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SCHAUDER BASES, SCHAUDER FUNCTIONS, AND THE GRAM-SCHMIDT PROCESS

A linearly independent subset of a vector space is a basis for the space, if every vector can be expressed as a linear combination of the basis vectors. In the finite–dimensional case, each basis contains the same (finite) number of elements, and the space coincides with the span of any one of these finite subsets, a property not enjoyed, however, by any infinite–dimensional space, $L^2[0,1]$ being a case in point. One possible modification of this concept of basis would allow the linear combinations to involve denumerably many vectors, but then, of course, one would need some notion of convergence for the series thus arising. This, in turn, leads one to conduct the discussion within the friendly confines of a distinguished class of vector spaces.

Let X be a Banach space; i.e., a complete, normed linear space, or, in other words, a normed vector space that is (Cauchy) complete in the topology engendered by its norm. A denumerable subset of X, $\{x_n : n = 1, 2, \ldots\}$, is a Schauder basis for X iff each x in X has a unique representation $x = \sum_{n=1}^{\infty} c_n x_n$, in the sense that $\lim_n \|x - \sum_{i=1}^{n} c_i x_i\| = 0$. Thus, by virtue of the Riesz–Fischer theorem, every complete orthonormal system associated with an interval $[a,b]$ is a Schauder basis for $L^2[a,b]$. Familiar examples of this type are the trigonometric system, for $L^2[-\pi,\pi]$; the Legendre system, for $L^2[-1,1]$; and the Haar system, for $L^2[0,1]$.

The name for this sort of basis derives from an article written by Schauder [9] in which the concept is defined, and, among other things, it is shown how to construct bases for $C[0,1]$, the space of all real–valued, continuous functions on $[0,1]$, a Banach space when endowed with the supremum norm

$$\|f\|_\infty = \sup\{|f(t)| : t \in [0,1]\}.$$

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A typical construction begins with a sequence $\{t_{nk}\}_{n=0}^{\infty}, {}_{k=0}^{2^n}$ such that $\{t_{nk} : n \geq 0, 0 \leq k \leq 2^n\}$ is dense in $[0,1]$; $t_{00} = 0$; $t_{01} = 1$; $t_{n+1,2k} = t_{nk}$, for all $n \geq 0$, $0 \leq k \leq 2^n$; and $t_{nk} < t_{n+1,2k+1} < t_{n,k+1}$, for all $n \geq 0$, $0 \leq k \leq 2^n - 1$. The first two Schauder functions are given by the equations $\varphi_{00}(t) = 1$, $\varphi_{01}(t) = t$, and, for $n > 0$, $1 \leq k \leq 2^{n-1}$, φ_{nk} is the function supported by $(t_{n,2k-2}, t_{n,2k})$, linear on $[t_{n,2k-2}, t_{n,2k-1}]$ and on $[t_{n,2k-1}, t_{n,2k}]$, such that $\varphi_{nk}(t_{n,2k-1}) = 1$. Finally, one changes the notation slightly, in order to arrange these functions in the lexicographical order, setting $\varphi_0 = \varphi_{00}$, $\varphi_1 = \varphi_{01}$, and, for $n \geq 0$, $1 \leq k \leq 2^n$, $\varphi_{2^n+k} = \varphi_{n+1,k}$. (The ordering of the elements proves to be a critical matter, since there are orderings for which the resulting system is not a Schauder basis.) The series expansion for an element f of $C[0,1]$ is determined in the most straightforward manner. Denoting the n^{th} partial sum of the series by $S_n f$, one defines these, inductively, by setting $S_0 f = f(0)\varphi_0$, $S_1 f = S_0 f + [f(1) - f(0)]\varphi_1$, and for $m = 2^n + k$, $n \geq 0$, $1 \leq k \leq 2^n$, $S_m f = S_{m-1} f + (f - S_{m-1} f)(t_{n+1,2k-1})\varphi_m$. The uniform convergence of $\{S_n f\}_{n=0}^{\infty}$ to f follows from the uniform continuity of f, and, because $\varphi_m(t) = 0$ whenever $\varphi_n(t) = 1$ and $m > n > 0$, this representation is unique.

Closely related to one of the Schauder systems is the Haar system, the definition of which involves the binary rationals. The first Haar function is identically 1 on the unit interval, and the remaining elements of the system are grouped into blocks the members of which are translates of one another. For $n \geq 0$, the n^{th} block contains 2^n functions, and the k^{th} element of block n is given by

$$h_{nk}(t) = \begin{cases} +2^{\frac{n}{2}} & \text{if } t \in [\frac{2k-2}{2^{n+1}}, \frac{2k-1}{2^{n+1}}) \\ -2^{\frac{n}{2}} & \text{if } t \in (\frac{2k-1}{2^{n+1}}, \frac{2k}{2^{n+1}}] \\ 0 & \text{otherwise.} \end{cases}$$

Haar created these functions in order to demonstrate the existence of an orthonormal system with respect to which every function continuous on $[0,1]$ has a development that converges uniformly to the function, a property not possessed by the trigonometric system or by other orthonormal systems arising from mathematical physics. Thus, the Haar system is much like a Schauder basis for $C[0,1]$, failing in this, of course, because most of the Haar functions are not themselves elements of the space.

In order to construct an orthonormal system that does not suffer from this basic deficiency, Franklin [3] proceeded in the following manner. Let $\{a_n\}_{n=1}^{\infty}$ be the sequence of binary rationals: $0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \ldots, \frac{1}{2^k}, \ldots, \frac{2^k-1}{2^k}, \ldots$; let v_0 be the constant function 1; for each $n \geq 1$, let

$$v_n(x) = \begin{cases} 0 & \text{if } x \leq a_n \\ x - a_n & \text{if } x \geq a_n \end{cases}$$

and let $V = \{v_n : n = 0, 1, \ldots\}$. Then V is linearly independent, and, thus, application of the ordinary Gram–Schmidt process yields an orthonormal family, GSV, of continuous functions. Franklin showed that GSV is a Schauder basis for $C[0,1]$ (as well as for $L^2[0,1]$.)

Subsequently Kaczmarz and Steinhaus [6] modified Franklin's approach by substituting, for the family V, one of Schauder's bases for $C[0,1]$, as described above. Application of the Gram–Schmidt process to any one of those Schauder systems will yield an orthonormal Schauder basis for $C[0,1]$; members of the resulting families are termed Franklin functions.

A final modification of the Franklin schema was made by Ciesielski [2], who began his seminal work in this area with the Schauder system based upon the binary–rational points of $[0,1]$. The Gram–Schmidt orthonormalization of this family is *the* Franklin system. (It is a curious historical anomaly that the functions that bear his name are not, in fact, the ones described by Franklin in the article cited above.) Ciesielski gave a brief, clever proof that the Franklin system is a Schauder basis for $C[0,1]$, and indicated how the argument could be extended to show that this system is a Schauder basis for each of the L^p–spaces, $p \geq 1$, as well.

This last conclusion also follows swiftly from two results due to Banach (see [1], Chapter VII) and the M. Riesz–Thorin interpolation theorem (see, for example, [16], Volume 2, Chapter XII). Let $Y = \{y_n : n = 1, 2, \ldots\}$ be an orthonormal set of functions continuous on $[0,1]$. Then, if Y is a Schauder basis for $L^p[0,1]$, for some p in $(1, +\infty]$ (for $p = +\infty$, let $L^p = C$), then Y is a basis for $L^q[0,1]$, $\frac{1}{p} + \frac{1}{q} = 1$, as well; and if $1 \leq p_1 < p_2 \leq +\infty$, and if Y is a Schauder basis for both $L^{p_1}[0,1]$ and $L^{p_2}[0,1]$, then Y is a basis for $L^p[0,1]$, for each p in $[p_1, p_2]$.

As a further consequence of the results last noted, one observes that if Ψ be any Schauder basis for $C[0,1]$, then exactly one of the following must obtain:

(i) $GS\Psi$ is a Schauder basis for every $L^p[0,1]$, $1 \leq p \leq +\infty$;

(ii) $GS\Psi$ is a Schauder basis for precisely those $L^p[0,1]$ with $1 < p < +\infty$;

(iii) there exists an α in $[2, +\infty)$ such that $GS\Psi$ is a Schauder basis for precisely those $L^p[0,1]$ with $\alpha/(\alpha - 1) \leq p \leq \alpha$;

(iv) there exists an α in $(2, +\infty)$ such that $GS\Psi$ is a Schauder basis for precisely those $L^p[0,1]$ with $\alpha/(\alpha - 1) < p < \alpha$.

Since Franklin's system, V, and the Schauder systems all yield Gram–Schmidt orthonormalizations of type (i), one well might question whether $GS\Psi$ is a Schauder basis for $C[0,1]$, whenever Ψ so serves. On the basis of Baxter's metatheorem, however, one would guess that the answer to this question is no, and, indeed, Szlenk [11] has resolved this matter, in the negative, by modifying one of the systems devised by Schauder.

Let $\{t_{nk}\}_{n=0}^{\infty}{}_{k=0}^{2^n}$ be a sequence that satisfies $t_{n,2^n-1} = 1 - 1/2^n$, for each $n \geq 1$, as well as the original conditions specified by Schauder (above), and let $\{\varepsilon_m\}_{m=1}^{\infty}$ be a sequence of positive numbers. For each $n = 2^m + 1$, with $m \geq 1$, let $\psi_n = \varphi_n + \varepsilon_m \varphi_{2n-2}$, and for the remaining values of n, let $\psi_n = \varphi_n$. Then, by virtue of the Krein–Milman–Rutman theorem on the stability of bases (see, for example, [7, pages 442–444]), $\Psi = \{\psi_n : n = 0, 1, \ldots\}$ will be a Schauder basis for $C[0,1]$ as long as the ε_m are taken to be sufficiently small. On the other hand, Szlenk showed that if Ψ be based upon a (Schauder) sequence for which $t_{n1} < \varepsilon_n^2/2^{n+6}$, then $GS\Psi$ will not be a Schauder basis for $C[0,1]$.

One notes, however, that each of the Szlenk systems is closed (the set of all finite linear combinations of elements of the set is dense) in each space $L^p[0,1]$, $p \geq 1$, because every linear combination of elements of the corresponding Φ can be written as a linear combination of elements of Ψ. Thus, it might be the case that $GS\Psi$ is a basis for the L^p–spaces even though it fails so to serve for $C[0,1]$. Of course, there must be an interval such that, for every p contained therein, $GS\Psi$ is a Schauder basis for $L^p[0,1]$, but this interval need not be $(1, +\infty)$. Indeed, Veselov [13] has shown that each of the outcomes (i)–(iv) is possible. In particular, one may perturb a Schauder system, a la Szlenk, so that the new system will be again a Schauder basis for $C[0,1]$, yet its Gram–Schmidt orthonormalization will be a basis only for $L^2[0,1]$.

Proceeding on a slightly different tack, suppose now that one were to delete some of the elements from a Schauder system, Φ, and then subject the residual system, Φ_ρ, to the Gram–Schmidt process. It would be too much to expect of it that $GS\Phi_\rho$ be a Schauder basis for $C[0,1]$, since Φ_ρ does not fulfill this function, but if Φ_ρ were a sufficiently thick subset of Φ, it is conceivable that $GS\Phi_\rho$ could be a basis for some of the L^p–spaces.

Several years ago, Goffman [4] showed that to every Lebesgue measurable function on $[0,1]$ there corresponds a Schauder series, with coefficients tending to 0, that converges almost everywhere to the function. In this work, Goffman used the dyadically based Schauder functions, but the arguments employed therein, mutatis mutandis, would apply equally well to any of the Schauder systems. Subsequently, he asked whether it be necessary to require all of the Schauder functions to be present in order that every measurable function be representable in this way. From a superficial examination of Goffman's work, it is clear that a corresponding representation theorem will be obtained if the Schauder system be replaced by any one of its cofinite subsets; thus, in analogy with questions treated by Talalyan [12] and by Goffman and Waterman [5], it is natural to ask whether it be possible to obtain a result of this type for a Schauder subsystem whose complement is infinite. This question was answered in the affirmative, in [14], where it was shown that a subsystem $\Phi_\rho = \{\psi_n : n = 1, 2, \ldots\}$ is sufficiently rich for the specified method of representation of measurable functions if (*) $\mu(\limsup_n E_n) = 1$, where μ is the Lebesgue measure, and E_n is the support of ψ_n. Moreover, in an observation peripheral to the work on this sort of representation of measurable functions, it was noted that Φ_ρ will be closed in each $L^p[0, 1]$, $p \geq 1$, if (*) be satisfied. Thus, for such a subsystem, $GS\Phi_\rho$ will be a Schauder basis for those L^p–spaces that correspond to values of p belonging to some interval that contains 2, and it seems likely that this interval should be nontrivial. The following example is instructive.

Let Φ_ρ be obtained by deleting from a Schauder system those members whose supports abut $\{0\}$. Let f be continuous on $[0, 1]$, and let $S_n f$ be the n^{th} partial sum of the Fourier expansion of f with respect to $GS\Phi_\rho$. Because of the triangular nature of the Gram–Schmidt process, $S_n f$ can be written as a linear combination of the elements of an initial segment of Φ_ρ; viz.,

$$S_n f = \sum_{i=2}^{m} \sum_{j=2}^{2^i} c_{ij}\varphi_{ij} + \sum_{j=2}^{k+1} c_{m+1,j}\varphi_{m+1,j},$$

where $n = 2^{m+1} - m + k$, $0 \leq m$ and $0 \leq k < 2^{m+1} - 1$.

Now any such sum, whatever be the coefficients, will be a polygonal function whose vertices have abscissae belonging to the set

$$S = \{t_{ij} : 0 \leq i \leq m, 1 \leq j \leq 2^i\} \cup \{t_{m+1,j} : 2 \leq j \leq 2k\}.$$

Thus, $S_n f$ will be 0 on $[0, t_{m1}]$ and at each point t_{q1}, with $q \leq m$. To every other t_{ij} in S, however, there corresponds one of the φ_{rs}, involved in $S_n f$, such that $\varphi_{rs}(t_{ij}) = 1$. It follows that a polygonal function built upon the first n members of Φ_ρ may assume any real value at such an interior point.

Consider one of the intervals $[t_{q+1,1}, t_{q1}] = Q$. Relabel the $t_{ij} \in S \cap Q$: $t_{q+1,1} = t_1 < t_2 < \ldots < t_k = t_{q1}$. Then, on Q, the polygonal function must have the form

$$\varphi(t) = \frac{y_{i+1} - y_i}{\delta_i}(t - t_i) + y_i, \quad i = 1, \ldots, k - 1,$$

where $\delta_i = |t_{i+1} - t_i|$, and $y_1 = y_k = 0$. The particular φ of this type, $S_n f$, will have its y_i, $2 \leq i \leq k - 1$, specified by the condition

$$\int_0^1 [f(t) - S_n f(t)]^2 dt = \min_{\varphi} I(\varphi),$$

where $I(\varphi) = \int_0^1 [f(t) - \varphi(t)]^2 dt$, and φ belongs to the set of polygonal functions here considered. Some of the necessary conditions for the minimality of the functional, I, are

$$\frac{\partial I}{\partial y_i} = 0, \quad i = 2, \ldots, k-1,$$

from which follow the estimates

$$2|y_i| \leq 3\|f\|_\infty + \|\varphi\|_\infty, \quad i = 2, \ldots, k-1.$$

These inequalities, together with the corresponding estimates for the other intervals $[t_{q'+1,1}, t_{q'1}]$, yield

$$\|S_n f\|_\infty = \|\varphi\|_\infty \leq 3\|f\|_\infty,$$

and thus, from a theorem of Orlicz [8], one also may conclude that

$$\|S_n g\|_p \leq 3\|g\|_p, \quad \forall g \in L^p[0,1], \quad \forall p \geq 1.$$

Hence, each of the sequences $\{\|S_n\|_p\}_{n=1}^\infty$ is bounded, and, since Φ_ρ is closed in each of these spaces, $GS\Phi_\rho$ is a Schauder basis for each $L^p[0,1]$, $1 \leq p < +\infty$.

This approach, which is due to Ciesielski, is applicable here, because those y_i, not required to be 0, are independent variables. This, in turn, is the case, because, whenever some $\varphi_{ij} \in \Phi_\rho$, one has also $\varphi_{rs} \in \Phi_\rho$, if the support of the latter is contained in the support of the former. Hence, one may apply this same argument to show that, if a subsystem, Φ_ρ, of a Schauder system, Φ, satisfies both (*) and

$$\varphi_{rs} \in \Phi_\rho, \text{ whenever } \varphi_{ij} \in \Phi_\rho \text{ and support } \varphi_{rs} \subset \text{support } \varphi_{ij}, \qquad (**)$$

then $GS\Phi_\rho$ is a Schauder basis for each $L^p[0,1]$, $1 \leq p < +\infty$.

For example, one might select those members of the standard Schauder system whose supports lie in

$$\left[\frac{1}{4}, \frac{3}{4}\right] \cup \left[\frac{1}{16}, \frac{3}{16}\right] \cup \left[\frac{13}{16}, \frac{15}{16}\right] \cup \left[\frac{1}{64}, \frac{3}{64}\right] \cup \ldots \cup \left[\frac{61}{64}, \frac{63}{64}\right] \cup \ldots.$$

The resulting Franklinesque system, obtained by performing the Gram-Schmidt maneuver on this collection, has the interesting property that each of its constituents vanishes on the "Cantor" set $C = \cap_{n=1}^\infty C_n$, where $C_1 = [0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$, $C_2 = [0, \frac{1}{16}] \cup [\frac{3}{16}, \frac{1}{4}] \cup [\frac{3}{4}, \frac{13}{16}] \cup [\frac{15}{16}, 1], \ldots$. Indeed, one may carry this idea one step further in order to obtain an orthonormal set that is a basis for each $L^p[0,1]$, $1 \leq p < +\infty$, each of whose members vanishes on a preassigned Cantor set.

On the other hand, what can one say about $GS\Phi_\rho$, if Φ_ρ satisfies (*) but not (**)? Here the situation will be more complex, because the approximating polynomials engendered by Φ_ρ will have vertices (t_i, y_i) with the y_i depending upon various other corner–values y_j. For example, if Φ_ρ be a subset of the standard Schauder system, and if φ_{11} and φ_{21} belong to Φ_ρ, but φ_{32} does not, then a polygonal function, φ, formed from an initial segment of Φ_ρ, will have $\varphi(\frac{3}{8})$ a linear combination of $\varphi(\frac{1}{4})$, $\varphi(\frac{1}{2})$ and, perhaps, also $\varphi(0)$ and $\varphi(1)$. Thus, the Ciesielski schema cannot be followed precisely. Although it has been shown [15], albeit not without a surprising amount of difficulty, that if one deletes any single function from the Schauder system, then the Gram–Schmidt orthonormalization of the residual system will be a Schauder basis for $L^p[0,1]$, $1 \le p < +\infty$, it is not known if the same result obtains when one makes an arbitrary finite number of deletions. The degree of complexity of the general problem seems to be a good bit higher.

Of course, it may be that this question can be resolved easily, if one but approach it in a different manner. For example, there could be some general principle(s) that would trivialize the problem. One possibility of this nature: suppose it were the case that from the inclusion $\Psi_1 \subset \Psi_2$ (where Ψ_1 and Ψ_2 are both closed in $L^p[0,1]$, $1 \le p < +\infty$), it followed that $GS\Psi_2$ is a Schauder basis at least for those L^p–spaces for which Ψ_1 so serves. Then, if Φ_δ were any finite subset of a Schauder system Φ, and if $\Phi_\varepsilon = \{\varphi_n \in \Phi : \exists m \ge n \ni \varphi_m \in \Phi_\delta\}$, then $\Phi \setminus \Phi_\varepsilon$ would satisfy (**), and, thus, would be a Schauder basis for each $L^p[0,1]$, $1 \le p < +\infty$. Since $\Phi \setminus \Phi_\varepsilon \subset \Phi \setminus \Phi_\delta$, the same property would be enjoyed by $\Phi \setminus \Phi_\delta$. Unfortunately, a counterexample to the proposed "principle" is rather easily obtained from the work of Szlenk. If in that construction one takes $\varepsilon_n = 1/2^{n^2+2}$ and $t_{n1} = \varepsilon_n^2/2^{n+6}$, then one finds that, for every $p > 2$, $\{\|S_n\|_p\}_{n=1}^{\infty}$, where S_n is the n^{th} partial–sum operator associated with $GS\Psi$, is unbounded, so that $GS\Psi$ cannot be a Schauder basis for $L^p[0,1]$, for these values of p. Yet, if one deletes from Ψ the elements ψ_n for which $n = 2^m + 1$, $m \ge 1$, one obtains a subsystem of a Schauder system that satisfies both (*) and (**). At the time of this writing, general principles seem to be in somewhat short supply.

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