# Fixed point theorems for set valued mappings in partially ordered $G$-metric space

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**Abstract**

The notion of $(X, \preceq)$ partially ordered set is well known and its study for fixed points is well entrenched in the literature. In this manuscript, we obtain sufficient conditions for the existence of common fixed point for two set valued mappings satisfying an implicit relation in complete $G$-metric space on partially ordered set $X$.

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## 1 Introduction and Preliminaries

Unless mentioned or defined otherwise, for all terminology and notation in this paper, the reader is referred to [6, 10, 11, 15].

There are several reasons for the acceleration of interest in fixed point theory. One way to study a fixed point is through set valued maps. For such fixed point study, Nadler [11] introduced a important notion of set valued contraction and proved a set valued version of the Banach contraction principle. In a related vein, several authors studied many fixed point results for set valued contraction mappings see [8, 9, 14, 15]. Popa [12, 13] initiated the study of fixed point for mappings satisfying implicit relations satisfying $\varphi$-map. After that Berinde [5] proved some constructive fixed point theorems for almost contractions for an implicit relation, which generalize related results (see [2, 3, 6, 7, 15]).

Throughout in this paper, let $(X, G)$ be $G$-metric space, $CB(X)$ denotes the collection of all non-empty closed bounded subsets of $X$. Let $H(.,.,.)$ be the Hausdorff $G$-distance on $CB(X)$, i.e, for $A, B, C \in CB(X)$ and $x \in X$

$$D_G(A, B, C) := \inf\{G(a, b, c) : a \in A, b \in B, c \in C\}$$
$$\delta_G(A, B, C) := \sup\{G(a, b, c) : a \in A, b \in B, c \in C\}$$

and in [9] Kaewcharoen and Kaewkhao defined Hausdorff $G$-metric as,

$$H_G(A, B, C) := \max\{\sup_{x\in A} G(x, B, C), \sup_{x\in B} G(x, C, A), \sup_{x\in C} G(x, A, B)\}$$

where,

$$G(x, B, C) = d_G(x, B) + d_G(B, C) + d_G(x, C)$$
$$d_G(x, B) = \inf\{d_G(x, y) : y \in B\}$$

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$$d_G(A,B) = \inf\{d_G(a,b) : a \in A, b \in B\}.$$

Note that

$$D_G(A,B,C) \le H_G(A,B,C) \le \delta_G(A,B,C).$$

Recently, Beg and Butt [4] obtained the sufficient conditions for the existence of common fixed point of set valued mapping satisfying an implicit relation in partially ordered metric space. In this manuscript, we provide a birds eye view on [4] to prove a fixed point theorems in partially ordered complete $G$-metric spaces for a set valued mapping satisfying an implicit relations.
It is necessary to present a formidable number of definitions in order to make available the basic concepts and terminology, which will be used in sequel.

Mustafa and Sims [10] introduced more appropriate notion of generalized metric space called $G$-metric spaces as follows.

**Definition 1.1.** [10] Let $X$ be a nonempty set, and let $G : X \times X \times X \to R^+ \cup \{0\}$ be a function satisfying the following axioms:

(G1) $G(x,y,z) = 0$, if $x = y = z$;

(G2) $G(x,x,y) > 0$, for all $x,y \in X$ with $x \neq y$;

(G3) $G(x,x,y) \le$ G(x, y, z), for all $x,y,z \in X$ with z $\neq$ y;

(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x) =$ (symmetry in all three variables);

(G5) $G(x,y,z) \le G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$ (rectangle inequality).

then the function $G$ is called a generalized metric, or, more specifically a $G$-metric on $X$ and the pair $(X,G)$ is called a $G$-metric space. Clearly, these properties are satisfied when $G(x,y,z)$ is perimeter of triangle with vertices $x,y$ and $z \in R^2$.

**Example 1.2.** Let $(X,d)$ be a metric space. The function $G : X \times X \times X \to [0,\infty)$, defined by
$G(x,y,z) = \max\{d(x,y), d(y,z), d(z,x)\}$, or
$G(x,y,z) = d(x,y) + d(y,z) + d(z,x)$, for all $x,y,z \in X$, is a $G$-metric on $X$.

**Proposition 1.3.** The following useful properties of $G$-metric are readily derived from the Proposition 1 of [10], for any $x,y,z,a \in X$, it follows that:

(G1) $G(x,y,z) = 0$ if $x = y = z$;

(G2) $G(x,y,z) \le G(x,x,y) + G(x,x,z)$;

(G3) $G(x,y,y) \le 2G(y,x,x)$;

(G4) $G(x,y,z) \le G(x,a,z) + G(a,y,z)$;

(G5) $G(x,y,z) = G(x,z,y) = G(y,z,x)$;

(G6) $G(x,y,z) \le \frac{2}{3}[G(x,y,a)) + G(x,a,z) + G(a,y,z)]$;

(G7) $G(x,y,z) \le [G(x,a,a) + G(y,a,a) + G(z,a,a)]$.

**Definition 1.4.** [10] Let $(X,G)$ be a $G$-metric space, and $\{x_n\}$ is sequence of points in $X$, one says that sequence $\{x_n\}$

i) is $G$-convergent to $x$ if, for any $\varepsilon > 0$, there exists an $x \in X$ and $L \in \mathbb{N}$ such that $G(x, x_n, x_m) < \varepsilon$, for all $n, m \geq L$;

ii) is $G$-Cauchy sequence if, for any $\varepsilon > 0$, there exists $L \in N$ such that $G(x_l, x_n, x_m) < \varepsilon$, for all $n, m, l \geq L$;

iii) is $G$-complete if every $G$-Cauchy sequence in $(X,G)$ is $G$-convergent in $X$.

**Proposition 1.5.** [10] Let $(X,G)$ be a $G$-metric space. Then the following are equivalent:

i) $\{x_n\}$ is $G$-convergent to $x$;

ii) $G(x_n, x_n, x) \to 0$ as $n \to \infty$;

iii) $G(x_n, x, x) \to 0$ as n $\to \infty$;

iv) $G(x_m, x_n, x) \to 0$ as $m, n \to \infty$.

**Definition 1.6.** [10] Let $(X,G)$ be a $G$-metric space, then for $x_0 \in X, r > 0$, the $G$-ball with center $x_0$ and radius $r$ is defined as

$$B_G(x_0, r) = \{y \in X : G(x_0, y, y) < r\}.$$

**Proposition 1.7.** [10] Let $(X,G)$ be a $G$-metric space then for $x_0 \in X, r > 0$ we have

i) If $G(x_0, y, y) < r$, then $x, y \in B_G(x_0, r)$;

ii) If $y \in B_G(x_0, r)$, then there exists a $\delta > 0$ such that $B_G(y, \delta) \subseteq B_G(x_0, r)$

**Definition 1.8.** Let $R_+$ be the set of non-negative real numbers and $\tau$ the set of real-valued functions $T : R_+^6 \to R$ satisfying the following conditions:
$\tau_0 : T(\liminf_{n\to\infty} p_n) \leq \liminf_{n\to\infty} T(p_n)$ for any $p_n \in R_+^6$, where $\liminf_{n\to\infty} p_n$ means components wise $\liminf$.
$\tau_1 : T(t_1, t_2 ..., t_6)$ is non-increasing in $t_2, t_3, ..., t_6$;
$\tau_2$ : there exists a continuous strictly increasing function $\psi : R_+ \to R_+$ with $\psi(t) < t$ for $t > 0$ and $\varepsilon > 0$ such that the inequalities
$u \leq w + \varepsilon$, and
$T(w, v, v, u, u + v, 0) \leq 0$
or
$T(w, v, u, v, 0, u + v) \leq 0$ implies $w \leq \psi(v)$;
$\tau_3$: $T(w, 0, v, 0, 0, v) \leq 0$ and $T(w, 0, 0, v, v, 0) \leq 0$ implies $w \leq \psi(v)$ where $\psi$ is function in $\tau_2$.

**Example 1.9.** [1] $T(t_1, t_2 ..., t_6) = t_1 - f(\max\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\})$, where $f : R_+ \to R_+$ is a continuous strictly increasing function with $f(t) < t$ for t $> 0$. $\tau_0$ and $\tau_1$ are obvious.
$\tau_2$: Let $u > 0$, then choose $\varepsilon > 0$ so that $f(u) + \varepsilon < u$ (this is possible since $f(u) < u$). Now let $u \leq w + \varepsilon$ and $T(w, v, v, u, u+v, 0) \leq 0 = w - f(max\{u, v\}) \leq 0$. If $u \geq v$ then $u \leq w + \varepsilon \leq f(u) + \varepsilon < u$, is a contradiction. Thus $u < v$ and $w \leq f(v)$. Similarly $u \leq w + \varepsilon$ and $T(w, v, u, v, 0, u + v) \leq 0$ imply $w \leq f(v)$. If $u = 0$ then $w \leq f(v)$. Thus $\tau_2$ is satisfied.
$\tau_3$: $T(w, 0, v, 0, 0, v) = T(w, 0, 0, v, v, 0) \leq 0 = w - f(v) \leq 0$ imply $w \leq f(v) = \psi(v)$.

**Example 1.10.** [1] $T(t_1, t_2..., t_6) = t_1 - \alpha \ \max\{t_2, t_3, t_4\} - (1-\alpha)(at_5 + bt_6)$, where $0 < \alpha < 1, a \geq 0, b \leq \frac{1}{2}$.
$\tau_0$ and $\tau_1$ are obvious.
$\tau_2$: Let $u > 0$, then choose $\varepsilon > 0$ so that $\max\{\alpha + 2a(1-\alpha), \alpha + 2b(1-\alpha)\}u + \varepsilon < u$(this is possible since $0 < \max\{\alpha + 2a(1-\alpha), \alpha + 2b(1-\alpha)\} < 1$). Now let $u \leq w + \varepsilon$ and $T(w, v, v, u, u+v, 0) = w - \alpha \max\{u, v\} - (1-\alpha)a(u+v) \leq 0$. If $u \geq v$ then $u \leq w + \varepsilon \leq [\alpha + 2a(1-\alpha)]u + \varepsilon < u$, is a contradiction. Thus $u < v$ and $w \leq \max\{\alpha + 2a(1-\alpha), \alpha + 2b(1-\alpha)\}v$. Similarly, $u \leq w + \varepsilon$ and $T(w, v, u, v, 0, u+v) \leq 0$ imply $w \leq \max\{\alpha + 2a(1-\alpha), \alpha + 2b(1-\alpha)\}v$. Thus, $\tau_2$ is satisfied.
$\tau_3 : T(w, 0, v, 0, 0, v) = w - \alpha v - (1-\alpha)bv \leq 0$ imply $w \leq [\alpha + b(1-\alpha)]v \leq \psi(v)$ and $T(w, 0, 0, v, v, 0) = w - \alpha v - (1-\alpha)av \leq 0$ implies $w \leq [\alpha + a(1-\alpha)]v \leq \psi(v)$.

Now we begin this section with the following theorem that gives the existence of fixed point in partially ordered $G$-metric space $X$ for a set valued mapping.

## 2 Main Theorems

**Theorem 2.1.** Let $(X, \preceq)$ be a partially ordered set and $G$ be a complete $G$-metric on $X$. Let $x_0 \in X$, $r > 0$ and $\acute{F}, \acute{G} : B_G(x_0, r) \to C(X)$. Suppose that for all $x, y \in B_G(x_0, r)$, $\acute{F}x$ and $\acute{G}y$ are bounded and satisfying
$T[H_G(\acute{F}x, \acute{G}y, \acute{G}y), G(x, y, y),$

$$G(x, \acute{F}x, \acute{F}x), G(y, \acute{G}y, \acute{G}y), G(x, \acute{G}y, \acute{G}y), G(y, \acute{F}x, \acute{F}x)] \leq 0 \tag{2.1}$$

for all comparable elements $x, y$ of $X$ and some $T \in \tau$. Also assume that the following conditions are satisfied:

1. For each $x \in X$, there exists $y \in \pounds x$ with $x \preceq y$ such that $G(x, y, y) \leq G(x, \pounds x, \pounds x) + \varepsilon$ for $\pounds \in \{\acute{F}, \acute{G}\}$.
2. $G(x_0, x_1, x_1) < r - \psi(r)$ for some $x_1 \in \acute{F}x_0$ with $x_0 \preceq x_1$ and $\psi$ is function defined in $\tau_2$.
3. $\sum_{i=1}^{\infty} \psi^i(r - \psi(r)) \leq \psi(r)$, where $\psi$ is function defined in $\tau_2$.
4. If $x_n \to x$ is a sequence in $B_G(x_0, r)$, whose consecutive terms are comparable, then $x_n \preceq x$ for all $n$. Then there exists $x \in B_G(x_0, r)$ with $x \in \acute{F}x \cap \acute{G}x$.

**Proof:** Using condition 2, one can choose $x_1 \in \acute{F}x_0$ with $x_0 \preceq x_1$ such that

$$G(x_0, x_1, x_1) < r - \psi(r) \tag{2.2}$$

Since $r - \psi(r) < r$, $\{G(x_0, x_1, x_1) < r\}$, now in view of equation (2.2), we have $x_1 \in B_G(x_0, r)$. Since $\psi$ is strictly increasing function, therefore we can choose $\varepsilon > 0$, such that

$$\psi(G(x_0, x_1, x_1)) + \varepsilon < \psi(r - \psi(r)) \tag{2.3}$$

Again by condition 1, there exists $x_2 \in \acute{G}x_1$ with $x_1 \preceq x_2$ s.t.

$$G(x_1, x_2, x_2) \leq G(x_1, \acute{G}x_1, \acute{G}x_1) + \varepsilon \leq H_G(\acute{F}x_0, \acute{G}x_1, \acute{G}x_1) + \varepsilon \tag{2.4}$$

Also, $x_0, x_1 \in B_G(x_0, r)$ therefore inequality (2.1) gives
$T[H_G(\acute{F}x_0, \acute{G}x_1, \acute{G}x_1), G(x_0, x_1, x_1), G(x_0, \acute{F}x_0, \acute{F}x_0), G(x_1, \acute{G}x_1, \acute{G}x_1),$

$G(x_0, \acute{G}x_1, \acute{G}x_1), G(x_1, \acute{F}x_0, \acute{F}x_0)] \leq 0$
from $\tau_1$, we get

$$T[H_G(\acute{F}x_0, \acute{G}x_1, \acute{G}x_1), G(x_0, x_1, x_1), G(x_0, x_1, x_1), G(x_1, x_2, x_2),$$

$$G(x_0, x_1, x_1) + G(x_1, x_2, x_2), 0] \leq 0,$$

which can be written as

$$T(w, v, v, u, u + v, 0) \leq 0,$$

where $w = H_G(\acute{F}x_0, \acute{G}x_1, \acute{G}x_1), u = G(x_1, x_2, x_2), v = G(x_0, x_1, x_1)$,
by $\tau_2$, we have $w \leq \psi(v)$ i.e.,

$$H_G(\acute{F}x_0, \acute{G}x_1, \acute{G}x_1) \leq \psi(G(x_0, x_1, x_1)) \tag{2.5}$$

Using (2.4), (2.5) together with (2.3), we have

$$G(x_1, x_2, x_2) \leq \psi(G(x_0, x_1, x_1)) + \varepsilon < \psi(r - \psi(r)).$$

Thus,

$$G(x_0, x_2, x_2) \leq G(x_0, x_1, x_1) + G(x_1, x_2, x_2),$$

by condition (3), which becomes

$$< r - \psi(r) + \psi(r - \psi(r))$$

$$< r - \psi(r) + \psi(r) = r.$$

$$\Rightarrow x_2 \in B_G(x_0, r).$$

Again choose $\delta > 0$, such that

$$\psi(G(x_1, x_2, x_2)) + \delta < \psi^2(r - \psi(r)). \tag{2.6}$$

On applying condition 2, we found that there exists a $x_3 \in \acute{F}x_2$ with $x_2 \preceq x_3$ such that

$$G(x_2, x_3, x_3) \leq G(x_2, \acute{F}x_2, \acute{F}x_2) + \delta < H_G(\acute{G}x_1, \acute{F}x_2, \acute{F}x_2) + \delta \tag{2.7}$$

Using 2.1 together with the fact that $x_1, x_2 \in B_G(x_0, r)$ and $x_1 \preceq x_2$ we get

$$T[H_G(\acute{F}x_2, \acute{G}x_1, \acute{G}x_1), G(x_2, x_1, x_1), G(x_2, \acute{F}x_2, \acute{F}x_2), G(x_1, \acute{G}x_1, \acute{G}x_1),$$

$$G(x_2, \acute{G}x_1, \acute{G}x_1), G(x_1, \acute{F}x_2, \acute{F}x_2)] \leq 0$$

from $\tau_1$, we obtain

$$T[H_G(\acute{F}x_2, \acute{G}x_1, \acute{G}x_1), G(x_1, x_2, x_2), G(x_2, x_3, x_3), G(x_1, x_2, x_2), 0,$$

$$G(x_1, x_2, x_2) + G(x_2, x_3, x_3)] \leq 0$$

which reduces to

$$T(w, v, u, v, 0, u + v) \leq 0,$$

where

$$w = H_G(\acute{F}x_2, \acute{G}x_1, \acute{G}x_1), u = G(x_2, x_3, x_3), v = G(x_1, x_2, x_2),$$

using $\tau_2$, clearly, $w \leq \psi(v)$ which gives

$$H_G(\acute{F}x_2, \acute{G}x_1, \acute{G}x_1) \leq \psi(G(x_1, x_2, x_2)) \tag{2.8}$$

On using (2.7), (2.8) and (2.6), we get

$$G(x_2, x_3, x_3) \leq \psi(G(x_1, x_2, x_2)) + \delta < \psi^2(r - \psi(r)).$$

Also from condition 3

$$\begin{aligned} G(x_0, x_3, x_3) &\leq G(x_0, x_1, x_1) + G(x_1, x_2, x_2) + G(x_2, x_3, x_3) \\ &< r - \psi(r) + \psi(r - \psi(r)) + \psi^2(r - \psi(r)) \\ &\leq r - \psi(r) + \sum_{i=1}^{\infty} \psi^i(r - \psi(r)) \\ &\leq r - \psi(r) + \psi(r) = r. \\ &\Rightarrow x_3 \in B_G(x_0, r) \end{aligned}$$

Proceeding in this manner, we obtain a sequence $\{x_n\} \subseteq B_G(x_0, r)$ with $x_n \preceq x_{n+1}$ such that $x_{2n+2} \in \acute{G}x_{2n+1}$ and $x_{2n+1} \in \acute{F}x_{2n}$ for $n \geq 0$ and

$$G(x_n, x_{n+1}, x_{n+1}) < \psi^n(r - \psi(r)) \tag{2.9}$$

using condition 3, and equation (2.9) we found that $\{x_n\}$ is Cauchy sequence. Now as $X$ complete $G$-metric, there exists some point $x \in B_G(x_0, r)$ such that $x_n \to x$ and applying condition 4, we get $x_n \preceq x$ for all $n$.

Now it remains to show $x \in \acute{F}x \cap \acute{G}x$, for this, we consider two cases for $'n'$

CASE 1: - $n$ is even

Since $x_n, x \in B_G(x_0, r)$ and $x_n \preceq x$, hence by equation 2.1, we have

$$T[H_G(\acute{F}x, \acute{G}x_{n-1}, \acute{G}x_{n-1}), G(x, x_{n-1}, x_{n-1}), G(x, \acute{F}x, \acute{F}x), G(x_{n-1}, \acute{G}x_{n-1}, \acute{G}x_{n-1}),$$

$$G(x, \acute{G}x_{n-1}, \acute{G}x_{n-1}), G(x_{n-1}, \acute{F}x, \acute{F}x)] \leq 0$$

Taking limit inferior as $n \to \infty$ and using $\tau_2$, together with the

$$G(x, \acute{G}x_{n-1}, \acute{G}x_{n-1}) \leq G(x, x_n, x_n) \to 0$$

$$G(x_{n-1}, \acute{G}x_{n-1}, \acute{G}x_{n-1}) \leq G(x_{n-1}, x_n, x_n) \to 0$$

and

$$G(x_n, \acute{G}x_{n-1}, \acute{G}x_{n-1}) \to 0.$$

Thus the expression turns out to be

$$T[\lim_{n\to\infty} \inf H_G(\acute{F}x, \acute{G}x_{n-1}, \acute{G}x_{n-1}), 0, G(x, \acute{F}x, \acute{F}x), 0, 0, G(x_{n-1}, x, x)$$

$$+G(x,\acute{F}x,\acute{F}x)] \le 0$$

.
implies

$$T[\lim_{n\to\infty}\inf H_G(\acute{F}x,\acute{G}x_{n-1},\acute{G}x_{n-1}),0,G(x,\acute{F}x,\acute{F}x),0,0,G(x,\acute{F}x,\acute{F}x)] \le 0.$$

Now using the condition 3, we get

$$\lim_{n\to\infty}\inf H_G(\acute{F}x,\acute{G}x_{n-1},\acute{G}x_{n-1}) \le \psi(G(x,\acute{F}x,\acute{F}x)).$$

Consider

$$\begin{aligned} G(x,\acute{F}x,\acute{F}x) &\le G(x,x_n,x_n)+G(x_n,\acute{F}x,\acute{F}x) \\ &\le G(x,x_n,x_n)+H_G(\acute{G}x_{n-1},\acute{F}x,\acute{F}x) \end{aligned}$$

Now taking limit inferior $n\to\infty$, in this inequality we get,

$$G(x,\acute{F}x,\acute{F}x) \le 0+\lim_{n\to\infty}\ inf\ H_G(\acute{G}x_{n-1},\acute{F}x,\acute{F}x) \le \psi(G(x,\acute{F}x,\acute{F}x))$$

$$\Rightarrow G(x,x,\acute{F}x) \le \psi(G(x,\acute{F}x,\acute{F}x)).$$

Also $\psi(t) < t$ for $t > 0$, which gives

$$G(x,\acute{F}x,\acute{F}x) = 0.$$

$$\Rightarrow x \in \acute{F}x. \tag{2.10}$$

Case II: - $n$ is odd

$$\begin{aligned} T[H_G(\acute{F}x_{n-1},\acute{G}x,\acute{G}x),G(x_{n-1},x,x),G(x_{n-1},\acute{F}x_{n-1},\acute{F}x_{n-1}),G(x,\acute{G}x,\acute{G}x), \\ G(x_{n-1},\acute{G}x,\acute{G}x),G(x,\acute{F}x_{n-1},\acute{F}x_{n-1})] \le 0 \end{aligned}$$

Proceeding in the similar vein, as in (case I) one can get,

$$\lim_{n\to\infty}\inf H_G(\acute{G}x,\acute{F}x_{n-1},\acute{F}x_{n-1}) \le \psi(G(x,\acute{G}x,\acute{G}x))$$

and therefore

$$\begin{aligned} G(x,\acute{G}x,\acute{G}x) &\le G(x,x_n,x_n)+G(\acute{F}x_{n-1},\acute{G}x,\acute{G}x) \\ &\le G(x,x_n,x_n)+H_G(\acute{G}x,\acute{F}x_{n-1},\acute{F}x_{n-1}) \\ &\le 0+\lim_{n\to\infty} inf H_G(\acute{G}x,\acute{F}x_{n-1},\acute{F}x_{n-1}) \end{aligned}$$

$$\Rightarrow\ G(x,\acute{G}x,\acute{G}x) \le H_G(\acute{G}x,\acute{F}x_{n-1},\acute{F}x_{n-1}) \le \psi(G(x,\acute{G}x,\acute{G}x))$$

$$\Rightarrow\ G(x,\acute{G}x,\acute{G}x) \le \psi(G(x,\acute{G}x,\acute{G}x)) \Rightarrow G(x,\acute{G}x,\acute{G}x) = 0$$

$$\Rightarrow x \in \acute{G}x \tag{2.11}$$

Hence, from (2.10) and (2.11), we conclude that

$$x \in \acute{F}x \cap \acute{G}x.$$

Hence the result.

The extension of the above Theorem 2.1 is given as:

**Theorem 2.2.** Let $(X, \preceq)$ be a partially ordered set and $G$ be a complete $G$-metric on $X$. Let $\acute{F}, \acute{G} : X \to C(X)$ be such that $\acute{F}x, \acute{G}y$ are bounded and satisfying the implicit relation 2.1 of Theorem 2.1 for all comparable element $x, y$ of $X$ and some $T \in \tau$. Also assume that the following conditions are satisfied:

1. For each $x \in X$, there exists $y \in \pounds x$ with $x \preceq y$ such that $G(x, y, y) \leq G(x, \pounds x, \pounds x) + \varepsilon$ for $\pounds \in \{\acute{F}, \acute{G}\}$.
2. there exist $r > 0$ such that
$$\sum_{i=1}^{\infty} \psi^i(r - \psi(r)) < \infty, where\ \psi\ isfunction\ defined\ in\ \tau_2.$$
3. If $x_n \to x$ is sequence in $X$, whose consecutive terms are comparable, then $x_n \preceq x$ for all $n$. Then there exists $x \in X$ with $x \in \acute{F}x \cap \acute{G}x$.

**Proof:** Proceeding in similar vein, as Theorem 2.1, first, we show that $\inf_{x \in X}(G(x, \acute{F}x, \acute{F}x)) = 0$ or $\inf_{x \in X}(G(x, \acute{G}x, \acute{G}x)) = 0$.
We shall show this result by contradiction.
Suppose $\inf_{x \in X}(G(x, \acute{F}x, \acute{F}x)) = \delta_1 > 0$ and $\inf_{x \in X}(G(x, \acute{G}x, \acute{G}x)) = \delta_2 > 0$. On the contrary, suppose that
$$\delta_1 \leq \delta_2.$$
Using the continuity of $\psi$, and $\psi(\delta_1) < \delta_1$ one can observe that there exists $\varepsilon > 0$ such that
$$\psi(t) < \delta_1,\ for\ t \in [\delta_1, \delta_1 + \varepsilon). \tag{2.12}$$
Choose $x \in X$ such that
$G(x, \acute{F}x, \acute{F}x) \geq \delta_1$ and $G(x, Px, Px) < \delta_1 + \varepsilon$. Now, by using the condition 1 there exist $y \in Px$ with $x \preceq y$, such that
$$\delta_1 \leq G(x, y, y) < \delta_1 + \varepsilon \tag{2.13}$$
In view of equation (2.1) of Theorem 2.1, we get
$$T[H_G(\acute{F}x, \acute{G}y, \acute{G}y), G(x, y, y), G(x, \acute{F}x, \acute{F}x), G(y, \acute{G}y, \acute{G}y),$$
$$G(x, \acute{G}y, \acute{G}y), G(y, \acute{F}x, \acute{F}x)] \leq 0.$$

From $\tau_1$ together with the facts that
$$G(x, \acute{F}x, \acute{F}x) \leq G(x, y, y) + G(y, \acute{F}x, \acute{F}x)$$
and
$$G(y, \acute{G}y, \acute{G}y) \leq G(x, y, y) + G(y, \acute{G}y, \acute{G}y)$$
thus
$$T[H_G(\acute{F}x, \acute{G}y, \acute{G}y), G(x, y, y), G(y, \acute{G}y, \acute{G}y), G(x, y, y) + G(y, \acute{G}y, \acute{G}y), 0] \leq 0.$$

i.e.,

$$T(w,v,v,u,u+v,0) \leq 0,$$

where

$$w = H_G(\acute{F}x, \acute{G}y, \acute{G}y), u = G(x, \acute{G}y, \acute{G}y), v = G(x,y,y),$$

from $\tau_2$, we have $w \leq \psi(v)$ this implies

$$H_G(\acute{F}x, \acute{G}y, \acute{G}y) \leq \psi(G(x,y,y)) \tag{2.14}$$

Since $G(y, \acute{G}y, \acute{G}y) \leq H_G(\acute{F}x, \acute{G}y, \acute{G}y)$ now from equation (2.12), (2.11) and (2.10), we get

$$G(y, \acute{G}y, \acute{G}y) \leq \psi(G(x,y,y) < \delta_1,$$

a contradiction to our assumption as $inf_{x\in X}(G(x, \acute{G}x, \acute{G}x)) = \delta_2 > \delta_1$,
Thus, $\quad inf_{x\in X}(G(x, \acute{F}x, \acute{F}x)) = 0$
implies there exists $x_0 \in X$ with $G(x_0, \acute{F}x_0, \acute{F}x_0) < r - \psi(r)$ and hence we have the existence of $x_1 \in \acute{F}x_0$ with $x_0 \preceq x_1$ such that $G(x_0, x_1, x_1) < r - \psi(r)$. Now following Theorem 2.1, we have the existence of sequences $x_{2n+2} \in Gx_{2n+1}$ and $x_{2n+1} \in \acute{F}x_{2n}$, for $n \geq 0$ with $x_n \preceq x_{n+1}$ such that

$$G(x_n, x_{n+1}, x_{n+1}) < \psi^n(r - \psi(r)),$$

Now with the help of condition 2, we found that $\{x_n\}$ is Cauchy sequence and hence convergent to some point $x$ in complete $G$-metric space $X$. The rest of the proof can be given by argument analogously to those used in Theorem 2.1, which gives that

$$x \in \acute{F}x \cap \acute{G}x.$$

Thus, the result follows.

It is also notice that in Theorem 1.3 [1] assumed the implicit relation for the comparable elements of partially ordered $G$-metric space.

**Corollary 2.3.** Let $(X, G)$ be a complete $G$-metric space with partial ordered set $(X, \preceq)$. Let $x_0 \in X, r > 0$ and $\acute{F} : B(x_0, r) \to C(X)$. Suppose that for all $x, y \in B(x_0, r)$, $\acute{F}x$ is bounded and satisfying
$T[H_G(\acute{F}x, \acute{F}y, \acute{F}y), G(x,y,y),$

$$G(x, \acute{F}x, \acute{F}x), G(y, \acute{F}y, \acute{F}y), G(x, \acute{F}y, \acute{F}y), G(y, \acute{F}x, \acute{F}x)] \leq 0 \tag{2.15}$$

for all comparable elements $x, y$ of $X$ and some $T \in \tau$. Also assume that the following conditions are satisfied:-

1. for each $x \in X$, there exists $y \in \acute{F}x$ with $x \preceq y$ such that $G(x,y,y) \leq G(x, \acute{F}x, \acute{F}x) + \varepsilon$;
2. $G(x_0, x_1, x_1) < r - \psi(r)$ for some $x \in \acute{F}x_0$ with $x_0 \preceq x_1$;
3. $\sum_{i=1}^{\infty} \psi^i(r - \psi(r)) \leq \psi(r)$;

4. If $x_n \to x$ is a sequence whose consecutive terms are comparable, then $x_n \preceq x$, for all $n$.

Then there exist $x \in B(x_0, r)$ with $x \in \acute{F}x$.

**Remark 2.4.** In assumption 1, 2 and 4 of Theorem 2.1, we need only comparability of the elements and there is no need of monotonicity of the terms of the sequence.

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