\left[1-\left(1+\frac{t}{\beta}\right)^{-\alpha}\right]\right\} \;=\; \sum_{n=0}^\infty\frac{-1}{n}\int_0^\infty\left(1+\frac{t}{\beta}\right)^{-\alpha\,n}\mathrm{e}^{-a\,t}\,dt.$$

Let $u=1+\frac{t}{\beta}$, then we have

$$\mathbb{E}\left\{\log\left[1-\left(1+\frac{t}{\beta}\right)^{-\alpha}\right]\right\} \;=\; -\sum_{n=0}^\infty\frac{\beta}{n}\,\mathrm{e}^{a\,\beta}\,\mathbb{E}_{a\,n}(a\,\beta). \tag{32}$$

The proof can easily follow from Eqs. (31), (32) and (22).

Estimation of the parameters EBIII{Lx} distribution. Let $X_1,X_2,...,X_n$ be the random sample from EBIII{Lx} distribution defined in Eq. (27), then the likelihood function is

$$\begin{aligned}\ell(\Theta) \;=\;& n\,\log\left(\frac{a\,c\,k}{\alpha}\right)-(c+1)\sum_{i=1}^n\log x_i-(k+1)\sum_{i=1}^n\log\left(1+x^{-c}\right)\\ &-\left(\frac{\alpha+1}{\alpha}\right)\sum_{i=1}^n\log\left\{1-\left(1+x^{-c}\right)^{-k}\right\}-a\,\sum_{i=1}^n\left[\left\{1-(1+x^{-c})^{-k}\right\}^{-\frac{1}{\alpha}}-1\right].\end{aligned}$$

The score vector are:

$$U_a \;=\; \frac{n}{a}-\sum_{i=1}^n\left[(1-B_{c,k}(x))^{-\frac{1}{\alpha}}-1\right],$$

$$U_\alpha \;=\; -\frac{n}{\alpha}+\frac{1}{\alpha^2}\sum_{i=1}^n\log\left\{1-\left(1+x^{-c}\right)^{-k}\right\}+a\sum_{i=1}^n\left\{1-(1+x^{-c})^{-k}\right\}^{-\frac{1}{\alpha}}\log\left\{1-(1+x^{-c})^{-k}\right\},$$

$$
\begin{aligned}
U_k &= \frac{n}{k} - \sum_{i=1}^{n} \log\left(1+x^{-c}\right) - \left(\frac{\alpha+1}{\alpha}\right) \sum_{i=1}^{n} \left[\frac{(1+x^{-c})^{-k}\, \log(1+x^{-c})}{1-(1+x^{-c})^{-k}}\right] \\
&\quad - \frac{a}{\alpha} \sum_{i=1}^{n} \left[\frac{\left\{1-(1+x^{-c})^{-k}\right\}^{-\frac{\alpha+1}{\alpha}}\, \log(1+x_i^{-c})}{(1+x_i^{-c})^{k}}\right], \\
U_c &= \frac{n}{c} - \sum_{i=1}^{n} \log x_i + (k+1) \sum_{i=1}^{n} \left[\frac{x_i^{-c}\, \log x_i}{1+x_i^{-c}}\right] - k \left(\frac{\alpha+1}{\alpha}\right) \sum_{i=1}^{n} \left[\frac{(1+x^{-c})^{-k-1}\, x_i^{-c}\, \log x_i}{1-(1+x^{-c})^{-k}}\right] \\
&\quad + \frac{a\,k}{\alpha} \sum_{i=1}^{n} \frac{x_i^{-c}\, \log x_i\, (1+x^{-c})^{-k-1}}{\left[1-(1+x^{-c})^{-k}\right]^{\frac{1}{\alpha}+1}}.
\end{aligned}
$$

Setting U_b, U_a,U_k and U_c equal to zero, and then solving these equations simultaneously yields the maximum likelihood estimates.

5 Simulation study

In this section, we investigate the performance of maximum likelihood method (MLM) for EBIII{Lx} distribution for different sample sizes n = 50, 100, 200, 300, 500. We simulate 1000 samples for the two parameters combinations; Set-1: c=1.5, k=2, a=3, α=4, Set-II:c=3, k=3, a=4, α=2 for EBIII{Lx} distribution. The mean estimates and mean square errors (MSEs) of the parameters are obtained, and are reported in Table 2. We noted that the MSEs decrease as the sample size increases. Thus, the MLM performs well in estimating the parameters of EBIII{Lx} distribution.

TABLE 2. Simulation results showing the means and MSEs of the EBIII{Lx} distribution.

	Set 1				Set 2			
N	$\hat{c}$	$\hat{k}$	$\hat{a}$	$\hat{\alpha}$	$\hat{c}$	$\hat{k}$	$\hat{a}$	$\hat{\alpha}$
50	1.9458	2.3702	5.5602	4.3846	6.1584	2.3163	4.3306	2.9353
	(0.25763)	(0.1895)	(0.7565)	(0.6382)	(0.8465)	(0.2266)	(0.7324)	(0.6092)
100	1.9458	2.3702	5.5602	4.3846	4.1671	3.0341	3.0271	5.8784
	(0.1821)	(0.1340)	(0.5349)	(0.4512)	(0.3474)	(0.1560)	(0.2496)	(0.5622)
200	1.7977	2.1270	5.0157	5.0501	4.6527	2.3920	3.2211	4.9456
	(0.0754)	(0.0746)	(0.1895)	(0.3853)	(0.2030)	(0.0929)	(0.1527)	(0.4010)
300	1.8222	1.9923	5.2490	5.0109	4.7080	2.2654	3.0147	4.2998
	(0.05381)	(0.0568)	(0.1287)	(0.3046)	(0.1529)	(0.0693)	(0.0992)	(0.3067)
500	1.6787	2.1535	5.0589	5.5814	4.7080	2.2654	3.0147	4.2998
	(0.0329)	(0.0455)	(0.0803)	(0.2627)	(0.1184)	(0.0537)	(0.0768)	(0.2376)

6 Applications

In this section, we check the performance of EBIII{Lx} distribution by applying to three real-life data sets (two for uncensored and one for censored). The model parameters are estimated by using MLM, and the well-known goodness-of-fit criterion Viz. Akaike information criterion (AIC), Anderson-Darling (A^*), Cramer-von Mises (W^*) and Kolmogrov-Smirnov (K-S) are used to compare the fitted models. In general, small values of these statistics and large p-value show good fit to the data. The plots of the PDFs and CDFs of the fitted distributions are also displayed for visual comparison. The required computations are carried out in the R-language.

6.1 Uncensored (or Complete) data sets

Data set 1. Weisberg (2005) reported this data set which represent the sum of skin folds of 202 athletes collected at the Australian Institute of Sports. The data set has recently been used by Alzaatreh et al. (2016). We fitted this data set to the EBIII{Lx}, WG{E}, WG{LL} and Cauchy-G{logistic} (CaG{L}) models reported in Alzaatreh et al. (2016). Table 3 lists the MLEs and their corresponding standard errors (in parentheses) of the model parameters. The results in Table 4 shows values of AIC and K-S, and large K-S p- value of all fitted distributions. For visual comparison, the histogram of the data set 1, fitted density functions, and estimated cumulative functions are displayed in Figure 5. We noted that the proposed EBIII{Lx} model provides the best fit as compared to other fitted models for first data set.

Data set 2. Proschan (1963) reported and studied this data set which consists of the number of successive failures for the air conditioning system of each member in a fleet of 13 Boeing 720 jet airplanes. Here, we compare EBIII{Lx} model with exponentiated Kumaraswamy-Dagum (EKD), Kumaraswamy-Dagum (KD), Dagum (D), beta-Burr III (BBIII) (Gomes et al., 2013), beta-Lomax (BLx) and Kumaraswamy Lomax (KLx) models reported in Huang and Oluyede (2014). Table 5 lists the MLEs and their corresponding standard errors (in parentheses) of the model parameters. The results in Table 6 show values of AIC, A^* and W^* for all fitted distributions. For visual comparison, the histogram of the data set 2, fitted density functions, and estimated cumulative functions are displayed in Figure 6. We noted that the proposed EBIII{Lx} model provides the better fit as compared to other fitted models for second data set.

Table 3. MLEs and their standard errors (in parentheses) for data set 1.

Distribution	c	k	a	α
EBlll{Lx}	1.5691	800.0536	3.4161	1.8467
	(0.2553)	(618.5830)	(2.8376)	(0.9089)
WG{E}	0.7291	2.6521	3.9319	17.3862
	(0.0404)	(0.0025)	(0.4010)	(0.0025)
WG{LL}	0.3184	10.4219	0.0219	13.4018
	(0.0518)	(2.8058)	(0.0131)	(2.7954)
CaG{L}	-0.9076	2.1914	3.2642	29.4223
	(0.3571)	(0.0236)	(0.3125)	(0.0203)

Table 4. The statistics AIC, K-S and K-S p-value for data set 1.

Distribution	$\hat{\ell}$	AIC	K-S	p-value (K-S)
EBlll{Lx}	952.8343	1913.6690	0.0622	0.3930
WG{E}	953.5709	1915.1420	0.0634	0.3921
WG{LL}	962.2296	1932.4590	0.0793	0.1578
CaG{L}	977.9650	1963.9300	0.1174	0.0076

Data set 3 (Censored data set). Lee and Wang (2003) reported this censored data set which consist of death times (in weeks) of patients with tongue cancer with aneuploid DNA profile. The asterisks

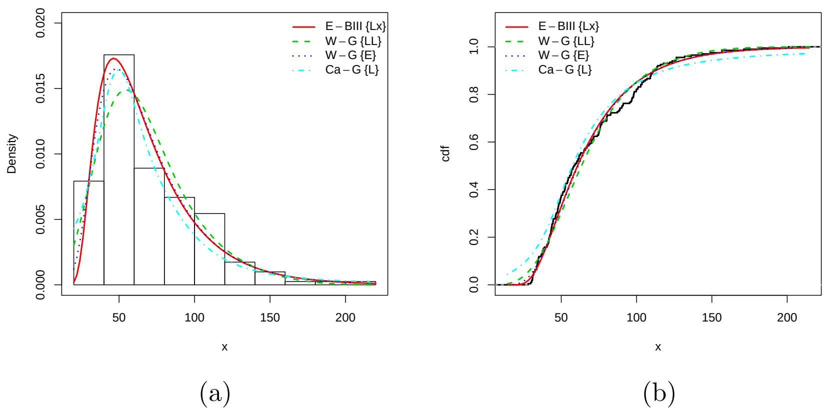


FIGURE 5. Plots of estimated PDFs and CDFs of fitted models for data set 1.

TABLE 5. MLEs and their standard errors (in parentheses) for the data set 2.

Distribution	c	k	a	b	β
EBlllLx	0.4032	10.4221	0.4403	1.5781	-
	(0.2234)	(2.8847)	(0.4220)	(2.5644)	-
EKD	0.1687	16.8331	18.9585	5.4991	0.6328
	(0.0142)	(0.0118)	(0.0029)	(0.0029)	(0.0127)
BBIII	0.5177	83.7384	0.1200	44.5743	-
	(0.1010)	(80.5487)	(0.1101)	(63.0834)	-
KLx	48.2990	0.0459	1.4297	91.6647	-
	(44.2857)	(0.0604)	(0.3104)	(166.1740)	-
BLx	146.0058	0.2395	1.2594	12.5473	-
	(95.1799)	(2.0863)	(0.2115)	(110.2322)	-
KD	5.0354	4.3846	0.3762	-	21.7047
	(2.1177)	(3.0727)	(0.1253)	-	(27.9167)
D	1.2390	94.1526	-	-	1.2626
	(0.1749)	(33.7549)	-	-	(0.0663)

in Table 7 denote the censoring times. This data set has recently been analysed by Oguntunde and Adejumo (2015) for analysing the performance of generalized inverted generalized exponential (GIGE) distribution. Here we compare the EBIII{Lx} model with Kumaraswamy-Lomax (KLx), beta-Lomax (BLx), beta-BurrIII models. We use AIC and BIC (Bayesian information criterion) statistics to compare the fits of these fitted models.

We consider a data set $D = (x, r)$, where $x = (x_1, x_2, ..., x_n)^T$ are the observed failure times and $r_i = (r_1, r_2, ..., r_n)^T$ are the censored failure times. r_i is equal to 1 if a failure is observed

Table 6. The statistics AIC, W^* and A^* for data set 2.

Distribution	$\hat{\ell}$	AIC	W^*	A^*
EBlll{Lx}	1032.423	2072.847	0.0297	0.2278
EKD	1032.272	2074.544	0.0356	0.2575
BBIII	1032.504	2073.007	0.0333	0.2530
KLx	1034.961	2077.922	0.0516	0.3888
BLx	1034.624	2077.247	0.0562	0.4087
KD	1036.672	2079.345	0.0992	0.5360
D	1038.672	2083.345	0.1066	0.7469

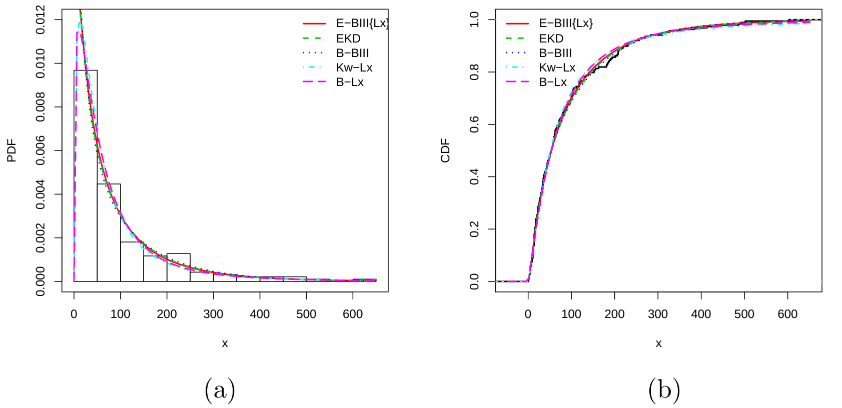


Figure 6. Plots of estimated PDFs and CDFs of fitted models for data set 2.

and 0 if otherwise. We suppose that the data are independently and identically distributed from a distribution with PDF given by Eq. (27). Let $\Theta = (c, k, \alpha, a)^T$ denote the vector of parameters. Then the likelihood function of Θ can be written as

$$\ell(D;\Theta) = \prod_{i=1}^{n} \left[f(x_i;\Theta)\right]^{r_i} \left[1 - F(x_i;\Theta)\right]^{1-r_i},$$

and the log-likelihood reduces to

$$\ell(\Theta) = r_i \sum_{i=1}^{n} \log\left[f(x_i;\Theta)\right] + (1 - r_i) \sum_{i=1}^{n} \log\left[1 - F(x_i;\Theta)\right]. \tag{33}$$

From Eq.s (26),(27) and (33), we have

$$\begin{aligned}\ell &= r_i \sum_{i=1}^{n} \Big[\log\left(\frac{a\,c\,k}{\alpha}\right) - \left(\frac{\alpha+1}{\alpha}\right) \log\left(1-(1+x^{-c})^{-k}\right) - (c+1)\log(x) - (k+1)\log\left(1+x^{-c}\right) \\ &\quad - \left\{a\left(\left\{1-(1+x^{-c})^{-k}\right\}^{-\frac{1}{\alpha}} - 1\right)\right\}\Big] + a\,(1-r_i)\sum_{i=1}^{n}\left\{\left(\left\{1-(1+x^{-c})^{-k}\right\}^{-\frac{1}{\alpha}} - 1\right)\right\}.\end{aligned}$$

The log-likelihood function can be maximized numerically to obtain the maximum likelihood estimates. Various routines are available in R, Mathematica and Matlab which can be used for numerical maximization. Here we use the Optimum routine using R-software.

TABLE 7. Data set 3 (censored): Death times (in weeks) of patients with tongue cancer.

1	3	3	4	10	13	13	16	16	24	26	27	28
30	30	32	41	51	61*	65	67	70	72	73	74*	77
79*	80*	81*	87*	87*	88*	89*	91	93	93*	96	97	100
101*	104	104*	108*	109*	120*	131*	150*	157	167	231*	240*	400*

TABLE 8. MLEs and their standard errors and statistics $\hat{\ell}$, AIC and BIC for the data set 3.

Model	Parameters	MLE	Standard error	$\hat{\ell}$	AIC	BIC
EBIII{Lx}	c	0.4288	0.3426	-155.5849	319.1699	326.9749
	k	6.5052	3.7037			
	α	0.2044	0.3640			
	a	0.1607	0.2281			
KLx	a	0.5071	0.5517	-156.7473	321.4945	329.2995
	b	3.4304	3.7028			
	c	1.5674	0.5484			
	d	28.9164	17.9781			
BBIII	a	0.5412	0.4206	-158.2267	324.4534	332.2584
	b	13.9600	10.6778			
	c	0.4290	0.1045			
	k	18.2258	10.9104			
BLx	a	1.8124	0.5767	-159.1005	326.201	334.006
	b	3.1859	2.3737			
	c	34.7539	25.1470			
	k	0.7193	0.5061			

Table 8 shows that MLEs and their standard errors, and the values of statistics AIC and BIC for data set 3. Oguntunde and Adejumo (2015) also compared GIGE model with other competitive models for data set 3, and reported AIC= 607.712. Our proposed model gives minimum AIC value as compared to GIGE model. Also our model EBIII{Lx} gives minimum AIC and BIC values as compared the competitive models KwLx, BBIII, and BLx. For visual comparison, the histogram of the data set 3 and estimated cumulative functions are displayed in Figure 7. We noted that EBIII{Lx} model gives smaller values as compared to other fitted models, enforcing us to say that the proposed model is better as compared other models for data set 3.

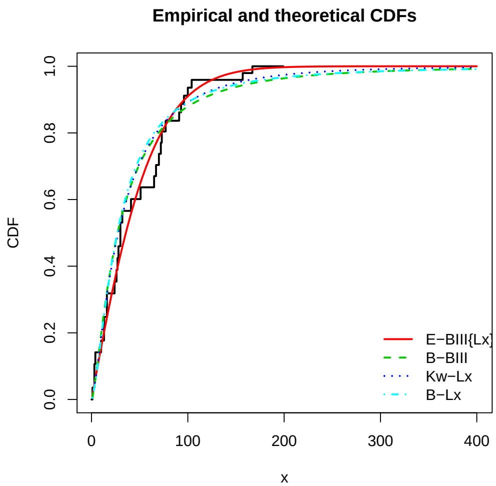


FIGURE 7. Plots of the estimated CDFs of fitted models for data set 3.

7 Final remarks

We proposed a new extended generalized Burr-III family, that is T-BurrIII{Y}, which hosts three sub-families. We study some general properties of the extended generalized Burr III family and Burr III sub-families including explicit expressions for their quantile functions, moments, Shannon entropy and mean deviations. We study the density and hazard rate plots of four special models of sub-families which shows the flexibility of this extended family. A simulation study is carried out to check the performance of maximum likelihood method for EBIII{Lx} model. Later, three real data sets were analysed to demonstrate the goodness-of-fit of the proposed EBIII{Lx} model to complete and censored data sets. As a whole, the EBIII{Lx} model performs better as compared to all other competitors to these three data sets.

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