The method of infinite ascent applied on $2^p A^6 + B^3 = C^2$

Susil Kumar Jena *

Department of Electronics & Telecommunication Engineering, KIIT University, Bhubaneswar 751024, Odisha, India
E-mail: susil_kumar@yahoo.co.uk

Abstract

In this paper, we prove that for any positive integer p, when $p \equiv 1 \pmod 6$ or, $p \equiv 3 \pmod 6$, the Diophantine equation: $2^p A^6 + B^3 = C^2$ has infinitely many co-prime integral solutions A, B, C. When $p = 0$, this equation has only four integral solutions with $(A, B, C) = (\pm 1, 2, \pm 3)$. For other integer values of p, the problem is open.

2000 Mathematics Subject Classification. **11D41**. 11D72.
Keywords. Diophantine equation $2^p A^6 + B^3 = C^2$, method of infinite ascent, higher order Diophantine equation.

1. Introduction

In the VII$^{\text{th}}$ Joint Meeting of the American Mathematical Society and the Sociedad Matematica Mexicana held in Zacatecas, Mexico, during May 23-26, 2007, in a talk titled : *Method of infinite ascent applied on* $A^6 + nB^3 = C^2$, I introduced a method called the *method of infinite ascent* for regenerating an infinite number of co-prime integral solutions A, B, C for a class of integers n. In a recent paper[1], I generalized this result. In another paper[2], this method has been used non-recursively to find infinitely many co-prime integral solutions A, B, C for all positive integral values of n. Other papers which use this new method include [3], [4], [5], [6] and [7]. We quote a paragraph from a paper of Jena[1] to explain the Method of Infinite Ascent (MIA) in brief.

"Sometimes a Diophantine equation possessing an infinite number of integral solutions does not exhibit this infinitude characteristics as seen in its original form. Putting into a slightly modified form - which we need to discover - this equation becomes regenerative, so that any set of solution for the equation will lead to the next set of solution for the same; the first set leading to the second, the second set leading to the third and so on without end. This is a regeneration technique which we wish to call the Method of Infinite Ascent (MIA), explicitly showing on how to generate the endless set of integral solutions for the Diophantine equation."

The main results of this paper are the following two theorems which are related to the Diophantine equation

$$mA^6 + nB^3 = C^2. \tag{1.1}$$

Theorem 1.1. For any positive integer q, the Diophantine equation

$$2^{6q-5} A^6 + B^3 = C^2 \tag{1.2}$$

has infinitely many co-prime integral solutions A, B, C.

*Thankful to my parents - my heavenly mother and revered father, for their subtle support.

Tbilisi Mathematical Journal 7(1) (2014), pp. 31–36.
Tbilisi Centre for Mathematical Sciences.

Received by the editors: 24 March 2013.
Accepted for publication: 15 May 2014.

Theorem 1.2. For any positive integer q, the Diophantine equation

$$2^{6q-3}A^6 + B^3 = C^2 \tag{1.3}$$

has infinitely many co-prime integral solutions A, B, C.

We will prove these two theorems in the next section after stating two lemmas. Finally, in the conclusion, we make some remarks about Mordell-Weil rank of elliptic curves, and the viewpoint of the theory of elliptic curves on the Diophantine equation of the title.

2. Main Results

We need the following two lemmas for proving Theorem 1.1 and Theorem 1.2.

Lemma (Jena [1]) 2.1. For any integer m, p and q,

$$m(2pq)^6 + (mp^6 - q^2)(9mp^6 - q^2)^3 = (27m^2p^{12} - 18mp^6q^2 - q^4)^2. \tag{2.1}$$

Lemma (Jena [1]) 2.2. If (A_t, B_t, C_t) is a solution of the Diophantine equation $mA^6 + nB^3 = C^2$ with m, n, A, B and C as integers then, $(A_{t+1}, B_{t+1}, C_{t+1})$ is also a solution of the same equation such that

$$(A_{t+1}, B_{t+1}, C_{t+1}) = \{(2A_tC_t), -B_t(9mA_t^6 - C_t^2), (27m^2A_t^{12} - 18mA_t^6C_t^2 - C_t^4)\} \tag{2.2}$$

and if mA_t, nB_t and C_t are pair-wise co-prime where nB_t is an odd integer and 3 is not a factor of C_t, then mA_{t+1},nB_{t+1} and C_{t+1} are also pair-wise co-prime where nB_{t+1} is an odd integer and 3 is not a factor of C_{t+1}. In addition to this, mA_{t+1} will be always an even integer and C_{t+1}, always an odd integer.

We need not prove Lemma 2.1 and Lemma 2.2 as they appear as *Theorem* 2.1 and *Theorem* 3.1 in paper [1].

2.1. Proof of Theorem 1.1

We will prove Theorem 1.1 in three steps. First, we have to establish that equation (1.2) has infinitely many co-prime integral solutions A, B, C when $q = 1$. Then, we will see how to use these co-prime solutions of first step to find the initial co-prime solutions A, B, C of equation (1.2) for other values of $q > 1$. Next, we will show that the conditions of generating infinite number of co-prime integral solutions, as proposed by Lemma 2.2, are applicable to (1.2) for each value of q.

Step I. Putting $q = 1$ in (1.2) we get

$$2^1A^6 + B^3 = C^2. \tag{2.3}$$

We will denote the i^{th} solution for (A, B, C) of (1.2) when $q = j$ as $(A_i, B_i, C_i)_{q=j}$ where i and j take positive integral values. Now, we know

$$2^7 + 17^3 = 71^2; \text{Or}, 2 \cdot 2^6 + 17^3 = 71^2. \tag{2.4}$$

Using the result of (2.4), we get the starting solution for (A, B, C) of equation (1.2) as

$$(A_1, B_1, C_1)_{q=1} = (2, 17, 71).$$

Comparing (2.3) with (1.1) we get the values of m and n to be 2 and 1 respectively. Now, the conditions of generating infinite number of co-prime integral solutions, as proposed by Lemma 2.2, are applicable for (2.3) because, the terms : mA_1 , nB_1 and C_1, taking values 2^2, 17 and 71 respectively, are pair-wise co-prime; nB_1 is an odd integer and 3 is not a factor of C_1. Thus, the repeated use of Lemma 2.2 will enable us to generate an infinite number of co-prime integral solutions A, B, C. Using (2.2) we get

$$\begin{aligned}(A_2, B_2, C_2)_{q=1} &= \{(2A_1C_1), -B_1(9mA_1^6 - C_1^2), (27m^2A_1^{12} - 18mA_1^6C_1^2 - C_1^4)\};\\ &= \{(2 \cdot 2 \cdot 71), -17 \cdot (9 \cdot 2 \cdot 2^6 - 71^2),\\ &\quad (27 \cdot 2^2 \cdot 2^{12} - 18 \cdot 2 \cdot 2^6 \cdot 71^2 - 71^4)\};\\ &= (2^2 \cdot 71, 66113, -36583777).\end{aligned}$$

Hence, taking the magnitudes of (A_2, B_2, C_2), we get

$$(A_2, B_2, C_2)_{q=1} = (2^2 \cdot 71, 66113, 36583777).$$

In general, using equation (2.2), we calculate the k^{th} solution of (2.3) as

$$(A_k, B_k, C_k)_{q=1} = (2^kA_k^{'}, B_k, C_k)_{q=1},$$

where integer $k \geq 1$, $A_k = 2^kA_k^{'}$ and the three terms: $A_k^{'}$, B_k and C_k are odd. Now, the repeated use of equation (2.2) will enable us to find any number of co-prime integral solutions A, B, C of equation (2.3).

Step II. The first solution A_1, B_1, C_1 of equation (2.3) is $(A_1, B_1, C_1)_{q=1} = (2, 17, 71)$. Using these values for (A, B, C) in (2.3) we get

$$2 \cdot 2^6 + 17^3 = 71^2. \text{Or}, 2^7 \cdot 1^6 + 17^3 = 71^2. \tag{2.5}$$

The second solution A_2, B_2, C_2 of equation (2.3) is

$$(A_2, B_2, C_2)_{q=1} = (2^2 \cdot 71, 66113, 36583777).$$

Using these values for (A, B, C) in (2.3) we get

$$2 \cdot 2^{12} \cdot 71^6 + 66113^3 = 36583777^2. \text{Or}, 2^{13} \cdot 71^6 + 66113^3 = 36583777^2. \tag{2.6}$$

The k^{th} solution for (A, B, C) of equation (2.3) is $(2^kA_k^{'}, B_k, C_k)$. Using these values for (A, B, C) in (2.3) we get

$$2^{6k+1}{A_k^{'}}^6 + B_k^3 = C_k^2. \tag{2.7}$$

When $q = 2$, from (2.5) we get the starting solution for (A, B, C) of equation (1.2) as

$$(A_1, B_1, C_1)_{q=2} = (1, 17, 71).$$

When $q = 3$, from (2.6) we get the starting solution for (A, B, C) of equation (1.2) as

$$(A_1, B_1, C_1)_{q=3} = (71, 66113, 36583777).$$

When $q = k$, from (2.7) we get the starting solution for (A, B, C) of equation (1.2) as

$$(A_1, B_1, C_1)_{q=k} = (A_k^{'}, B_k, C_k).$$

Step III. In Step I, we proved Theorem 1.1 for $q = 1$. Putting $q = 2$ in (1.2) we get

$$2^7 A^6 + B^3 = C^2. \tag{2.8}$$

Now, for each integral value of $q > 1$, there is a starting solution for (A, B, C) for equation (1.2) as we showed in Step II. Since the values of B and C in these starting solutions are the same values which are generated by the subsequent solutions of equation (2.3), they should be co-prime; B and C are odd integers; and 3 is not a factor of C. Hence, for any integer $q > 1$, the statement of Theorem 1.1 should also be valid, because the conditions of generating an infinite number of co-prime integral solutions, as proposed by Lemma 2.2, are satisfied. This completes the proof of Theorem 1.1.

The initial numerical solutions A_1,B_1,C_1 of the Diophantine equation (1.2) for the first five integral values of q have been given in the Table 2.1.

Table 2.1

q	A_1	B_1	C_1
1	2	17	71
2	1	17	71
3	71	66113	36583777
4	2597448167	-535925530724803712767	2661377178406628694765 981631103
5	6912789273747929683869 501481664157538201	7728055556225610857542 8473064932087654075947 9924781966253867217010 713775046113600513	6793711119839793669503 4193688655551123572122 6683940182435309400379 4029662880013377760398 8937460368896088450165 0478073311169023

2.2. Proof of Theorem 1.2

Since, $2^9 + (-7)^3 = 13^2$ or, $2^3 \cdot 2^6 + (-7)^3 = 13^2$, we get the first co-prime solution for (A, B, C) of the Diophantine equation (1.3) when $q = 1$ as

$$(A_1, B_1, C_1)_{q=1} = (2, -7, 13).$$

Using Lemma 2.2 we get

$$(A_2, B_2, C_2)_{q=1} = (2^2 \cdot 13, 31073, 5491823).$$

The steps in the proof of Theorem 1.1 should guide us in establishing Theorem 1.2. The initial numerical solutions A_1,B_1,C_1 of the Diophantine equation (1.3) for the first five integral values of q have been given in the Table 2.2.

Table 2.2

q	A_1	B_1	C_1
1	2	-7	13
2	1	-7	13
3	13	31073	5491823
4	71393699	892933489418780033	9948222845453398761776 87617
5	7102404274132234697252 7100244125283	-134806646659202787164 9406221155988955366877 2724390241549956741560 14340607	3844277980252270256032 1005431420380951456584 3387403850379940429742 8760934340309413757406 752322913294757212417

3. Conclusion

Sometimes it is possible to transform a Diophantine equation to an elliptic curve, and depending upon the Mordell-Weil rank of that elliptic curve one can tell if the corresponding Diophantine equation has “zero”, “finitely many”, or “infinitely many” co-prime integral solutions. Infact, there are standard techniques available in the literature[10] on elliptic curves which allow us to explicitly find some of these solutions. Since our paper is based on an elementary approach we have not told anything on this matter. How ever, to make the paper complete, we want to make a few remarks on them.

The theory of elliptic curves has links with many branches of mathematics. Given an elliptic curve E over a field K, the Mordell-Weil group of E, denoted as $E(K)$, is the group of K-rational points of E. A theorem of Mordell (later generalized by Weil to abelian varieties) states that, if K is a number field, then $E(K)$ is a finitely generated abelian group. Its rank is called the Mordell-Weil rank of E.

Some comments relating to the equation $ax^6 + by^3 + cz^2 = 0$ can be found in ([8], p. 396). The general super-Fermat equation is the equation $Ax^p + By^q + Cz^r = 0$ for given nonzero integers A, B, C and integral exponents $p, q, r \geq 2$. For the parabolic case, when $1/p + 1/q + 1/r = 1$, we can have infinitely many or finitely many solutions of this super-Fermat equation, depending on A, B, C-refer Proposition 14.6.1 of ([9], p. 481) for more details.

The elliptic curves

$$E_1 : y^2 = x^2 + 1, E_2 : y^2 = x^2 + 2, E_8 : y^2 = x^2 + 8$$

has rank $E_1(\mathbb{Q}) = 0$, rank $E_2(\mathbb{Q}) = 1$, rank $E_8(\mathbb{Q}) = 1$, respectively. So, we conclude that the Diophantine equations

$$A^6 + B^3 = C^2, 2A^6 + B^3 = C^2, 8A^6 + B^3 = C^2$$

has “finitely many solutions (in fact $(A, B, C) = (\pm 1, 2, \pm 3)$)”, “infinitely many co-prime integer solutions”, “infinitely many co-prime integer solutions”, respectively.

Acknowledgement

The author expresses his thanks to the referee for many valuable suggestions which improved the presentation of the paper. He thanks Dr. A. Samanta, the founder Chancellor of KIIT University and Sj. L. Pattajoshi for their continued support and encouragemant.

References

[1] S. K. Jena, *Method of infinite ascent applied on* $mA^6 + nB^3 = C^2$, Math. Student, Vol. 77 (2008), 239–246.

[2] S. K. Jena, *Method of infinite ascent applied on* $A^4 \pm nB^2 = C^3$, Math. Student, Vol. 78 (2009), 233–238.

[3] S. K. Jena, *Method of infinite ascent applied on* $mA^3 + nB^3 = C^2$, Math. Student, Vol. 79 (2010), 187–192.

[4] S. K. Jena, *Beyond the method of infinite descent*, J. Comb. Inf. Syst. Sci., Vol 35 (2010), 501–511.

[5] S. K. Jena, *Method of infinite ascent applied on* $mA^3 + nB^3 = 3C^2$, Math. Student, Vol. 81 (2012), 151–160.

[6] S. K. Jena, *The method of infinite ascent applied on* $A^4 \pm nB^3 = C^2$, Czech. Math. J. 63 (138) (2013), No. **2**, 369-374.

[7] S. K. Jena, *Method of infinite ascent applied on* $A^3 \pm nB^2 = C^3$, Notes on Number Theory and Discrete Mathematics, Vol. 19 (2013), No. **2**, 233–238.

[8] H. Cohen, *Number Theory-Volume I: Tools and Diophantine Equations*, Springer-Verlag, GTM 239, 2007.

[9] H. Cohen, *Number Theory-Volume II: Analytic and Modern Tools*, Springer-Verlag, GTM 240, 2007.

[10] J. H. Silverman, *The Arithmetic of Elliptic Curves*, Springer-Verlag, GTM 106, 1986; Expanded 2nd ed., 2009.