Generalized Hermite -Hadamard type integral inequalities for functions whose 3rd derivatives are s-convex

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Abstract

In this paper, we have established Hermite-Hadamard type inequalities for functions whose 3rd derivatives are s-convex depending on a parameter. These results have generalized some relationships with [4].

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1 Introduction

Definition 1.1. The function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$, is said to be convex if the following inequality holds

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$. We say that f is concave if $(-f)$ is convex.

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are very important in the literature (see, e.g.,[6],[10, p.137]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, [1, 2, 6, 7, 10]) and the references cited therein.

Definition 1.2. [3] Let s be a real numbers, $s \in (0, 1]$. A function $f : [0, \infty) \to [0, \infty)$ is said to be $s-$convex (in the second sense), or that f belongs to the class K_s^2, if f

$$f(\alpha x + (1 - \alpha)y) \leq \alpha^s f(x) + (1 - \alpha)^s f(y)$$

for all $x, y \in [0, \infty)$ and $\alpha \in [0, 1]$.

An $s-$convex function was introduced in Breckner's paper [3] and a number of properties and connections with $s-$convexity in the first sense are discussed in paper [8]. Of course, $s-$convexity means just convexity when $s = 1$.

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Lemma 1.3. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a three times differentiable function on I° with $a, b \in I$ and $a < b$. If $f''' \in L[a,b]$, then

$$\begin{aligned} & \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \frac{b-a}{12}\left[f'(b) - f'(a)\right] \\ = & \frac{(b-a)^3}{12}\int_0^1 t(1-t)(2t-1) f'''\left[tb+(1-t)a\right]dt. \qquad (1.2) \end{aligned}$$

In [4], Chun and Qi establish the following inequalities:

Theorem 1.4. Let $f : [a,b] \to \mathbb{R}$ be a three times differentiable mapping on (a,b) with $0 \le a < b$. If $|f'''|^q$ is s-convex on $[a,b]$ for same fixed $s \in (0,1]$ and $q \ge 1$, then

$$\begin{aligned} & \left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \frac{b-a}{12}\left[f'(b) - f'(a)\right]\right| \\ \le & \frac{(b-a)^3}{192}\left(\frac{2^{2-s}\left(6+s+2^{s+2}s\right)}{(s+2)(s+3)(s+4)}\left[\left|f'''(a)\right|^q + \left|f'''b\right|^q\right]\right)^{\frac{1}{q}}. \qquad (1.3) \end{aligned}$$

Theorem 1.5. Let $f : [a,b] \to \mathbb{R}$ be a three times differentiable mapping on (a,b) with $0 \le a < b$. If $|f'''|^q$ is s-convex on $[a,b]$ for same fixed $s \in (0,1]$ and $q > 1$, then

$$\begin{aligned} & \left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \frac{b-a}{12}\left[f'(b) - f'(a)\right]\right| \\ \le & \frac{(b-a)^3}{96}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2^{1-s}\left(s2^s+1\right)}{(s+1)(s+2)}\left[\left|f'''(a)\right|^q + \left|f'''b\right|^q\right]\right)^{\frac{1}{q}} \qquad (1.4) \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 1.6. Let $f : [a,b] \to \mathbb{R}$ be a three times differentiable mapping on (a,b) with $0 \le a < b$. If $|f'''|^q$ is s-convex on $[a,b]$ for same fixed $s \in (0,1]$ and $q > 1$, then

$$\begin{aligned} & \left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \frac{b-a}{12}\left[f'(b) - f'(a)\right]\right| \\ \le & \frac{(b-a)^3}{24}\left(\frac{1}{(p+1)(p+3)}\right)^{\frac{1}{p}}\left(\frac{2}{(s+2)(s+3)}\left[\left|f'''(a)\right|^q + \left|f'''b\right|^q\right]\right)^{\frac{1}{q}} \qquad (1.5) \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

For more information and recent developments on this topic, please refer to [4, 5, 9, 11, 12].

The aim of this paper is to establish generalized Hermite-Hadamard's inequalities for function whose 3rd derivatives in absolute value at certain powers are s-convex functions and these results have generalized some relationships with [4].

2 Main Results

We give a important identity for three times differentiable convex functions:

Lemma 2.1. Let $f : [a,b] \to \mathbb{R}$ be a three times differentiable mapping on (a,b) with $a < b$. If $f''' \in L[a,b]$, then the following equality holds:

$$\begin{aligned} & \frac{(1-2\lambda)^2 (b-a)}{12} \left[f'(\lambda a + (1-\lambda) b) - f'(\lambda b + (1-\lambda) a) \right] \\ & -\frac{(1-2\lambda)}{2} \left[f(\lambda a + (1-\lambda) b) + f(\lambda b + (1-\lambda) a) \right] + \frac{1}{b-a} \int_{\lambda b + (1-\lambda) a}^{\lambda a + (1-\lambda) b} f(x)dx \\ = & \frac{(1-2\lambda)^4 (b-a)^3}{12} \int_0^1 t(1-t)(2t-1) f''' \left[t(\lambda a + (1-\lambda) b) + (1-t)(\lambda b + (1-\lambda) a) \right] dt \end{aligned} \tag{2.1}$$

where $\lambda \in [0,1] \setminus \{\frac{1}{2}\}$.

Proof. It suffices to note that

$$\begin{aligned} I &= \int_0^1 t(1-t)(2t-1) f''' \left[t(\lambda a + (1-\lambda) b) + (1-t)(\lambda b + (1-\lambda) a) \right] dt \\ &= -2 \int_0^1 t^3 f''' \left[t(\lambda a + (1-\lambda) b) + (1-t)(\lambda b + (1-\lambda) a) \right] dt \\ & \quad +3 \int_0^1 t^2 f''' \left[t(\lambda a + (1-\lambda) b) + (1-t)(\lambda b + (1-\lambda) a) \right] dt \\ & \quad - \int_0^1 t f''' \left[t(\lambda a + (1-\lambda) b) + (1-t)(\lambda b + (1-\lambda) a) \right] dt \\ &= -2I_1 + 3I_2 - I_3. \end{aligned}$$

Integrating by parts

$$\begin{aligned} I_1 &= \int_0^1 t^3 f''' \left[t(\lambda a + (1-\lambda) b) + (1-t)(\lambda b + (1-\lambda) a) \right] dt \\ &= \frac{f''(\lambda a + (1-\lambda) b)}{(1-2\lambda)(b-a)} - \frac{3f'(\lambda a + (1-\lambda) b)}{(1-2\lambda)^2 (b-a)^2} \\ & \quad + \frac{6f(\lambda a + (1-\lambda) b)}{(1-2\lambda)^3 (b-a)^3} - \frac{6}{(1-2\lambda)^4 (b-a)^4} \int_{\lambda b + (1-\lambda) a}^{\lambda a + (1-\lambda) b} f(x)dx, \end{aligned}$$

similarly,

$$\begin{aligned} I_2 &= \int_0^1 t^2 f'''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt \\ &= \frac{f''(\lambda a+(1-\lambda)b)}{(1-2\lambda)(b-a)}-\frac{2f'(\lambda a+(1-\lambda)b)}{(1-2\lambda)^2(b-a)^2} \\ &\quad +\frac{2f(\lambda a+(1-\lambda)b)}{(1-2\lambda)^3(b-a)^3}-\frac{2f(\lambda b+(1-\lambda)a)}{(1-2\lambda)^3(b-a)^3} \end{aligned}$$

and

$$\begin{aligned} I_3 &= \int_0^1 t f'''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt \\ &= \frac{f''(\lambda a+(1-\lambda)b)}{(1-2\lambda)(b-a)}-\frac{f'(\lambda a+(1-\lambda)b)}{(1-2\lambda)^2(b-a)^2}+\frac{f'(\lambda b+(1-\lambda)a)}{(1-2\lambda)^2(b-a)^2}. \end{aligned}$$

Hence, we get

$$\begin{aligned} I &= \int_0^1 t(1-t)(2t-1)f'''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt \\ &= \frac{f'(\lambda a+(1-\lambda)b)-f'(\lambda b+(1-\lambda)a)}{(1-2\lambda)^2(b-a)^2} \\ &\quad -\frac{6\left[f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)\right]}{(1-2\lambda)^3(b-a)^3}+\frac{12}{(1-2\lambda)^4(b-a)^4}\int_{\lambda b+(1-\lambda)a}^{\lambda a+(1-\lambda)b} f(x)dx. \end{aligned}$$

This completes the proof. Q.E.D.

Remark 2.2. If we take $\lambda = 1$ or $\lambda = 0$ in Lemma 2.1, then the identity (2.1) reduces the identity (1.2) which is proved in [4].

Now, we state the main results:

Theorem 2.3. Let $f : [a, b] \to \mathbb{R}$ be a three times differentiable mapping on (a, b) with $a < b$. If

$|f'''|^q$ is s-convex on $[a,b]$ for same fixed $s\in(0,1]$ and $q\geq 1$, then the following inequality holds:

$$\begin{aligned}
&\left|\frac{(1-2\lambda)^2(b-a)}{12}\left[f'(\lambda a+(1-\lambda)b)-f'(\lambda b+(1-\lambda)a)\right]\right.\\
&\left.-\frac{(1-2\lambda)}{2}\left[f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)\right]+\frac{1}{b-a}\int_{\lambda b+(1-\lambda)a}^{\lambda a+(1-\lambda)b}f(x)dx\right|\\
\leq\ &\frac{(1-2\lambda)^4(b-a)^3}{192}\left(\frac{2^{2-s}\left(6+s+2^{s+2}s\right)}{(s+2)(s+3)(s+4)}\right)^{\frac{1}{q}}\\
&\times\left(|f'''(\lambda a+(1-\lambda)b)|^q+|f'''(\lambda b+(1-\lambda)a)|^q\right)^{\frac{1}{q}} \qquad (2.2)
\end{aligned}$$

where $\lambda\in[0,1]\setminus\{\frac{1}{2}\}$.

Proof. Using Lemma 2.1, s-convexity of $|f'''|^q$ and well-known Hölder's inequality, we obtain

$$\begin{aligned}
&\left|\frac{(1-2\lambda)^2(b-a)}{12}\left[f'(\lambda a+(1-\lambda)b)-f'(\lambda b+(1-\lambda)a)\right]\right.\\
&\left.-\frac{(1-2\lambda)}{2}\left[f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)\right]+\frac{1}{b-a}\int_{\lambda b+(1-\lambda)a}^{\lambda a+(1-\lambda)b}f(x)dx\right|\\
\leq\ &\frac{(1-2\lambda)^4(b-a)^3}{12}\int_0^1 t(1-t)|2t-1|\left|f'''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]\right|dt\\
\leq\ &\frac{(1-2\lambda)^4(b-a)^3}{12}\left(\int_0^1 t(1-t)|2t-1|\,dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1 t(1-t)|2t-1|\left|f'''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]\right|^q dt\right)^{\frac{1}{q}}\\
\leq\ &\frac{(1-2\lambda)^4(b-a)^3}{12}\left(\frac{1}{16}\right)^{1-\frac{1}{q}}\left(|f'''(\lambda a+(1-\lambda)b)|^q\int_0^1 t^{s+1}(1-t)|2t-1|\,dt\right.\\
&\left.+|f'''(\lambda b+(1-\lambda)a)|^q\int_0^1 t(1-t)^{s+1}|2t-1|\,dt\right)^{\frac{1}{q}}.
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(1-2\lambda)^4 (b-a)^3}{12} \left(\frac{1}{16}\right)^{1-\frac{1}{q}} \\
&\quad \times \left(\frac{6+s+2^{s+2}s}{(s+2)(s+3)(s+4)} \left[|f'''(\lambda a+(1-\lambda)b)|^q + |f'''(\lambda b+(1-\lambda)a)|^q \right] \right)^{\frac{1}{q}} \\
&= \frac{(1-2\lambda)^4 (b-a)^3}{192} \left(\frac{2^{2-s}\left(6+s+2^{s+2}s\right)}{(s+2)(s+3)(s+4)} \right)^{\frac{1}{q}} \\
&\quad \times \left(|f'''(\lambda a+(1-\lambda)b)|^q + |f'''(\lambda b+(1-\lambda)a)|^q \right)^{\frac{1}{q}}
\end{aligned}
$$

The proof of Theorem 2.3 is comleted. Q.E.D.

Remark 2.4. If we take $\lambda = 1$ or $\lambda = 0$ in Theorem 2.3, then the inequality (2.2) reduces the inequality (1.3) which is proved in [4].

Theorem 2.5. Let $f : [a,b] \to \mathbb{R}$ be a three times differentiable mapping on (a,b) with $a < b$. If $|f'''|^q$ is s-convex on $[a,b]$ for same fixed $s \in (0,1]$ and $q > 1$, then the following inequality holds:

$$
\begin{aligned}
&\left| \frac{(1-2\lambda)^2 (b-a)}{12} \left[f'(\lambda a+(1-\lambda)b) - f'(\lambda b+(1-\lambda)a) \right] \right. \\
&\left. -\frac{(1-2\lambda)}{2} \left[f(\lambda a+(1-\lambda)b) + f(\lambda b+(1-\lambda)a) \right] + \frac{1}{b-a} \int_{\lambda b+(1-\lambda)a}^{\lambda a+(1-\lambda)b} f(x)dx \right| \\
&\leq \frac{(1-2\lambda)^4 (b-a)^3}{96} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{2^{1-s}(s2^s+1)}{(s+1)(s+2)} \right)^{\frac{1}{q}} \\
&\quad \times \left(|f'''(\lambda a+(1-\lambda)b)|^q + |f'''(\lambda b+(1-\lambda)a)|^q \right)^{\frac{1}{q}} \qquad (2.3)
\end{aligned}
$$

where $\lambda \in [0,1] \setminus \{\frac{1}{2}\}$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using Lemma 2.1, s-convexity of $|f'''|^q$ and well-known Hölder's inequality, we obtain

$$
\begin{aligned}
&\left| \frac{(1-2\lambda)^2 (b-a)}{12} \left[f'(\lambda a+(1-\lambda)b) - f'(\lambda b+(1-\lambda)a) \right] \right. \\
&\left. -\frac{(1-2\lambda)}{2} \left[f(\lambda a+(1-\lambda)b) + f(\lambda b+(1-\lambda)a) \right] + \frac{1}{b-a} \int_{\lambda b+(1-\lambda)a}^{\lambda a+(1-\lambda)b} f(x)dx \right|
\end{aligned}
$$

$$\leq \frac{(1-2\lambda)^4(b-a)^3}{12}\int_0^1 t(1-t)|2t-1|\left|f'''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]\right|dt$$

$$\leq \frac{(1-2\lambda)^4(b-a)^3}{12}\left(\int_0^1 t^p(1-t)^p|2t-1|\,dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 |2t-1|\left|f'''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{(1-2\lambda)^4(b-a)^3}{12}\left(\frac{1}{2^{2p+1}(p+1)}\right)^{\frac{1}{p}}$$

$$\times\left(|f'''(\lambda a+(1-\lambda)b)|^q\int_0^1|2t-1|\,t^s dt+|f'''(\lambda b+(1-\lambda)a)|^q\int_0^1|2t-1|(1-t)^s\,dt\right)^{\frac{1}{q}}$$

$$\leq \frac{(1-2\lambda)^4(b-a)^3}{12}\left(\frac{1}{2^{2p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{s2^s+1}{2^s(s+1)(s+2)}\right)^{\frac{1}{q}}$$

$$\times\left(|f'''(\lambda a+(1-\lambda)b)|^q+|f'''(\lambda b+(1-\lambda)a)|^q\right)^{\frac{1}{q}}$$

$$\leq \frac{(1-2\lambda)^4(b-a)^3}{96}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2^{1-s}(s2^s+1)}{(s+1)(s+2)}\right)^{\frac{1}{q}}$$

$$\times\left(|f'''(\lambda a+(1-\lambda)b)|^q+|f'''(\lambda b+(1-\lambda)a)|^q\right)^{\frac{1}{q}}$$

which is the inequality (2.3). Q.E.D.

Remark 2.6. If we choose $\lambda = 0$ or $\lambda = 1$ in Theorem 2.5, then the inequality (2.3) reduces the inequality (1.4).

Theorem 2.7. Let $f : [a, b] \to \mathbb{R}$ be a three times differentiable mapping on (a, b) with $a < b$. If $|f'''|^q$ is s-convex on $[a, b]$ for same fixed $s \in (0, 1]$ and $q \geq 1$, then the following inequality holds:

$$\left|\frac{(1-2\lambda)^2(b-a)}{12}\left[f'(\lambda a+(1-\lambda)b)-f'(\lambda b+(1-\lambda)a)\right]\right.$$

$$\left.-\frac{(1-2\lambda)}{2}\left[f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)\right]+\frac{1}{b-a}\int_{\lambda b+(1-\lambda)a}^{\lambda a+(1-\lambda)b}f(x)dx\right|$$

$$\leq \frac{(1-2\lambda)^4(b-a)^3}{24}\left(\frac{1}{(p+1)(p+3)}\right)^{\frac{1}{p}}\left(\frac{2}{(s+2)(s+3)}\right)^{\frac{1}{q}}$$

$$\times\left(|f'''(\lambda a+(1-\lambda)b)|^q+|f'''(\lambda b+(1-\lambda)a)|^q\right)^{\frac{1}{q}} \tag{2.4}$$

where $\lambda \in [0,1] \setminus \{\frac{1}{2}\}$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 2.1, s-convexity of $|f'''|^q$ and well-known Hölder's inequality, we have

$$
\begin{aligned}
& \left| \frac{(1-2\lambda)^2 (b-a)}{12} \left[f'\left(\lambda a+(1-\lambda) b\right) - f'\left(\lambda b+(1-\lambda) a\right) \right] \right. \\
& \left. -\frac{(1-2\lambda)}{2} \left[f\left(\lambda a+(1-\lambda) b\right) + f\left(\lambda b+(1-\lambda) a\right) \right] + \frac{1}{b-a} \int\limits_{\lambda b+(1-\lambda)a}^{\lambda a+(1-\lambda)b} f(x)dx \right| \\
\leq\ & \frac{(1-2\lambda)^4 (b-a)^3}{12} \int\limits_0^1 t(1-t)\left|2t-1\right| \left| f'''\left[t(\lambda a+(1-\lambda) b) + (1-t)\left(\lambda b+(1-\lambda) a\right)\right] \right| dt \\
\leq\ & \frac{(1-2\lambda)^4 (b-a)^3}{12} \left(\int\limits_0^1 t(1-t)\left|2t-1\right|^p dt \right)^{\frac{1}{p}} \\
& \times \left(\int\limits_0^1 t(1-t) \left| f'''\left[t(\lambda a+(1-\lambda) b) + (1-t)\left(\lambda b+(1-\lambda) a\right)\right] \right|^q dt \right)^{\frac{1}{q}} \\
\leq\ & \frac{(1-2\lambda)^4 (b-a)^3}{12} \left(\frac{1}{2(p+1)(p+3)} \right)^{\frac{1}{p}} \\
& \times \left(\left| f'''\left(\lambda a+(1-\lambda) b\right) \right|^q \int\limits_0^1 t^{s+1}(1-t)\,dt + \left| f'''\left(\lambda b+(1-\lambda) a\right) \right|^q \int\limits_0^1 t(1-t)^{s+1} dt \right)^{\frac{1}{q}} \\
=\ & \frac{(1-2\lambda)^4 (b-a)^3}{12} \left(\frac{1}{2(p+1)(p+3)} \right)^{\frac{1}{p}} \left(\frac{1}{(s+2)(s+3)} \right)^{\frac{1}{q}} \\
& \times \left(\left| f'''\left(\lambda a+(1-\lambda) b\right) \right|^q + \left| f'''\left(\lambda b+(1-\lambda) a\right) \right|^q \right)^{\frac{1}{q}} \\
=\ & \frac{(1-2\lambda)^4 (b-a)^3}{24} \left(\frac{1}{(p+1)(p+3)} \right)^{\frac{1}{p}} \left(\frac{2}{(s+2)(s+3)} \right)^{\frac{1}{q}} \\
& \times \left(\left| f'''\left(\lambda a+(1-\lambda) b\right) \right|^q + \left| f'''\left(\lambda b+(1-\lambda) a\right) \right|^q \right)^{\frac{1}{q}}.
\end{aligned}
$$

Q.E.D.

Remark 2.8. If we take $\lambda = 0$ or $\lambda = 1$ in Theorem 2.7, then the inequality (2.4) becomes the inequality (1.5).

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