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ON THE CARATHÉODORY SUPERPOSITION

Abstract

In this paper we investigate properties of Carathéodory's superposition and show some of their applications to differential equations.

Throughout this article $(X, \mathcal{M}, \mu)$ denotes a totally σ-finite measure space and $(\Phi, \Rightarrow)$ will denote a differentiation basis for this space. This means ([1]) that $\Phi \subset \mathcal{M}$ is a family of sets of finite positive measure μ and $\Rightarrow$ is a notion of contraction of nets (generalized sequences) of sets in Φ to points of X, such that the following two conditions are satisfied:

(i) if $x \in X$, there exists a net (I_α) of elements of Φ contracting (in the sense of $\Rightarrow$) to x; in symbols, $I_\alpha \Rightarrow x$;

(ii) any subnet of a net contracting to a point x also contracts to x.

Let $A \in \mathcal{M}$ and let $x \in X$. We define the upper and lower densities of A at x with respect to $(\Phi, \Rightarrow)$ by

$$D^u(A,x) = \sup\{\limsup \mu(A \cap I_\alpha)/\mu(I_\alpha)\}$$

and

$$D_l(A,x) = \inf\{\liminf \mu(A \cap I_\alpha)/\mu(I_\alpha)\},$$

where the limits superior and inferior are taken over a net (I_α) contracting to x and the supremum and infimum are taken over the family of all such nets.

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Section I

Let (Y, ϱ) be a metric space with the metric ϱ and let $g : X \longrightarrow Y$ be a function. Let $\gamma \in (0, 1]$ be a number. We say that the function g has the property:

$(S_{u,\gamma})$ (with respect to $(\Phi, \Rightarrow)$) if for every open set $U \subset Y$ and for every x with $g(x) \in U$, there is a set $B \subset g^{-1}(U)$ such that $B \in \mathcal{M}$ and $D^u(B, x) \geq \gamma$;

$(S_{l,\gamma})$ if for every open set $U \subset Y$ and for every x with $g(x) \in U$ there is a set $B \in \mathcal{M}$ such that $D_l(B, x) \geq \gamma$.

Theorem 1 *Let $f : X \times Y \longrightarrow Y$ be a function. Suppose that:*

(1) all sections $f^y(x) = f(x, y)(x \in X, y \in Y)$ have the property $(S_{u,\gamma})$ $((S_{l,\gamma}))$, where $\gamma \in (0, 1]$;

(2) for every point $(x, y) \in X \times Y$ there is a set $A(x, y) \in \mathcal{M}$ such that all sections $f_t(u) = f(t, u)$ $(t \in A(x, y), u \in Y)$ are equicontinuous at a point y and $D_l(A(x, y), x) = 1$.

Then for every function $g : X \longrightarrow Y$ having the property $(S_{l,1})$ the Carathédory superposition

$$h(x) = f(x, g(x)), x \in X,$$

has the property $(S_{u,\gamma})$ $((S_{l,\gamma}))$.

PROOF. Fix $x \in X$ and an open set $U \subset Y$ such that $h(x) \in U$. Let $\varepsilon > 0$ be such that $\{u \in Y; \varrho(u, h(x)) \leq \varepsilon\} \subset U$. By (2) there is a set $A(x, g(x)) \in \mathcal{M}$ and a positive number $\delta > 0$ such that $D_l(A(x, g(x)) = 1$ and $\varrho(f(t, y), f(t, g(x))) < \varepsilon/2$ for $t \in A(x, g(x))$ and $y \in Y$ with $\varrho(y, g(x)) < \delta$. Since the section $t \longrightarrow f(t, g(x))$ has the property $(S_{u,\gamma})$ $((S_{l,\gamma}))$, there is a set $C \in \mathcal{M}$ such that $D^u(C, x) \geq \gamma$ $(D_l(C, x) \geq \gamma)$ and $\varrho(f(t, g(x)), f(x, g(x)))$ $< \varepsilon/2$ for every $t \in C$. Analogously, since the function g has the property $(S_{l,1})$, there is a set $E \in \mathcal{M}$ such that $D_l(E, x) = 1$ and $\varrho(g(t), g(x)) < \delta$ for every $t \in E$. Observe that the set $B = C \cap E \cap A(x, g(x)) \in \mathcal{M}$ and $D^u(B, x) \geq \gamma$ $(D_l(B, x) \geq \gamma)$. If $t \in B$, then

$$\begin{aligned}
\varrho(h(t), h(x)) &= \varrho(f(t, g(t)), f(x, g(x))) \\
&\leq \varrho(f(t, g(t)), f(t, g(x))) + \varrho(f(t, g(x)), f(x, g(x))) \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

So, $B \in \mathcal{M}, D^u(B, x) \geq \gamma$ $(D_l(B, x) \geq \gamma)$ and $B \subset h^{-1}(U)$. This completes the proof.

Remark 1 *If $(X, \mathcal{T})$ is a topological space with the topology $\mathcal{T}$ such that $\mathcal{T} \subset \mathcal{M}$, then we can define the properties $(S'_{u,\gamma})$ and $(S'_{l,\gamma})$ assuming that the set $B \subset g^{-1}(U)$ belongs to $\mathcal{T}$.*

Then we have the following:

Theorem 2 *Let $f : X \times Y \longrightarrow Y$ be a function. Suppose that:*

(1') all sections $f^y, y \in Y$, have the property $(S'_{u,\gamma})$ $((S'_{l,\gamma}))$, $\gamma \in (0, 1]$;

(2') for every point $(x, y) \in X \times Y$ there is a set $A(x, y) \in \mathcal{T}$ such that all sections f_t, $t \in A(x, y)$, are equicontinuous at y and $D_l(A(x, y), x) = 1$.

Then for every function $g : X \longrightarrow Y$ having the property $(S'_{l,1})$ the superposition $h(x) = f(x, g(x)), x \in X$, has the property $S'_{u,\gamma})$ $((S'_{l,\gamma}))$.

Section II

In this section we suppose that Y is a separable Banach space with the norm $\|\cdot\|$. Let $g : X \longrightarrow Y$ be a function which is integrable (in the Bochner sense) on every set $I \in \Phi$. The function g is called a derivative at a point $x \in X$ (with respect to $(\Phi, \Rightarrow)$) ([1]) if for every net $I_\alpha \Rightarrow x$ we have

$$\lim_\alpha \int_{I_\alpha} g(t)dt/\mu(I_\alpha) = g(x).$$

Theorem 3 *Let $f : X \times Y \longrightarrow Y$ be a bounded function. Suppose that f satisfies the condition (2) from Theorem 1 and the following conditions:*

(3) all sections $f^y, y \in Y$, are derivatives;

(4) for every function $g : X \longrightarrow Y$ having the property $(S_{l,1})$ the superposition $x \longrightarrow f(x, g(x))$ is μ-measurable, i.e. for every open set $U \subset Y$ the preimage $\{x; f(x, g(x)) \in U\} \in \mathcal{M}$.

Then for every function $g : X \longrightarrow Y$ having the property $(S_{l,1})$ Carathéodory's superposition $h(x) = f(x, g(x)), x \in X$, is a derivative.

PROOF. First, we observe that h is μ-measurable and bounded, so integrable (in the Bochner sense) on every set $I \in \Phi$ ([9]). Fix $x \in X$, a net $I_\alpha \Rightarrow x$ and $\varepsilon > 0$. Let $a > 0$ be such that $\|f(t, y)\| < a$ for each $(t, y) \in X \times Y$. By (2) there is a set $A(x, g(x)) \in \mathcal{M}$ and $\delta > 0$ such that $D_l(A(x, g(x)), x) = 1$ and $\|f(t, y) - f(t, g(x))\| < \varepsilon/3$ for all $t \in A(x, g(x))$ and $y \in Y$ with $\|y - g(x)\| < \delta$. Since g has the property $(S_{l,1})$, there is a set $E \in \mathcal{M}$ such that $D_l(E, x) = 1$ and $\|g(t) - g(x)\| < \delta$ for every $t \in E$. Observe that the set

$B = E \cap A(x, g(x)) \in \mathcal{M}$ and $D_l(B, x) = 1$. Since the section $t \longrightarrow f(t, g(x))$ is a derivative at x, we have

$$\lim_{\alpha} \int_{I_\alpha} f(t, g(x)) dt/\mu(I_\alpha) = f(x, g(x)).$$

There exists an index β such that for $\alpha > \beta$ we have:

(5) $\left\| \int_{I_\alpha} f(t, g(x)) dt/\mu(I_\alpha) - f(x, g(x)) \right\| < \varepsilon/3$;

(6) $\mu(B \cap I_\alpha)/\mu(I_\alpha) > 1 - \varepsilon/6a$.

Consequently, by (5) and (6) we have for $\alpha > \beta$,

$$\begin{aligned}
&\left\| \int_{I_\alpha} h(t) dt/\mu(I_\alpha) - h(x) \right\| \\
&= \left\| \int_{I_\alpha} f(t, g(t)) dt/\mu(I_\alpha) - f(x, g(x)) \right\| \\
&= \left\| \int_{I_\alpha} (f(t, g(t)) - f(x, g(x))) dt \right\| /\mu(I_\alpha) \\
&\leq \int_{I_\alpha \cap B} \| f(t, g(t)) - f(t, g(x)) \| dt/\mu(I_\alpha) \\
&\quad + \int_{I_\alpha \cap B} \| f(t, g(x)) - f(x, g(x)) \| dt/\mu(I_\alpha) \\
&\quad + \int_{I_\alpha - B} \| f(t, g(t)) - f(x, g(x)) \| dt/\mu(I_\alpha) \\
&< \varepsilon\mu(I_\alpha \cap B)/3\mu(I_\alpha) + \varepsilon\mu(I_\alpha \cap B)/3\mu(I_\alpha) \\
&\quad + 2a\mu(I_\alpha - B)/\mu(I_\alpha) < \varepsilon/3 + \varepsilon/3 + 2a\varepsilon/6a = \varepsilon.
\end{aligned}$$

This shows that

$$\lim_{\alpha} \int_{I_\alpha} h(t) dt/\mu(I_\alpha) = h(x)$$

and finishes the proof.

Remark 2 *A particular case of Theorem 3 is proved in [7].*

Remark 3 *In the general case there can exist nonmeasurable derivatives having the property* $(S_{l,1})$. *For example, if* $X = \mathbb{R}$ *(the set of all reals),* μ *is the Lebesgue measure and* Φ *is the family of open intervals then there exists a* μ*-nonmeasurable set* $A \subset [0, 1]$. *Let* $I_k \Rightarrow x \in \mathbb{R} - [0, 1]$ *mean that* $x \in I_k \subset \mathbb{R} - [0, 1]$ *for all* $k = 1, 2, \ldots$ *and* $\lim_{k \to \infty} d(I_k) = 0$ $(d(I_k)$ *denotes* *the diameter of* I_k), *and for* $x \in A$ *let* $I_k \Rightarrow x$ *mean that* $I_k \Rightarrow 2$, *and for* $x \in [0,1] - A$ *let* $I_k \Rightarrow x$ *mean that* $I_k \Rightarrow -2$. *Then the function*

$$g(x) = \begin{cases} 1 & \text{for } x \in A \cup (1, \infty) \\ 0 & \text{otherwise} \end{cases}$$

is a μ*-nonmeasurable derivative having the property* $(S_{l,1})$. *Some additional assumptions for the differentiation basis* $(\Phi, \Rightarrow)$ *(for example, the density property [1]) imply the* μ*-measurability of all derivatives and all functions having the property* $(S_{u,\gamma})$, $\gamma \in (0,1]$.

Remark 4 *Some counterexamples concerning Theorems 1 and 3 are contained in [5], [7], and [6].*

Section III

In this section we assume that (Y, ϱ) is a metric space, $(Y, \mathcal{M}_1, \mu_1)$ is a totally σ-finite complete measure space and that $(\Phi_1, \Rightarrow)$ is a differentiation basis in Y.

Theorem 4 *Let* $f : X \times Y \longrightarrow Y$ *be a function such that:*

(7) for every $(x,y) \in X \times Y$ *and for every* $\varepsilon > 0$ *there are sets* $A(x,y) \in \mathcal{M}$, $B(x,y) \in \mathcal{M}_1$ *such that* $D_l(A(x,y),x) = D_l(B(x,y),y) = 1$, $y \in B(x,y)$, *and* $\varrho(f(t,u), f(t,y)) < \varepsilon$ *for every* $t \in A(x,y)$ *and every* $u \in B(x,y)$;

(8) all sections $f^y, y \in Y$, *have the property* $(S_{l,\gamma})$ $((S_{u,\gamma}))$, *where* $\gamma \in (0,1]$.

Then for every function $g : X \longrightarrow Y$ *satisfying*

(9) if $g(x) \in U \in \mathcal{M}_1$ *and* $D_l(U, g(x)) = 1$, *then there is a set* $C \in \mathcal{M}$ *such that* $x \in C \subset g^{-1}(U)$ *and* $D_l(C,x) = 1$,

the superposition $h(x) = f(x, g(x)), x \in X$, *has the property* $(S_{l,\gamma})$ $((S_{u,\gamma}))$.

PROOF. Fix $x \in X$, and an open set $U \subset Y$ such that $h(x) \in U$. Let $\varepsilon > 0$ be such that

$$\{u \in Y; \varrho(u, h(x)) \leq \varepsilon\} \subset U.$$

By (7) there are sets $A(x, g(x)) \in \mathcal{M}$ and $B(x, g(x)) \in \mathcal{M}_1$ such that

$$g(x) \in B(x, g(x)),$$

$$D_l(A(x, g(x)), x) = D_l(B(x, g(x)), g(x)) = 1 \text{ and}$$

$$\varrho(f(t,u), f(t,g(x))) < \varepsilon/2$$

for every $t \in A(x, g(x))$ and every $u \in B(x, g(x))$. By (9) there is a set $C \in \mathcal{M}$ such that $x \in C \subset g^{-1}(B(x, g(x)))$ and $D_l(C, x) = 1$. Since the section $t \longrightarrow f(t, g(x))$ has the property $(S_{l,\gamma})$ $((S_{u,\gamma}))$, there is a set $E \in \mathcal{M}$ such that $D_l(E, x) \geq \gamma$ $(D^u(E, x) \geq \gamma)$ and $\varrho(f(t, g(x)), f(x, g(x))) < \varepsilon/2$ for each $t \in E$. Observe that the set $G = C \cap E \cap A(x, g(x)) \in \mathcal{M}$ and $D_l(G, x) \geq \gamma$ $(D^u(G, x) \geq \gamma)$. Moreover, we have for $t \in G$,

$$\begin{aligned} \varrho(h(t), h(x)) &= \varrho(f(t, g(t)), f(x, g(x))) \\ &\leq \varrho(f(t, g(t)), f(t, g(x))) + \varrho(f(t, g(x)), f(x, g(x))) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

This completes the proof.

Section IV

In this section we assume that $(Y, \|\cdot\|)$ is a separable Banach space and $(Y, \mathcal{M}_1, \mu_1)$ is a totally σ-finite complete measure space and $(\Phi_1, \Rightarrow)$ is a differentiation basis in Y.

Theorem 5 *Let $f : X \times Y \longrightarrow Y$ be a bounded function satisfying the condition (7) from Theorem 3 such that*

(10) all sections f^y, $y \in Y$, are derivatives;

(11) for every function $g : X \longrightarrow Y$ satisfying the condition (9) from Theorem 4 the superposition $h(x) = f(x, g(x)), x \in X$, is μ-measurable.

Then for every function $g : X \longrightarrow Y$ satisfying the condition (9) from Theorem 4 the superposition $h(x) = f(x, g(x)), x \in X$, is a derivative.

PROOF. Fix $x \in X$, a net $I_\alpha \Rightarrow x$, and $\varepsilon > 0$. By (7) there are sets $A(x, g(x)) \in \mathcal{M}$ and $B(x, g(x)) \in \mathcal{M}_1$ such that $g(x) \in B(x, g(x))$, $D_l(A(x, g(x)), x) = D_l(B(x, g(x)), g(x)) = 1$ and $\|f(t, u) - f(t, g(x))\| < \varepsilon/3$ for every $t \in A(x, g(x))$ and every $u \in B(x, g(x))$. By (9) there is a set $C \in \mathcal{M}$ such that $x \in C \subset g^{-1}(B(x, g(x)))$ and $D_l(C, x) = 1$. Observe that the set $E = A(x, g(x)) \cap C \in \mathcal{M}$ and

(12) $D_l(E, x) = 1$.

By (10) and (12) there is an index β such that for $\alpha > \beta$ we have:

(13) $\left\| \int_{I_\alpha} f(t, g(x)) dt / \mu(I_\alpha) - f(x, g(x)) \right\| < \varepsilon/3$;

(14) $\mu(I_\alpha \cap E)/\mu(I_\alpha) > 1 - \varepsilon/6a$, where $a > 0$ is such that $\|f(t,u)\| \le a$ for all $(t,u) \in X \times Y$.

If $\alpha > \beta$ then, by (13) and (14),

$$\begin{aligned}
&\left\| \int_{I_\alpha} h(t)dt/\mu(I_\alpha) - h(x) \right| \\
&= \left\| \int_{I_\alpha} f(t,g(t))dt/\mu(I_\alpha) - f(x,g(x)) \right\| \\
&= \left\| \int_{I_\alpha} (f(t,g(t)) - f(x,g(x)))dt \right\| /\mu(I_\alpha) \\
&\le (\int_{I_\alpha \cap E} \|f(t,g(t)) - f(t,g(x))\|dt \\
&\quad + \int_{I_\alpha - E} \|f(t,g(t)) - f(t,g(x))\|dt \\
&\quad + \int_{I_\alpha} \|f(t,g(x)) - f(x,g(x))\|dt/\mu(I_\alpha)) \\
&< \varepsilon\mu(I_\alpha \cap E)/3\mu(I_\alpha) + 2\varepsilon a/6a + \varepsilon/3 \\
&\le \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.
\end{aligned}$$

So,

$$\lim_\alpha \int_{I_\alpha} h(t)dt/\mu(I_\alpha) = h(x),$$

and the proof is finished.

Section V

Now we will show some applications of Theorems 1, 2, and 3 to the differential equations. In this section $X = \mathbb{R}, \mu$ is the Lebesgue measure and Φ denotes the family of all open intervals. Let $I_k \Rightarrow x$ mean that $x \in I_k$ for all k and $\lim_{k\to\infty} d(I_k) = 0$. Let Y be a separable Banach space. Observe that in this case all derivatives and all functions having the property $(S_{u,\gamma})$, $\gamma \in (0,1]$ are measurables in the Lebesgue sense. The functions with the property $(S_{l,1})$ $((S'_{l,1}))$ are called approximately continuous [2] (a.e. continuous [10]). Moreover, in this case every bounded function g having the property $(S_{l,1})$ is a derivative ([2], and [8]).

Let $D \subset \mathbb{R} \times Y$ be a nonempty open set and let $f : D \longrightarrow Y$ be a function. A continuous function $g : I \longrightarrow Y$, where I is a nondegenerate interval, is called a Carathéodory solution of the Cauchy problem

$$y'(x) = f(x, y(x)), y(x_0) = y_0, \tag{1}$$

if $g'(x) = f(x, g(x))$ almost everywhere (with respect to the Lebesgue measure) on I, $x_0 \in I$, and $g(x_0) = y_0$. Obviously, if the superposition $x \longrightarrow f(x, g(x)), x \in I$, is a derivative and f is bounded, then the Carathéodory solution g of the problem (1) is an ordinary solution of (1), i.e. $g'(x) = f(x, g(x))$ everywhere on I ([7]).

From the above, by Theorems 1, 2, and 3, and by Theorem 1 from [4], p. 7 we have the following:

Theorem 6 *Let $Y = \mathbb{R}^k, D = [t_0, t_0 + a] \times \{y \in \mathbb{R}^k; |y - y_0| < b\}(a, b > 0)$, and $f : D \longrightarrow \mathbb{R}^k$ be a locally bounded function satisfying the condition (2) ((2)) [(2')] { (2') } and such that:*

- *all sections f^y are derivatives having the property $(S_{l,\gamma})$ ($(S_{u,\gamma})$) [$(S'_{l,\gamma})$] { $(S'_{u,\gamma})$ }, where $\gamma \in (0, 1]$;*

- *almost all sections f_x are continuous;*

- *there is an integrable function $h : [t_0, t_0 + a] \longrightarrow \mathbb{R}$ such that $|f(t, y)| \leq h(t)$ for every $(t, y) \in D$.*

Let

$$g(u) = \int_{t_0}^{u} h(t)dt, u \in [t_0, t_0 + a].$$

Then for every d such that $0 < d \leq a$ and $g(t_0 + d) \leq b$ there is a solution y of the Cauchy problem (1) (where $t_0 = x_0$) defined on $[t_0, t_0 + d]$ and such that its derivative y' has the property $(S_{l,\gamma})$ ($(S_{u,\gamma})$) [$(S'_{l,\gamma})$] { $(S'_{u,\gamma})$ }.

Remark 5 *Observe that all sections $f_x, x \in [t_0, t_0 + a]$, of the function f from Theorem 6 are continuous. Indeed, if a section f_x is not continuous at some point y then there is $s > 0$ such that $oscf_x(y) \geq s$. By (2) there is a set $A \in \mathcal{M}$ and $r > 0$ such that $D_l(A, x) = 1$ and $|f(t, u) - f(t, y)| < s/4$ for each $t \in A$ and $u \in Y$ with $|u - y| < r$. Since every section f^u is a derivative and $|f(t, u) - f(t, y)| < s/4$ for $t \in A$ and $u \in Y$ with $|u - y| < r$, we have that $|f(x, u) - f(x, y)| \leq s/4$, a contradiction.*

Theorem 7 *Let $Y = \mathbb{R}^k, D = [0, 1] \times U$, where U is an open ball in $\mathbb{R}^k$ with center y_0 and radius $r_0 > 0$. Let $f : D \longrightarrow \mathbb{R}^k$ be a locally bounded function satisfying the condition (2) ((2)) [(2')] { (2') } such that:*

- *all sections $f^y, y \in U$, are derivatives having the property $(S_{l,\gamma})$ ($(S_{u,\gamma})$) [$(S_{l,\gamma})$] { $(S'_{u,\gamma})$}, where $\gamma \in (0, 1]$;*

- *almost all sections f_x, $x \in [0, 1]$, are continuous;*

- *there is an integrable function* $h : [0,1] \longrightarrow \mathbb{R}$ *such that* $|f(t,y)| \leq h(t)$ *for each* $(t,y) \in D$.

Let $J = [0,T]$, *where* $0 < T \leq 1$ *be such that* $\int_0^T (h(t)+1)dt < r_0$. *Then the set of all solutions* y *of the Cauchy problem* (1) *defined on* J *and such that their derivatives* y' *have the property* $(S_{l,\gamma})$ $((S_{u,\gamma}))$ $[(S'_{l,\gamma})]$ $\{(S'_{u,\gamma})\}$ *is an* R_δ*-set in the space* $C(J,\mathbb{R}^k)$ *of all continuous functions from* J *to* $\mathbb{R}^k$ *with the norm of uniform convergence.(Recall that a subset of a metric space is called an* R_δ*-set if it is the intersection of a decreasing sequence of (nonempty) compact absolute retracts.)*

The proof of Theorem 7 follows from de Blasi's and Myjak's Theorem in [3] and from our Theorems 1, 2, 3.

Now, we assume that Y is an infinite-dimensional separable Banach space and we recollect the following notions:

- for a bounded set $A \subset Y$, $\alpha(A)$ denotes the Kuratowski α-index of the set A, i.e. the greatest lower bound of the set of such numbers r that A can be covered by a finite number of sets with the diameter not larger than r;

- we shall call a Kamke function every function $\omega : [0,a] \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that all sections ω_t, $0 \leq t \leq a$, are continuous, all sections ω^y, $y \geq 0$, are measurable (in the Lebesgue sense), $\omega(t,0) = 0$ for $0 \leq t \leq a$, and $u(t) = 0$ for $0 \leq t \leq a$ is the only continuous solution of the inequality $u(t) \leq \int_0^t \omega(s,u(s))ds$ satisfying the condition $u(0) = 0$.

Fix a Kamke function ω. Then we have:

Theorem 8 *Let* D *be a rectangle* $[0,a] \times \{y \in Y; \|y - y_0\| < b\}(a,b > 0)$. *Let* $f : D \longrightarrow Y$ *be a bounded function such that:*

- *all sections* f_x, $0 \leq x \leq a$, *are continuous;*

- *all sections* f^y, $y \in Y$ *and* $\|y - y_0\| < b$, *are derivatives having the property* $(S_{l,\gamma})$ $((S_{u,\gamma}))$ $[(S'_{l,\gamma})]$ $\{(S'_{u,\gamma})\}$, $\gamma \in (0,1]$;

- *there is* $c > 0$ *such that* $\|f(x,y)\| \leq c$ *for each* $(x,y) \in D$;

- *for each bounded set* $A \subset Y$ *and for almost every* $x \in I$,

$$\lim_{s\to 0} \alpha(f(I_{x,s},A)) \leq \omega(x,\alpha(A)),$$

where $I = [0,\beta]$, $\beta = \min(a,b/c)$, $I_{x,s} = (x-s,x+s)$;

– *f satisfies the condition (2) ((2)) [(2')] { (2') }.*

Then there exists at least one solution y of the Cauchy problem (1) *defined on* $[0, \beta]$ *such that its derivative y' has the property* $(S_{l,\gamma})$ *(*$(S_{u,\gamma})$*) [*$(S'_{l,\gamma})$*]* { $(S'_{u,\gamma})$ }.

The proof of this Theorem follows from Theorems 1, 2, 3 and from Pianigiani's Theorem in [11].

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