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SOME CONSEQUENCES OF THE FREILING – HUMKE RESULT ON THE DENSITY PROPERTY

Abstract

Recently, Freiling and Humke answered a question of the present author concerning the density property of certain sets. In this paper we examine how their work affects two different but related theorems. The first theorem concerns the monotonicity of approximately semicontinuous functions and the second, the existence of approximate maxima of approximately continuous functions.

1. Background Information and Results

In this paper we wish to discuss some of the implications of the recent results by Chris Freiling and Paul Humke [1]. In [3], the present author proved a density property of certain sets:

> If $A \subset (0,1]$ is an F_σ set which has left density 1 at all its points, then there is a point $x_0 \in B = [0,1) \setminus A$ such that A has right density 1 at x_0.

This result was used to prove a result about the monotonicity of Baire class 1 functions, (see Theorem A below). In Query 7 of Vol. 16, No.1 (1990-1991), page 376 of the *Real Analysis Exchange* the present author asks whether the F_σ condition is necessary.

Before the Freiling-Humke result the best work along these lines was the joint paper [2], where, among other interesting things it was shown that:

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Corollary 4. *Let β strictly between* 0 *and* 1 *be given and let M be a nonempty measurable subset of* $(0,1)$ *which is not an interval (of arbitrary type). Let the upper right density of M at all it points be greater than β.*

Then there is a point $z \in (0,1)\setminus M$ for which the lower left density of $M \geq \beta$.

The above result has the advantage of generality in the set M and also in requiring upper density instead of density, but doesn't answer the query. Though it could be thought of as the limiting case when $\beta \to 1$.

The work of Freiling and Humke answers Query 7 as follows :

1) Theorem 6 of [1] shows that the F_σ assumption cannot be arbitrarily dropped.

2) Theorem 5 shows that it can be replaced by the weaker condition that A be $G_{\delta\sigma}$.

We will show how their work affects two different but related theorems, Theorem A of [3] and Theorem B of [4]. Theorem A deals with monotonicity of functions which are approximately semicontinuous in a certain sense. Theorem B deals with the existence of approximate maxima of approximately continuous functions. We will assume the reader has a working knowledge of the definitions associated with metric density and approximate limits of various kinds in $\mathbb{R}$.

Theorem A *Let f be a function on* $[0,1]$ *satisfying:*

1) f is Baire class 1,

2) ap lim sup $_{x\to x_0^-} f(x) \leq f(x_0) \leq$ ap lim sup $_{x\to x_0^+} f(x)$, for every x_0.

3) interior$[f(\{x : AD^+ f(x) \leq 0\})] = \emptyset$,

Then f is nondecreasing.[1]

Since condition 1 of Theorem A is only used to insure that certain sets are F_σ, it is natural to desire to delete that condition.

Theorem B *Let f be an approximately continuous function on* $[0,1]$. *Then there is a point $x_0 \in [0,1]$ at which f has an approximate maximum. That is, at x_0 the set $\{x : f(x) > f(x_0)\}$ has density* 0.

For Theorem B, it is natural to wish to replace the requirement of approximate continuity by the weaker condition of approximate uppersemicontinuity, as is possible in the classical case for upper semicontinuous functions.

[1]In [3], functions satisfying conditions 1 and 2 are called type *.

We will show, using the example of Theorem 6 of Freiling and Humke, neither change is possible in the strict sense. However, we will also show how Theorem B can be modified to apply to certain kinds of approximately upper semicontinuous functions, i.e. those which are uniformly positively approached from below. This in turn sheds a different light on the basic question contained in Query 7, and allows us to state a new monotonicity result similar to proposition 4 of [2]. We will end by showing that Theorem 5 of [1] has a simple corollary similar to Corollary 4 of [2].

2. New Results and Examples

Let A be the set of Theorem 6 which has left density 1 at all its points and for which its complement has positive right upper density at all its points. We may assume that $1 \in A$ and $0 \notin A$.

We define a function f on $[0,1]$ as follows:

$$f(x) = \begin{cases} x+2 & \text{if } x \notin A \\ x & \text{if } x \in A. \end{cases}$$

The reader can easily verify that this function satisfies condition 2 of Theorem A. Further, for every x_0 we have that $\mathrm{AD}^+ f(x_0) \geq 1$, so that condition 3) is satisfied. However, the function is clearly not nondecreasing. Thus condition 2 alone is not sufficient to imply the validity of Theorem A. However, the main theorem of [2] can be thought of as a natural extension of A in a way that will be clearer after we show how we can extend Theorem B.

For Theorem B we note that the example of Theorem 6 actually allows us to assume that A has density 1 at all its points and still not have a point in $[0,1) \setminus A$ at which A has right density 1. For such an A, assuming again that $1 \in A$ and $0 \notin A$, we redefine our function on A:

$$f(x) = -x \; if \; x \in A.$$

Again the reader should be able to verify easily that this f will be approximately upper semicontinuous but has no approximate maxima. Thus we cannot just switch to the weaker continuity property and expect approximate maxima.

At this point we would like to indicate how weakening is possible for the approximate maxima result. The key is that positive upper density at x is not sufficient; instead we must require a certain uniformity. Namely:

Let f be such that there is a $\alpha > 0$ such that, $\{x : f(x) > y\}$ has upper density $> \alpha$ at all its points for every $y \in \mathbb{R}$. Such a function we will say is uniformly positively approached from below.

Then we may prove:

Theorem 1 *If f is approximately upper semicontinuous on $[0,1]$ and uniformly positively approached from below, then there is an $x_0 \in [0,1]$ at which f has an approximate maximum.*[2]

PROOF. Rather than copy large sections of the proof of the original theorem in [3], we will only indicate where four major changes are involved.

First, for any closed subinterval of $[0,1]$, we previously said that the image under f would be a nondegenerate interval which would not contain its right endpoint. Now since our function is not necessarily Darboux, we can only say that the image is at least two points and does not contain its supremum.

Second, the original proof constructed two sequences of strictly nested subintervals, $[a_n, b_n]$,$[c_k, d_k]$ of which the second sequence was used to determine the next term of the first sequence. We must introduce a third sequence,$[r_n, s_n]$, whose terms also depend on this $[c_k, d_k]$ sequence.

More precisely, we will have a strictly increasing sequence of numbers y_n and two sequences of closed intervals $[a_n, b_n]$, $[r_n, s_n]$ for which we have:

1) $[a_{n+1}, b_{n+1}] \subset (a_n, b_n)$,

2) $b_{n+1} - a_{n+1} \leq .5(b_n - a_n)$,

3) $\lambda(\{x : f(x) > y_n\} \cap [r_n, s_n]) > .5\alpha(r_n - s_n)$
where α is the constant of uniformity mentioned above,

4) $((a_{n+1}, b_{n+1}) \subset (r_n, s_n) \subset (a_n, b_n)$,

5) $\lambda(\{x : f(x) > y_{n+1}\} \cap (c,d)) \leq \frac{\alpha}{2^{n+1}}(d-c)$
for all (c,d) with $(c,d) \subset [a_n, b_n]$ and (c,d) containing
either a_{n+1} or b_{n+1}.

The sequence (r_n, s_n) is introduced at the point in the proof on page 79 where we previously said:

> *Therefore, there is a $\delta > 0$ such that for all open intervals J of length less than δ and containing x_1 we have*
>
> $$\lambda(\{x : f(x) > y_1\} \cap J) > \frac{1}{2}\lambda(J).$$

Now we only know there is a sequence of intervals J satisfying the desired properties that $\lambda(J)$ is small enough and satisfies the inequality. We pick an appropriate one to be (r_n, s_n) and then pick $[c_k, d_k]$ contained in that J interior and set it as (a_{n+1}, b_{n+1}).

[2] *It should be noted that approximately continuous functions satisfy the hypotheses.*

Thirdly, as might already have been noticed, we use $\alpha/2^n$ in place of $1/2^n$.

Finally, the lemma on page 76 must be modified to this new α sequence and we note that the $\lambda(H_k(y))$ function will only be nondecreasing rather than strictly decreasing. (Though its limit as y increases will still be 0.)

Any other changes are minor and the reader should be able to follow the new proof.

It is somewhat surprising that a criteria of uniform approach from below is required to get this result, but perhaps it makes clearer why the authors of [2] used a similar condition in their Theorem 1. We may also see that the following result has a standard proof and should be viewed in connection with Proposition 4 of [2]:

Theorem 2 *If f is approximately upper semicontinuous on $[0,1]$, uniformly positively approached from below, and for every x in $[0,1)$ $AD^+f(x)$ is positive, then f is increasing.*

PROOF. We need only remark that as in the original proof of Theorem B we have that:

> There is a sequence of points x_n such that f has an approximate maximum at each x_n and
>
> $$\sup\{f(x_n) : \ n = 1, 2, \cdots\} = \sup\{f(x) : \ x \in [0,1]\}.$$

From the above it is clear that the example of Theorem 6 of [1] must have the property that for every β strictly between 0 and 1 there is a point in $[0,1) \setminus A$ at which A has right lower density $\geq \beta$.

It should also be pointed out that Theorem 5 can be used to show this is always true. Namely:

Corollary 1 *Let A be any set having left density 1 at all its points, $A \subset (0,1]$; let $1 > \beta > 0$ be given. Then there is a point $x_0 \in [0,1) \setminus A$ at which A has lower right density $> \beta$.*[3]

PROOF. Suppose not. Then let $A^\beta = \{x : A$ has lower right density $> \beta$ at $x\}$. Then A^β is a $G_{\delta\sigma}$ set contained in A and of the same measure as A. Hence it has left density 1 at all its points. But then by Theorem 5 there is a point outside of A_β where it has right density 1. But then A has right density 1 at this point also. However this would require that the point belong to A_β by definition. This contradiction proves the stated result.

Thus for a general extension of the the original density property the above Corollary 4 seems the best possible result.

[3] *This corollary is obviously also implied by Corollary 4 of [2].*

References

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