# Stability of the general form of quadratic-quartic functional equations in non-Archimedean $\mathcal{L}$-fuzzy normed spaces

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**Abstract**

In this paper, we introduce and obtain the general solution of a new generalized mixed quadratic and quartic functional equation and investigate its stability in non-Archimedean $\mathcal{L}$-fuzzy normed spaces.

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## 1 Introduction

For the last 40 years, fuzzy theory which was introduced by Zadeh [39], has become very active area of research and a lot of development has been made in the theory of fuzzy sets to find the fuzzy analogues of the classical set theory. This branch finds a wide range of application in the field of science and engineering. Katsaras [22] introduced an idea of fuzzy norm on a linear space in 1984. In the same year, Wu and Fang [35] introduced a notion of fuzzy normed space to give a generalization of the Kolmogoroff normalized theorem for fuzzy topological linear spaces. In 1991, Biswas [5] defined and studied fuzzy inner product spaces in linear space. In 1992, Felbin [18] introduced an alternative definition of a fuzzy norm on a linear topological structures of a fuzzy normed linear spaces. In 1994, Cheng and Mordeson [13] introduced a notion of fuzzy norm on a linear space in such a manner that the corresponding induced fuzzy metric is of Kramosil and Michalek [23]. This concept was modified in [4] by removing a regular condition.

Stability problem of a functional equation was first posed by Ulam [34] and that was partially answered by Hyers [21] and then generalized by Aoki [1] and Rassias [27] for additive mappings and linear mappings, respectively. In 1994, a generalization of Rassias theorem was obtained by Găvruta [19], who replaced $\varepsilon(||x||^p+||y||^p)$ by a general control function $\varphi(x,y)$. This idea is known as generalized Hyers-Ulam-Rassias stability. After that, the general stability problems of various functional equations such as quadratric, cubic, quartic and mixed type of such functional equations with more general domains and ranges have been investigated by a number of authors. We refer the interested readers for some results regarding to the stability of various forms of mixed functional equations to [7], [8], [9], [10], [11], [37] and [38].

One of the problems in $\mathcal{L}$-fuzzy topology is to obtain an appropriate concept of $\mathcal{L}$-fuzzy metric spaces and $\mathcal{L}$-fuzzy normed spaces. In 2004, Park [26] introduced and studied the notion of intuitionistic fuzzy metric spaces. In 2006, Saadati and Park [32] introduced and studied the notion of intuitionistic fuzzy normed spaces. Then, Deschrijver et al. [15] and Saadati [33] generalized

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the concept of intuitionistic fuzzy metric (normed) spaces and introduced and studied a notion of $\mathcal{L}$-fuzzy metric spaces and $\mathcal{L}$-fuzzy normed spaces. The generalized Hyers-Ulam stability of different functional equations in intuitionistic fuzzy normed spaces has been studied by a number of the authors; for example, see [6], [12], [29], [30] and [36].

In [16], Gordj et al. obtained the general solution and investigated the Ulam stability problem for the following mixed quadratic and quartic functional equation

$$\begin{aligned} &f(nx+y)+f(nx-y) \\ &\quad = n^2\{f(x+y)+f(x-y)\}+2f(nx)-2n^2f(x)-2(n^2-1)f(y) \end{aligned} \tag{1.1}$$

in quasi-$\beta$-normed spaces; see also [24] and [37]. A different form of a mixed quadratic and quartic functional equation which is introduced in [17] is as follows:

$$\begin{aligned} &f(nx+y)+f(nx-y) \\ &\quad = n^2\{f(x+y)+f(x-y)\}+\frac{n^2(n^2-1)}{6}(f(2x)-4f(x))-2(n^2-1)f(y). \end{aligned} \tag{1.2}$$

In this work, we consider the functional equation which is a generalization of (1.1) and (1.2) as follows:

$$\begin{aligned} &f(nx+my)+f(nx-my) \\ &\quad = (mn)^2\{f(x+y)+f(x-y)\}+2f(nx)+2f(my)-2(mn)^2\{f(x)+f(y)\}. \end{aligned} \tag{1.3}$$

It is easily verified that the function $f(x)=\alpha x^4+\beta x^2$ is a solution of the functional equation (1.3). We obtain the general solution and study the Hyers-Ulam-Rassias stability of the equation (1.3) in non-Archimedean $\mathcal{L}$-fuzzy normed spaces for fixed integers $m$ and $n$ such that $m\neq 0, n\neq 0, m+n\neq 0$.

## 2 Preliminary notations

In this section, we restate the usual terminology, notations and conventions of the theory of intuitionistic fuzzy normed space, as in [26], [28], [29], [30] and [31]. In general, the definition of an intuitionistic fuzzy set is given in [3] for the first time.

**Definition 2.1.** Let $(L,\leq_L)$ be a complete lattice and $U$ be a non-empty set called the universe. An $L$-fuzzy set in $U$ is defined as a mapping $\mathcal{F}: U \longrightarrow L$. For each $u$ in $U$, $\mathcal{F}(u)$ represents the degree (in $L$) to which $u$ is an element of $\mathcal{F}$.

Let $\leq_{L^*}$ be a order relation on the set $L^*=\{(x_1,x_2):(x_1,x_2)\in[0,1]^2, x_1+x_2\leq 1\}$ defined by

$$(x_1,x_2)\leq_L (y_1,y_2) \Longleftrightarrow x_1\leq y_1, y_2\leq x_2$$

for all $(x_1,x_2),(y_1,y_2)\in L^*$. Then, $(L^*,\leq_{L^*})$ is a complete lattice [14]. We denote the units of $L^*$ by $0_{L^*}=(0,1)$ and $1_{L^*}=(1,0)$. Recall that the above order relation is a well-known definition due to Atanassov [2].

**Definition 2.2.** An intuitionistic fuzzy set $\mathcal{F}_{\mu,\nu}$ in a universal set $U$ is an object $\mathcal{F}_{\mu,\nu}=\{(\mu_{\mathcal{F}}(u),\nu_{\mathcal{F}}(u)): u\in U\}$, where $\mu_{\mathcal{F}}(u)$ and $\nu_{\mathcal{F}}(u)$ belong to $[0,1]$ for all $u\in U$ with $\mu_{\mathcal{F}}(u)+\nu_{\mathcal{F}}(u)\leq 1$. The numbers $\mu_{\mathcal{F}}(u)$ and $\nu_{\mathcal{F}}(u)$ are called the membership degree and the non-membership degree, respectively, of $u$ in $\mathcal{F}_{\mu,\nu}$.

**Definition 2.3.** Let $\mathcal{L} = (L, \leq_L)$ be a lattice. A triangular norm ($t$-norm) on $\mathcal{L}$ is a mapping $\mathcal{T} : L \times L \longrightarrow L$ satisfying the following conditions:

(i) $\mathcal{T}(x, 1_{\mathcal{L}}) = x$ (boundary condition) $\quad (x \in L)$;

(ii) $\mathcal{T}(x, y) = \mathcal{T}(y, x)$ (commutativity) $\quad (x, y \in L)$;

(iii) $\mathcal{T}(x, \mathcal{T}(y, z)) = \mathcal{T}(\mathcal{T}(x, y), z)$ (associativity) $\quad (x, y, z \in L)$;

(iv) $x_1 \leq_L y_1$ and $x_2 \leq_L y_2 \Longrightarrow \mathcal{T}(x_1, x_2) \leq_L \mathcal{T}(y_1, y_2)$ (monotonicity)
$(x_1, x_2, y_1, y_2 \in L)$.

A $t$-norm $\mathcal{T}$ on $L$ is said to be *continuous* if, for any $x, y \in L$ and any sequences $\{x_n\}$ and $\{y_n\}$ which converge to $x$ and $y$, respectively, then $\lim_{n\to\infty} \mathcal{T}(x_n, y_n) = \mathcal{T}(x, y)$.

**Example 2.4.** Let $x = (x_1, x_2), y = (y_1, y_2) \in L$. Then $\mathcal{T}(x, y) = (x_1y_1, \min\{x_2 + y_2, 1\})$ and $\mathcal{M}(x, y) = (\min\{x_1, y_1\}, \max\{x_2, y_2\})$ are continuous $t$-norm [36].

Here, we define a sequence $\mathcal{T}^n$, recursively by $\mathcal{T}^1 = \mathcal{T}$ and

$$\mathcal{T}^n\left(x^{(1)}, x^{(1)}, \cdots, x^{(n+1)}\right) = \mathcal{T}\left(\mathcal{T}^{n-1}\left(x^{(1)}, x^{(1)}, \cdots, x^{(n)}\right), x^{(n+1)}\right)$$

for all $n \geq 2$ and $x^{(j)} \in L$. A $t$-norm $\mathcal{T}$ can also be extended to a countable operation by taking, for any sequence $\mathcal{T}_j^\infty\left(x^{(j)}\right) = \lim_{r\to\infty} \mathcal{T}_j^r\left(x^{(j)}\right)$. The limit on the right side of this equation exists since the sequence $\{\mathcal{T}_j^\infty\left(x^{(j)}\right)\}$ is non-increasing and bounded below.

**Definition 2.5.** A negator on $L$ is a decreasing mapping $\mathfrak{N} : L \longrightarrow L$ satisfying $\mathfrak{N}(0_{\mathcal{L}}) = 1_{\mathcal{L}}$ and $\mathfrak{N}(1_{\mathcal{L}}) = 0_{\mathcal{L}}$. If $\mathfrak{N}(\mathfrak{N}(x)) = x$, for all $x \in L$, then $\mathfrak{N}$ is called an involutive negator. A negator on $[0, 1]$ is a decreasing mapping $\mathcal{N} : L \longrightarrow L$ satisfying $\mathcal{N}(0) = 1$ and $\mathcal{N}(1) = 0$. The standard negator on $[0, 1]$ is defined by $\mathcal{N}_s(x) = 1 - x$ for all $x \in [0, 1]$.

**Definition 2.6.** The triple $(X, \mathcal{P}, \mathcal{T})$ is called an *$\mathcal{L}$-fuzzy normed space* if $X$ is a vector space, $\mathcal{T}$ is a continuous $t$-norm on $L$ and $\mathcal{P}$ is an $\mathcal{L}$-fuzzy set on $X \times (0, \infty)$ satisfying the following conditions:

(i) $0 <_{\mathcal{L}} \mathcal{P}(x, t)$;

(ii) $\mathcal{P}(x, t) = 1_{\mathcal{L}}$ if and only if $x = 0$;

(iii) $\mathcal{P}(\alpha x, t) = \mathcal{P}\left(x, \frac{t}{|\alpha|}\right)$ for all $\alpha \neq 0$;

(iv) $\mathcal{T}(\mathcal{P}(x, t), \mathcal{P}(y, s)) \leq_{\mathcal{L}} \mathcal{P}(x + y, t + s)$;

(v) The map $\mathcal{P}(x, \cdot) : (0, \infty) \longrightarrow \mathcal{L}$ is continuous;

(vi) $\lim_{t\to 0} \mathcal{P}(x, t) = 0_{\mathcal{L}}$ and $\lim_{t\to\infty} \mathcal{P}(x, t) = 1_{\mathcal{L}}$;

for all $x, y \in X$ and all $t, s > 0$. In this case $\mathcal{P}$ is called $\mathcal{L}$-fuzzy norm. If $\mathcal{P} = \mathcal{P}_{\mu,\nu}$ (see Definition 2.2) is an intuitionistic fuzzy set, then the triple $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is said to be an *intuitionistic fuzzy normed space* (briefly, IFN-space). In this case, $\mathcal{P}_{\mu,\nu}$ is called an *intuitionistic fuzzy norm* on $X$; some examples of IFN-space are provided in [30].

Note that, if $\mathcal{P}$ is an $L$-fuzzy norm on $X$, then the following statements hold:

(i) $\mathcal{P}(x,t)$ is nondecreasing with respect to $t$ for all $x \in X$;

(ii) $\mathcal{P}(x-y,t) = \mathcal{P}(y-x,t)$ for all $x, y \in X$ and $t > 0$.

**Definition 2.7.** Let $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ be an IFN-space.

(1) A sequence $\{x_n\}$ in $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is said to be *convergent* to a point $x$ if $\mathcal{P}_{\mu,\nu}(x_n - x, t) \to 1_{\mathcal{L}}$ as $n \to \infty$ for all $t > 0$;

(2) A sequence $\{x_n\}$ in $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is called a *Cauchy sequence* if, for every $t > 0$ and $0 < \varepsilon < 1$, there exists a positive integer $N$ such that $(N_s(\varepsilon), \varepsilon) \leq_{\mathcal{L}} \mathcal{P}_{\mu,\nu}(x_n - x_m, t)$ for all $m, n > N$, where $N_s$ is the standard negator;

(3) $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is said to be *complete* if and only if every Cauchy sequence in $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is convergent to a point in $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$. A complete intuitionistic fuzzy normed space is called an *intuitionistic fuzzy Banach space*.

In [20], Hensel introduced a field with a valuation in which it does not have the Archimedean property. By a non-Archimedean field, we mean a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot|$ from $\mathbb{K}$ to $[0,\infty)$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r||s|$ and $|r+s| \leq \max\{|r|,|s|\}$ for all $r, s \in \mathbb{K}$. Clearly, $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. Note that $|n| \leq 1$ for each integer $n$. From now on, we assume that $|\cdot|$ is non-trivial, i.e., there exists an $a_0 \in \mathbb{K}$ such that $|a_0| \neq 0, 1$.

**Definition 2.8.** [25] Let $X$ be a vector space over a non-Archimedean scalar field $\mathbb{K}$ with a valuation $|\cdot|$. A function $\|\cdot\| : X \longrightarrow [0,\infty)$ is a *non-Archimedean norm* if it satisfies for all $r \in \mathbb{K}$ and $x, y \in X$

(i) $\|x\| = 0$ if and only if $x = 0$,

(ii) $\|rx\| = |r|\|x\|$,

(iii) $\|x+y\| \leq \max\{\|x\|, \|y\|\}$ (the strong triangle inequality).

Then, $(X, \|\cdot\|)$ is called a *non-Archimedean normed space*.

**Definition 2.9.** A non-Archimedean $\mathcal{L}$-fuzzy normed space is a triple $(\mathcal{V}, \mathcal{P}, \mathcal{T})$, where $\mathcal{V}$ is a vector space over a non-Archimedean field $\mathbb{K}$, $\mathcal{T}$ is a continuous $t$-norm on $\mathcal{L}$ and $\mathcal{P}$ is an $\mathcal{L}$-fuzzy set on $\mathcal{V} \times (0,+\infty)$ satisfying the following conditions: for all $x, y \in \mathcal{V}$ and $t, s \in (0,+\infty)$,

(a) $0_{\mathcal{L}} \leq_{\mathcal{L}} \mathcal{P}(x,t)$;

(b) $\mathcal{P}(x,t) = 1_{\mathcal{L}}$ if and only if $x = 0$;

(c) $\mathcal{P}(\alpha x, t) = \mathcal{P}\left(x, \frac{t}{|\alpha|}\right)$ for all $\alpha \neq 0$;

(d) $\mathcal{T}(\mathcal{P}(x,t), \mathcal{P}(y,s)) \leq_{\mathcal{L}} \mathcal{P}(x+y, \max(t,s))$

(e) $\mathcal{P}(x,\cdot) : (0,\infty) \longrightarrow \mathcal{L}$ is continuous;

(f) $\lim_{t\to 0}\mathcal{P}(x,t)=0_{\mathcal{L}}$ and $\lim_{t\to 0}\mathcal{P}(x,t)=1_{\mathcal{L}}$.

**Example 2.10.** [30] Let $(X,\|\cdot\|)$ be a non-Archimedean normed linear space. Then, the triple $(X,\mathcal{P},\min)$, where

$$\mathcal{P}(x,t)=\begin{cases} 0, & t\le \|x\|\,, \\ 1, & t>\|x\|\,, \end{cases}$$

is a non-Archimedean $\mathcal{L}$-fuzzy normed space in which $\mathcal{L}=[0,\,1]$.

**Example 2.11.** [30] Let $(X,\|\cdot\|)$ be a non-Archimedean normed linear space. Denote $\tau_m(a,b)=(\min\{a_1,b_1\},\max\{a_2,b_2\})$ for all $a=(a_1,a_2), b=(b_1,b_2)\in L^*$ and let $\mathcal{P}_{\mu,\nu}$ be the intuitionistic fuzzy set on $X\times(0,+\infty)$ defined as follows:

$$\mathcal{P}_{\mu,\nu}(x,t)=\left(\frac{t}{t+\|x\|},\frac{\|x\|}{t+\|x\|}\right)$$

for all $t\in\mathbb{R}^+$. Then, $(X,\mathcal{P}_{\mu,\nu},\tau_M)$ is a non-Archimedean intuitionistic fuzzy normed space.

## 3 Solution and stability of (1.3)

In this section, we prove the generalized Hyers-Ulam-Rassias stability of the mixed type quadratic and quartic functional equation (1.3). We firstly find out the general solution of (1.3).

**Lemma 3.1.** *Let $X$ and $Y$ be real vector spaces. A mapping $f:X\longrightarrow Y$ satisfies the functional equation (1.3) if and only if $f$ satisfies (1.1).*

*Proof.* Assume that $f$ satisfies the functional equation (1.1). Letting $x=0$ in (1.1), we get $f(y)=f(-y)$. Replacing $(x,y)$ by $(-x,-y)$ and using the eveness of $f$, we obtain

$$\begin{aligned} &f(y+nx)+f(y-nx)\\ &\quad=n^2\{f(x+y)+f(x-y)\}+2f(nx)-2n^2f(x)-2(n^2-1)f(y)\end{aligned}\tag{3.1}$$

for all $x,y\in X$. Switching $x$ and $y$, and then interchanging $n$ into $m$ in (3.1), we have

$$\begin{aligned} &f(x+my)+f(x-my)\\ &\quad=m^2\{f(x+y)+f(x-y)\}+2f(my)-2m^2f(y)-2(m^2-1)f(x)\end{aligned}\tag{3.2}$$

for all $x,y\in X$. Substituting $y$ by $my$ in (1.1), we deduce that

$$\begin{aligned} &f(nx+my)+f(nx-my)\\ &\quad=n^2\{f(x+my)+f(x-my)\}+2f(nx)-2n^2f(x)-2(n^2-1)f(my)\end{aligned}\tag{3.3}$$

for all $x,y\in X$. Plugging (3.2) into (3.3), one can check that the equality (1.3) holds for all $x,y\in X$ and all $m\neq 0, n\neq 0, m+n\neq 0$. The converse is clear. Q.E.D.

**Proposition 3.2.** *Let $X$, $Y$ be real vector spaces and a mapping $f:X\longrightarrow Y$ satisfies the functional equation (1.3). Then, the mappings $g:X\longrightarrow Y$, defined by $g(x)=f(2x)-16f(x)$ and $h:X\longrightarrow Y$, defined via $h(x)=f(2x)-4f(x)$, are quadratic and quartic, respectively.*

*Proof.* The result follows from Lemma 3.1 and the proof of [16, Lemma 2.1]. Q.E.D.

Here and subsequently, let $\mathbb{K}$ be a non-Archimedean field, $X$ a vector space over $\mathbb{K}$ and $(Y, \mathcal{P}, \mathcal{T})$ a non-Archimedean $\mathcal{L}$-fuzzy Banach space over $\mathbb{K}$. We define an $\mathcal{L}$-fuzzy approximately quadratic mapping. Let $\Psi$ be an $\mathcal{L}$-fuzzy set on $X \times X \longrightarrow [0,\infty)$ such that $\Psi(x, y, \cdot)$ is non-decreasing, $\Psi(cx, cx, t) \geq_{\mathcal{L}} \Psi\left(x, x, \frac{t}{|c|}\right)$, and $\lim_{t\to\infty}(x, y, t) = 1_{\mathcal{L}}$, for all $x, y \in X, t > 0$ and $c \in \mathbb{K}\backslash\{0\}$.

Now before taking up the main subject, given $f : X \longrightarrow Y$, we define the difference operator $\Delta_{m,n} f : X \times X \longrightarrow Y$ by

$$\begin{aligned}\Delta_{m,n} f(x, y) &= f(nx + my) + f(nx - my) - (mn)^2\{f(x + y) + f(x - y)\}\\ &\quad - 2f(nx) + 2f(my) + 2(mn)^2\{f(x) + f(y)\}\end{aligned}$$

for all $x, y \in X$ and $t > 0$ and for fixed integers $m$ and $n$ such that $m \neq 0, n \neq 0, m + n \neq 0$.

A mapping $f : X \longrightarrow Y$ is said to be $\Psi$-*approximately quadratic-quartic* if

$$\mathcal{P}\big(\Delta_{m,n} f(x, y), t\big) \geq_{\mathcal{L}} \Psi(x, y, t) \tag{3.4}$$

for all $x, y \in X$ and $t > 0$.

**Theorem 3.3.** Let $f : X \longrightarrow Y$ be $\Psi$-approximately quadratic-quartic such that $f(0) = 0$. If there exists an $\alpha \in (0,\infty)$ and an integer $k \geq 2$ with $|2^k| < \alpha$ and $|2| \neq 0$ such that

$$\Psi(2^{-k}x, 2^{-k}y, t) \geq_{\mathcal{L}} \Psi(x, y, \alpha t), \tag{3.5}$$

$$\lim_{n\to\infty} \mathcal{T}_{j=n}^{\infty} M\left(x, \frac{\alpha^j t}{|2|^{kj}}\right) = 1_{\mathcal{L}},$$

for all $x, y \in X$ and $t > 0$, then there exists a unique quadratic mapping $\mathcal{Q} : X \longrightarrow Y$ such that

$$\mathcal{P}(g(x) - \mathcal{Q}(x), t) \geq_{\mathcal{L}} \mathcal{T}_{j=1}^{\infty} M\left(x, \frac{\alpha^{j+1} t}{|2|^{kj}}\right) \tag{3.6}$$

for all $x \in X$ and $t > 0$, where $g(x) = f(2x) - 16f(x)$ and

$$M(x, t) := \mathcal{T}(\Psi(x, x, n^4 t), \Psi(2x, 2x, n^4 t), \cdots, \Psi(2^{k-1}x, 2^{k-1}x, n^4 t)). \tag{3.7}$$

*Proof.* Firstly, we show, by induction on $j$ that, for all $x \in X, t > 0$ and $j \geq 1$,

$$\mathcal{P}(g(2^j x) - 4^j g(x), t) \geq_{\mathcal{L}} M_j(x, t) = \mathcal{T}(\Psi(x, x, n^4 t), \cdots, \Psi(2^{j-1}x, 2^{j-1}x, n^4 t)) \tag{3.8}$$

for all $x \in X$ and $t > 0$. Replacing $y$ by $x$ and $m$ by $n$ in (3.4), we have

$$\mathcal{P}(f(2nx) - n^4 f(2x) - 4f(nx) + 4n^4 f(x), t) \geq_{\mathcal{L}} \Psi(x, x, t) \tag{3.9}$$

for all $x \in X$ and $t > 0$. Substituting $x$ by $2x$ in (3.9), we obtain

$$\mathcal{P}(f(4nx) - n^4 f(4x) - 4f(2nx) + 4n^4 f(2x), t) \geq_{\mathcal{L}} \Psi\left(x, x, \frac{t}{2}\right) \tag{3.10}$$

for all $x \in X$ and $t > 0$. Adding the equations (3.9) and (3.10), we get

$$\begin{aligned}\mathcal{P}\big(f(2nx) + f(4nx) + 3n^4 f(2x) - 4f(nx) + 4n^4 f(x)\\ - n^4 f(4x) - 4f(2nx), t\big) \geq_{\mathcal{L}} \mathcal{T}\left(\Psi(x,x,t), \Psi\left(x,x,\frac{t}{2}\right)\right)\end{aligned} \tag{3.11}$$

for all $x \in X$ and $t > 0$. Interchanging $y$ into $x$ and $m$ by $3n$ in (3.4), we find

$$\begin{aligned}\mathcal{P}\big(-f(4nx) - f(2nx) + 9n^4 f(2x) + 2f(nx) + 2f(3nx)\\ - 36n^4 f(x), t\big) \geq_{\mathcal{L}} \Psi(x,x,t)\end{aligned} \tag{3.12}$$

for all $x \in X$ and $t > 0$. Plugging the equations (3.11) into (3.12) to obtain

$$\begin{aligned}\mathcal{P}(12n^4 f(2x) - 2f(nx) - 32n^4 f(x) - 4f(2nx) + 2f(3nx) - n^4 f(4x), t)\\ \geq_{\mathcal{L}} \mathcal{T}\left(\Psi(x,x,t), \Psi\left(x,x,\frac{t}{2}\right)\right)\end{aligned} \tag{3.13}$$

for all $x \in X$ and $t > 0$. Replacing $y$ by $x$ and $m$ by $2n$ in (3.4), we arrive at

$$\begin{aligned}\mathcal{P}\big(-2f(3nx) + 8n^4 f(2x) + 2f(nx) + 4f(2nx) - 32n^4 f(x), t\big)\\ \geq_{\mathcal{L}} \Psi\left(x,x,\frac{t}{2}\right)\end{aligned} \tag{3.14}$$

for all $x \in X$ and $t > 0$. Adding the equations (3.13) and (3.14), we get

$$\mathcal{P}(g(2x) - 4g(x), t) \geq_{\mathcal{L}} \Psi(x,x,n^4 t) \tag{3.15}$$

for all $x \in X$ and $t > 0$, where $g(x) = f(2x) - 16f(x)$. This proves (3.8) for $j = 1$. Let (3.8) hold for some $j > 1$. Proceeding as same as from the equation (3.4) by replacing $2^j x$ instead of $y$, we have

$$\mathcal{P}(g(2^{j+1}x) - 4g(2^j x), t) \geq_{\mathcal{L}} \Psi(2^j x, 2^j x, n^4 t) \tag{3.16}$$

for all $x \in X$ and $t > 0$. Since $|2| < 1$, it follows that

$$\begin{aligned}\mathcal{P}\left(g(2^{j+1}x) - 4^{j+1}g(x), t\right) &\geq_{\mathcal{L}} \mathcal{T}\left(\mathcal{P}(g(2^{j+1}x) - 4g(2^j x), t), \mathcal{P}(4g(2^j x) - 4^{j+1}g(x), t)\right)\\ &= \mathcal{T}\left(\mathcal{P}(g(2^{j+1}x) - 4g(2^j x), t), \mathcal{P}\left(g(2^j x) - 4^j g(x), \frac{t}{|4|}\right)\right)\\ &\geq_{\mathcal{L}} \mathcal{T}\left(\mathcal{P}(g(2^{j+1}x) - 4g(2^j x), t), \mathcal{P}(g(2^j x) - 4^j g(x), t)\right)\\ &\geq_{\mathcal{L}} \mathcal{T}\left(\Psi(2^j x, 2^j x, n^4 t), M_j(x,t)\right)\\ &= M_{j+1}(x,t)\end{aligned}$$

for all $x \in X$ and $t > 0$. Thus, the relation (3.8) holds for all $j \leq 1$. In particular, we have

$$\mathcal{P}(g(2^k x) - 4^k g(x), t) \geq_{\mathcal{L}} M(x,t) \tag{3.17}$$

for all $x \in X$ and $t > 0$. Replacing $x$ by $2^{-(kn+k)}x$ in (3.17) and using the inequality (3.5), we obtain

$$\mathcal{P}\left(g\left(\frac{x}{2^{kn}}\right) - 4^k g\left(\frac{x}{2^{kn+k}}\right), t\right) \geq_{\mathcal{L}} M\left(\frac{x}{2^{kn+k}}, t\right) \geq_{\mathcal{L}} M\left(x, \alpha^{n+1}t\right)$$

for all $x \in X,\ t > 0$, $n \geq 0$, and so

$$\begin{aligned}&\mathcal{P}\left((2^{2k})^n g\left(\frac{x}{(2^k)^n}\right) - \left(2^{2k}\right)^{n+1} g\left(\frac{x}{(2^k)^{n+1}}\right), t\right)\\ &\quad\geq_{\mathcal{L}} M\left(x, \frac{\alpha^{n+1}}{|(2^{2k})^n|}t\right) \geq_{\mathcal{L}} M\left(x, \frac{\alpha^{n+1}}{|(2^k)^n|}t\right)\end{aligned}$$

for all $x \in X,\ t > 0$, $n \geq 0$. The above relation implies that

$$\begin{aligned}&\mathcal{P}\left((2^{2k})^n g\left(\frac{x}{(2^k)^n}\right) - \left(2^{2k}\right)^{n+p} g\left(\frac{x}{(2^k)^{n+p}}\right), t\right)\\ &\quad\geq_{\mathcal{L}} \mathcal{T}_{j=n}^{n+p-1} \mathcal{P}\left(\left(2^{2k}\right)^j g\left(\frac{x}{(2^k)^j}\right) - \left(2^{2k}\right)^{j+1} g\left(\frac{x}{(2^k)^{j+1}}\right), t\right)\\ &\quad\geq_{\mathcal{L}} \mathcal{T}_{j=n}^{n+p-1} M\left(x, \frac{\alpha^{j+1}}{|(2^k)^j|}t\right)\end{aligned}$$

for all $x \in X,\ t > 0$, $n \geq 0$. Since $\lim_{n\to\infty} \mathcal{T}_{j=n}^{\infty} M(x, \frac{\alpha^{j+1}}{|(2^k)^j|}t) = 1_{\mathcal{L}}$, the sequence $\left\{\left(2^{2k}\right)^n g(\frac{x}{(2^k)^n})\right\}$ is Cauchy in the non-Archimedean $\mathcal{L}$-fuzzy Banach space $(Y, \mathcal{P}, \mathcal{T})$. Hence, we can define a mapping $\mathcal{Q} : X \longrightarrow Y$ such that

$$\lim_{n\to\infty} \mathcal{P}\left((2^{2k})^n g\left(\frac{x}{(2^k)^n}\right) - \mathcal{Q}(x), t\right) = 1_{\mathcal{L}}, \quad \forall x \in X, t > 0. \tag{3.18}$$

Next, for all $n \geq 1, x \in X$ and $t > 0$, we have

$$\begin{aligned}\mathcal{P}\left(g(x) - \left(2^{2k}\right)^n g(\frac{x}{(2^k)^n}), t\right) &= \mathcal{P}\left(\sum_{i=0}^{n-1} \left(2^{2k}\right)^j g\left(\frac{x}{(2^k)^i}\right) - \left(2^{2k}\right)^{i+1} g\left(\frac{x}{(2^k)^{i+1}}\right), t\right)\\ &\geq_{\mathcal{L}} \mathcal{T}_{i=0}^{n-1}\left(\mathcal{P}\left(\left(2^{2k}\right)^i g\left(\frac{x}{(2^k)^i}\right) - \left(2^{2k}\right)^{i+1} g\left(\frac{x}{(2^k)^{i+1}}\right), t\right)\right)\\ &\geq_{\mathcal{L}} \mathcal{T}_{i=0}^{n-1} M\left(x, \frac{\alpha^{i+1}t}{|2^k|^i}\right)\end{aligned}$$

and so

$$\begin{aligned}&\mathcal{P}(g(x) - \mathcal{Q}(x), t)\\ &\quad\geq_{\mathcal{L}} \mathcal{T}\left(\mathcal{P}\left(g(x) - \left(2^{2k}\right)^n g\left(\frac{x}{(2^k)^n}\right), t\right), \mathcal{P}\left(\left(2^{2k}\right)^n g\left(\frac{x}{(2^k)^n}\right) - \mathcal{Q}(x), t\right)\right)\\ &\quad= \mathcal{P}\left(\mathcal{T}_{i=0}^{n-1} M\left(x, \frac{\alpha^{i+1}t}{|2^k|^i}\right), \mathcal{P}\left(\left(2^{2k}\right)^n g\left(\frac{x}{(2^k)^n}\right) - \mathcal{Q}(x), t\right)\right)\end{aligned} \tag{3.19}$$

for all $x \in X$ and $t > 0$. Taking the limit as $n \to \infty$ in (3.19), we obtain

$$\mathcal{P}(g(x) - \mathcal{Q}(x), t) \geq \mathcal{T}_{i=0}^{\infty} M\left(x, \frac{\alpha^{i+1}t}{|2^k|^i}\right)$$

which proves (3.6). Replacing $x, y$ by $2^{-kn}x, 2^{-kn}y$, respectively in equations (3.4) and (3.5), we get

$$\begin{aligned}
&\mathcal{P}\left(2^{2kn}\Delta_{m,n}g(2^{-kn}x, 2^{-kn}y), t\right)\\
&\quad = \mathcal{T}\left(\mathcal{P}\left(2^{2kn}\Delta_{m,n}f(2^{-(kn-1)}x, 2^{-(kn-1)}y), t\right), \mathcal{P}\left(2^{2kn}(-16)\Delta_{m,n}f(2^{-kn}x, 2^{-kn}y), t\right)\right)\\
&\quad \geq_{\mathcal{L}} \mathcal{T}\left(\Psi\left(2^{-(kn-1)}x, 2^{-(kn-1)}y, \frac{t}{|2^{2k}|^n}\right), \Psi\left(2^{-kn}x, 2^{-kn}y, \frac{t}{16\,|2^{2k}|^n}\right)\right)\\
&\quad \geq_{\mathcal{L}} \mathcal{T}\left(\Psi\left(2^{-kn}x, 2^{-kn}y, \frac{t}{2\,|2^{2k}|^n}\right), \Psi\left(2^{-kn}x, 2^{-kn}y, \frac{t}{16\,|2^{2k}|^n}\right)\right)\\
&\quad \geq_{\mathcal{L}} \Psi\left(x, y, \frac{\alpha^n t}{|2^k|^n}\right)
\end{aligned}$$

for all $x, y \in X$ and $t > 0$. Since $\lim_{n\to\infty} \Psi(x, y, \frac{\alpha^n}{|(2^k)^n|}t) = 1_{\mathcal{L}}$, we infer that $\mathcal{Q}$ is a quadratic mapping. For the uniqueness of $\mathcal{Q}$, let $\mathcal{Q}' : X \longrightarrow Y$ be another quadratic mapping such that

$$\mathcal{P}(\mathcal{Q}'(x) - g(x), t) \geq_{\mathcal{L}} M(x, t),$$

for all $x \in X$ and $t > 0$. Then

$$\begin{aligned}
&\mathcal{P}\left(\mathcal{Q}(x) - \mathcal{Q}'(x), t\right)\\
&\quad \geq_{\mathcal{L}} \mathcal{T}\left(\mathcal{P}\left(\mathcal{Q}(x) - \left(2^{2k}\right)^n g\left(\frac{x}{(2^k)^n}\right), t\right), \mathcal{P}\left(\left(2^{2k}\right)^n g\left(\frac{x}{(2^k)^n}\right) - \mathcal{Q}'(x)\right), t\right)
\end{aligned}$$

for all $x \in X$ and $t > 0$. Now, from the equation (3.18) we conclude that $\mathcal{Q} = \mathcal{Q}'$. This completes the proof. Q.E.D.

We have the following result which is analogous to Theorem 3.3 for another case of $f$. The proof is similar but we include it for the sake of completeness.

**Theorem 3.4.** Let $f : X \longrightarrow Y$ be $\Psi$-approximately quadratic-quartic such that $f(0) = 0$. If there exists an $\alpha \in (0, \infty)$ and an integer $k \geq 2$ with $|2^k| < \alpha$ and $|2| \neq 0$ such that

$$\Psi(2^{-k}x, 2^{-k}y, t) \geq_{\mathcal{L}} \Psi(x, y, \alpha t), \tag{3.20}$$

$$\lim_{n\to\infty} \mathcal{T}_{j=n}^{\infty} M\left(x, \frac{\alpha^j t}{|2|^{kj}}\right) = 1_{\mathcal{L}}, \tag{3.21}$$

for all $x, y \in X$ and $t > 0$, then there exists a unique quartic mapping $\mathfrak{Q} : X \longrightarrow Y$ such that

$$\mathcal{P}(h(x) - \mathfrak{Q}(x), t) \geq_{\mathcal{L}} \mathcal{T}_{j=1}^{\infty} M\left(x, \frac{\alpha^{j+1}t}{|2|^{kj}}\right) \tag{3.22}$$

for all $x \in X$ and $t > 0$, where $h(x) = f(2x) - 4f(x)$ and $M(x, t)$ is defined in *(3.7)*.

*Proof.* Similar to the proof of Theorem 3.3, we wish to show that

$$\mathcal{P}(h(2^jx)-16^jh(x),t)\geq_{\mathcal{L}} M_j(x,t)=\mathcal{T}(\Psi(x,x,n^4t),...,\Psi(2^{j-1}x,2^{j-1}x,n^4t)) \tag{3.23}$$

for all $x\in X, t>0$ and $j\geq 1$. The same relations (3.9)-(3.15) in Theorem 3.3 can be repeated to obtain

$$\mathcal{P}(n^4f(4x)-20n^4f(2x)+64n^4f(x),t)\geq_{\mathcal{L}}\mathcal{T}\left(\Psi(x,x,t),\Psi\left(x,x,\frac{t}{2}\right)\right) \tag{3.24}$$

for all $x\in X$ and $t>0$. Assuming $h(x)=f(2x)-4f(x)$ in (3.24), we get

$$\mathcal{P}(h(2x)-16h(x),t)\geq_{\mathcal{L}}\Psi(x,x,n^4t) \tag{3.25}$$

for all $x\in X$ and $t>0$. This proves (3.23) for $j=1$. Let (3.23) holds for some $j>1$. Again, replacing $y$ by $2^jx$ in the equation (3.4), we get

$$\mathcal{P}(h(2^{j+1}x)-16h(2^jx),t)\geq_{\mathcal{L}}\Psi(2^jx,2^jx,n^4t) \tag{3.26}$$

for all $x\in X$ and $t>0$. It follows from $|2|<1$ that

$$\begin{aligned}
&\mathcal{P}\left(h(2^{j+1}x)-16^{j+1}h(x),t\right)\\
&\geq_{\mathcal{L}}\mathcal{T}\left(\mathcal{P}(h(2^{j+1}x)-16h(2^jx),t),\mathcal{P}(16h(2^jx)-16^{j+1}h(x),t)\right)\\
&=\mathcal{T}\left(\mathcal{P}(h(2^{j+1}x)-16h(2^jx),t),\mathcal{P}\left(h(2^jx)-16^jh(x),\frac{t}{|16|}\right)\right)\\
&\geq_{\mathcal{L}}\mathcal{T}\left(\mathcal{P}(h(2^{j+1}x)-16h(2^jx),t),\mathcal{P}(h(2^jx)-4^jh(x),t)\right)\\
&\geq_{\mathcal{L}}\mathcal{T}\left(\Psi(2^jx,2^jx,n^4t),M_j(x,t)\right)\\
&=M_{j+1}(x,t)
\end{aligned}$$

for all $x\in X$ and $t>0$. Thus, (3.23) holds for all $j\leq 1$. In particular, we have

$$\mathcal{P}(h(2^kx)-16^kh(x),t)\geq_{\mathcal{L}} M(x,t) \tag{3.27}$$

for all $x\in X$ and $t>0$. Replacing $x$ by $2^{-(kn+k)}x$ in (3.27) and applying inequality (3.20), we obtain

$$\mathcal{P}\left(h\left(\frac{x}{2^{kn}}\right)-16^kh\left(\frac{x}{2^{kn+k}}\right),t\right)\geq_{\mathcal{L}} M\left(\frac{x}{2^{kn+k}},t\right)\geq_{\mathcal{L}} M\left(x,\alpha^{n+1}t\right)$$

for all $x\in X, t>0$ and $n\geq 0$, and so

$$\begin{aligned}
&\mathcal{P}\left((2^{2k})^nh\left(\frac{x}{(2^k)^n}\right)-\left(2^{2k}\right)^{n+1}h\left(\frac{x}{(2^k)^{n+1}}\right),t\right)\\
&\quad\geq_{\mathcal{L}} M\left(x,\frac{\alpha^{n+1}}{|(2^{2k})^n|}t\right)\geq_{\mathcal{L}} M\left(x,\frac{\alpha^{n+1}}{|(2^k)^n|}t\right)
\end{aligned}$$

for all $x \in X,\ t > 0$, $n \geq 0$. Hence, it follows that

$$\begin{aligned}
&\mathcal{P}\left((2^{2k})^n h\left(\frac{x}{(2^k)^n}\right) - \left(2^{2k}\right)^{n+p} h\left(\frac{x}{(2^k)^{n+p}}\right), t\right)\\
&\quad \geq_{\mathcal{L}} \mathcal{T}_{j=n}^{n+p-1} \mathcal{P}\left(\left(2^{2k}\right)^j h\left(\frac{x}{(2^k)^j}\right) - \left(2^{2k}\right)^{j+1} h\left(\frac{x}{(2^k)^{j+1}}\right), t\right)\\
&\quad \geq_{\mathcal{L}} \mathcal{P}_{j=n}^{n+p-1} M\left(x, \frac{\alpha^{j+1}}{|(2^k)^j|} t\right)
\end{aligned}$$

for all $x \in X, t > 0$ and $n \geq 0$. The equation (3.21) shows that $\left\{\left(2^{4k}\right)^n h\left(\frac{x}{(2^k)^n}\right)\right\}$ is a Cauchy sequence in the non-Archimedean $\mathcal{L}$-fuzzy Banach space $(Y, \mathcal{P}, \mathcal{T})$ and hence it convergences to a mapping $\mathfrak{Q} : X \longrightarrow Y$ such that

$$\lim_{n\to\infty} \mathcal{P}\left((2^{4k})^n h\left(\frac{x}{(2^k)^n}\right) - \mathfrak{Q}(x), t\right) = 1_{\mathcal{L}}, \quad (x \in X, t > 0). \tag{3.28}$$

For each $n \geq 1, x \in X$ and $t > 0$, we have

$$\begin{aligned}
&\mathcal{P}\left(h(x) - \left(2^{4k}\right)^n h\left(\frac{x}{(2^k)^n}\right), t\right)\\
&\quad = \mathcal{P}\left(\sum_{j=0}^{n-1} \left(2^{4k}\right)^j h\left(\frac{x}{(2^k)^j}\right) - \left(2^{4k}\right)^{j+1} h\left(\frac{x}{(2^k)^{j+1}}\right), t\right)\\
&\quad \geq_{\mathcal{L}} \mathcal{T}_{j=0}^{n-1}\left(\mathcal{P}\left(\left(2^{4k}\right)^j h\left(\frac{x}{(2^k)^j}\right) - \left(2^{4k}\right)^{j+1} h\left(\frac{x}{(2^k)^{j+1}}\right), t\right)\right)\\
&\quad \geq_{\mathcal{L}} \mathcal{T}_{j=0}^{n-1} M\left(x, \frac{\alpha^{j+1} t}{|2^k|^j}\right)
\end{aligned}$$

and thus

$$\begin{aligned}
&\mathcal{P}(h(x) - \mathfrak{Q}(x), t)\\
&\quad \geq_{\mathcal{L}} \mathcal{T}\left(\mathcal{P}\left(h(x) - \left(2^{4k}\right)^n h\left(\frac{x}{(2^k)^n}\right), t\right), \mathcal{P}\left(\left(2^{4k}\right)^n h\left(\frac{x}{(2^k)^n}\right) - \mathfrak{Q}(x), t\right)\right)\\
&\quad = \mathcal{P}\left(\mathcal{T}_{j=0}^{n-1} M\left(x, \frac{\alpha^{j+1} t}{|2^k|^j}\right), \mathcal{P}\left(\left(2^{4k}\right)^n h\left(\frac{x}{(2^k)^n}\right) - \mathfrak{Q}(x), t\right)\right).
\end{aligned} \tag{3.29}$$

Letting the limit as $n \to \infty$ in (3.29), we obtain

$$\mathcal{P}(h(x) - \mathfrak{Q}(x), t) \geq \mathcal{T}_{j=0}^{\infty} M\left(x, \frac{\alpha^{j+1} t}{|2^k|^j}\right)$$

which proves (3.22). The proof of being quartic and unique for $\mathfrak{Q}$ is similar to the proof of Theorem 3.3.

Q.E.D.

In the upcoming result, we show that the quadratic-quartic functional equation (1.3) can be stable.

**Theorem 3.5.** Let $f : X \longrightarrow Y$ be $\Psi$-approximately quadratic-quartic such that $f(0) = 0$. If there exists an $\alpha \in (0, \infty)$ and an integer $k \geq 2$ with $|2^k| < \alpha$ and $|2| \neq 0$ such that

$$\Psi(2^{-k}x, 2^{-k}y, t) \geq_{\mathcal{L}} \Psi(x, y, \alpha t), \tag{3.30}$$

$$\lim_{n\to\infty} \mathcal{T}_{j=n}^{\infty} M\left(x, \frac{2\alpha^j t}{|2|^{kj}}\right) = 1_{\mathcal{L}},$$

for all $x, y \in X$ and $t > 0$, then there exists a unique quadratic mapping $\mathcal{Q} : X \longrightarrow Y$ and a unique quartic mapping $\mathfrak{Q} : X \longrightarrow Y$ such that

$$\begin{aligned} &\mathcal{P}(f(x) - \mathcal{Q}(x) - \mathfrak{Q}(x), t) \\ &\geq_{\mathcal{L}} \mathcal{T}\left(\mathcal{T}_{j=0}^{\infty} M\left(x, \frac{2\alpha^{j+1}t}{|2|^{kj}}\right), \mathcal{T}_{j=0}^{\infty} M\left(x, \frac{2\alpha^{j+1}t}{|2|^{kj}}\right)\right) \end{aligned} \tag{3.31}$$

for all $x \in X$ and $t > 0$, where $M(x, t)$ is defined in *(3.7)*.

*Proof.* By Theorem 3.3 and Theorem 3.4, there exists a unique quadratic function $\mathcal{Q}_0 : X \longrightarrow Y$ and a unique quartic function $\mathfrak{Q}_0 : X \longrightarrow Y$ satisfying

$$\mathcal{P}(f(2x) - 16f(x) - \mathcal{Q}_0(x), t) \geq_{\mathcal{L}} \mathcal{T}_{j=1}^{\infty} M\left(x, \frac{\alpha^{j+1}t}{|2|^{kj}}\right) \tag{3.32}$$

and

$$\mathcal{P}(f(2x) - 4f(x) - \mathfrak{Q}_0(x), t) \geq_{\mathcal{L}} \mathcal{T}_{j=1}^{\infty} M\left(x, \frac{\alpha^{j+1}t}{|2|^{kj}}\right) \tag{3.33}$$

for all $x \in X$ and $t > 0$. It follows from (3.32) and (3.33) that

$$\begin{aligned} &\mathcal{P}(f(x) - \mathcal{Q}(x) - \mathfrak{Q}(x), t) \\ &\geq_{\mathcal{L}} \mathcal{T}\left(\mathcal{T}_{j=0}^{\infty} M\left(x, \frac{2\alpha^{j+1}t}{|2|^{kj}}\right), \mathcal{T}_{j=0}^{\infty} M\left(x, \frac{2\alpha^{j+1}t}{|2|^{kj}}\right)\right) \end{aligned}$$

for all $x \in X$ and $t > 0$, where $\mathcal{Q}(x) = \frac{-1}{12}\mathcal{Q}_0(x)$ and $\mathfrak{Q}(x) = \frac{1}{12}\mathfrak{Q}_0(x)$. Q.E.D.

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