Real Analysis Exchange Vol.16 (1990–91)
Richard D. Mabry, Department of Mathematics, Louisiana State University in Shreveport, Shreveport, LA 71115

Sets which are Well-Distributed and Invariant Relative to All Isometry Invariant Total Extensions of Lebesgue Measure

1 Introduction

In this paper we discuss subsets A of the real line having the property

$$\mu(A \cap J) = \alpha\ \mu(J), \tag{1}$$

for any interval J of the real line, where $0 < \alpha < 1$ and μ is an isometry-invariant extension of the usual Lebesgue measure λ on the real line. In [18], Simoson considers the notion of a set having this property, but with μ replaced by the Lebesgue outer measure λ^*. Simoson calls such a set a *comb*, and goes on to show that no comb exists. The purpose of this paper is to show that such sets do exist if the outer measure is replaced by suitable extensions of the Lebesgue measure. In particular, for any $\alpha \in (0,1)$, there are sets A, which we shall call *α-shadings* of $\mathbf{R}$, or *combs of shade α*, which have the property that for **any** finitely-additive isometry invariant extension μ of λ to $2^{\mathbf{R}}$, one has

$$\mu(A \cap E) = \alpha\ \lambda(E),$$

for any Lebesgue measurable set E. In fact, many different types of such sets are shown to exist, some having appeared in the literature as examples of non-Lebesgue measurable sets. For instance, one of the classic examples of a non-measurable set is discussed by Halmos [6], and many of the sets in this paper are generalizations of this set. Another set is due to Sierpiński [16], which was shown by Hewitt and Stromberg [8] to satisfy $\lambda^*(A \cap J) \geq \frac{1}{2}\lambda(J)$, for intervals $J \subset \mathbf{R}$. Other results concerning some of these sets have been of the form $\lambda^*(A \cap J) = \lambda(J)$, and the reader is referred to Pu [13] and Simoson [19]. The notion of an α-shading will then be generalized to that of an f-shading, where f is any continuous function mapping **R** into the closed unit interval $[0, 1]$, and these sets will also be shown to exist.

We consider sets having property (1) to be "well-distributed" in the sense of W. Rudin [14], who used the term to describe certain Lebesgue measurable sets A of reals having the property

$$0 < \lambda(A \cap V) < \lambda(V)$$

for every nonempty open set $V \subset [0, 1]$. The use of the term "shade" is a heuristic one, which is suggested by property (1) and the expression "shades of grey", where we can consider a set to be black if $\alpha = 1$, white if $\alpha = 0$, and grey if $0 < \alpha < 1$. What makes these sets especially interesting is that the parameter α associated with the shade does not depend on the particular extension of the Lebesgue measure (so long as it is an isometry invariant *total* or *universal* extension of λ, i.e., an extension to the entire power set of **R**), so such sets have a certain *invariance* property. The Soviet mathematician A.B. Harazishvili has done extensive investigation into countably-additive extensions of Lebesgue measure, and invariance relative to such measures. While most of this paper concerns itself with finitely-additive extensions, we will discuss some relations to invariance in countably-additive extensions.

We shall construct several types of shadings, including shadings that have the "Bernstein property", where the set and its complement intersect every uncountable closed subset of **R**. We will also explore some other interesting properties of these sets, and suggest some problems for future research.

2 Notation

In what follows, **Z** will denote the set of integers, **N** the set of positive integers, **R** the real numbers, **Q** the rationals in **R**, **H** the irrationals in **R**. We let λ and λ^* denote the Lebesgue measure and outer measure, respectively. For a set X, 2^X denotes the power set of X, that is, the set of subsets of X. We denote by $\mathcal{M}$ the set of all finitely-additive extensions of λ to $2^{\mathbf{R}}$ that are isometry invariant, that is, if $\mu \in \mathcal{M}$, then $\mu(A) = \mu(B)$ whenever A and B are isometric. These measures are known to exist as a consequence of the Hahn-Banach theorem (see, for example, [12]). Since so much of what we do in this paper relies on the properties of measures on disjoint unions of sets, we use the symbol $\uplus$ to emphasize disjoint unions, that is, $X \uplus Y$ denotes the set $X \uplus Y$, where $X \cap Y = \emptyset$. The complement of a set $X \subset \mathbf{R}$ is denoted by X^c. The symmetric difference of two sets X and Y is denoted by $X \triangle Y$. For two sets X and Y, we write $X \doteq Y$ if card $(X \triangle Y) < 2^{\aleph_0}$.

For $X \subset \mathbf{R}$, $Y \subset \mathbf{R}$, and $t \in \mathbf{R}$, we define $X + t$, $X + Y$, and tX as follows:

$$X + t = \{x + t : x \in X\},$$
$$tX = \{tx : x \in X\},$$
$$X + Y = \{x + y : x \in X, y \in Y\}.$$

If $S \subset \mathbf{R}$, we denote by χ_S the usual characteristic function of S.

We will have occasion to make use of the binary, or base-2 expansion of real numbers. For $x \in [0,1]$ we shall write

$$x = (.x_1x_2x_3\cdots)_2$$

when

$$x = \sum_{i=1}^{\infty} x_i 2^{-i},$$

where each x_i is either 0 or 1.

3 Definition and Existence of Shadings

We start by stating two well known facts that can be found in [6, p.69].

Lemma 3.1 If $h \in \mathbf{H}$, then the set $h\mathbf{Z} + \mathbf{Z}$ is dense in $\mathbf{R}$.

Lemma 3.2 For $h \in \mathbf{H}$, the relation $\sim$ given by

$$x \sim y \Leftrightarrow x - y \in h\mathbf{Z} + \mathbf{Z}$$

is an equivalence relation.

Definition 3.3 The above equivalence relation partitions $\mathbf{R}$ into distinct equivalence classes, so by the axiom of choice we can choose one element γ from each such class to form a set Γ. For each $h \in \mathbf{H}$, we let $\mathcal{E}(h)$ denote the class of all the different index sets that can be so constructed. The equivalence class containing γ is $\gamma + h\mathbf{Z} + \mathbf{Z}$ and we have

$$\mathbf{R} = \biguplus_{\gamma \in \Gamma} (\gamma + h\mathbf{Z} + \mathbf{Z}).$$

Definition 3.4 Let $\Gamma \in \mathcal{E}(h)$. For $M \subset \mathbf{Z}$, we define

$$K(h,\Gamma;M) = \biguplus_{\gamma\in\Gamma}(\gamma + hM + \mathbf{Z}) = \Gamma + hM + \mathbf{Z}.$$

We will abbreviate this to $K(M)$ when h and $\Gamma \in \mathcal{E}(h)$ are fixed.

For $a \in \mathbf{N}, b \in \mathbf{Z}$, we define

$$M_{a,b} = a\mathbf{Z} + b$$

and

$$K_{a,b}(h,\Gamma) = K(h,\Gamma;M_{a,b}).$$

For h and Γ fixed, we denote the latter by $K_{a,b}$.

We note some simple, yet fundamental properties of the sets $K_{a,b}$ in the following theorem, the proof of which is omitted.

Theorem 3.5 Let h and $\Gamma \in \mathcal{E}(h)$ be fixed. The following properties are satisfied:

1) $K(\mathbf{Z}) = \mathbf{R}$,

2) $K(M_1 \uplus M_2) = K(M_1) \uplus K(M_2)$,

3) $M_{a,b} = M_{a,b\,\mathrm{mod}\,a}, \quad \forall\, a \in \mathbf{N}, b \in \mathbf{Z}$,

4) $\biguplus_{0\le b<a} M_{a,b+c} = \mathbf{Z}, \quad \forall\, a \in \mathbf{N}, c \in \mathbf{Z}$

5) $\biguplus_{0\le b<d} M_{cd,cb} = M_{c,0}, \quad \forall\, c,d \in \mathbf{N}$,

6) $K_{a,b} + (ma+c)h + n = K_{a,b+c}, \quad \forall\, a \in \mathbf{N}, b,c,m,n \in \mathbf{Z}$.

Property 6) shows that the set $K_{a,b}$ is invariant under a dense set of translates, namely, those in $ah\mathbf{Z} + \mathbf{Z}$ (which is dense in $\mathbf{R}$ by lemma 3.1). In [7], a set which is invariant under a group G of isometries is called G-invariant. In [19], Simoson defines an *Archimedean set* to be a set A such that $A+r = A$ for densely many $r \in \mathbf{R}$, so we see that $K_{a,b}$ is an Archimedean set. He then shows that if such a set A has positive Lebesgue outer measure, then $\lambda^*(A \cap J) = \lambda(J)$ for any interval $J \subset \mathbf{R}$. In [18] he states, *"It would seem plausible that some exotic manipulation of the Cantor set or some wild invocation of the axiom of choice ought to yield a comb. But no such scheme exists."*

We will see, however, that all of the examples he gives of Archimedean sets in [19] are, in our sense, combs. In fact all of the examples of Archimedean sets in this paper will be seen to be combs, causing us to wonder whether or not *every* Archimedean set is necessarily a comb (see Sec. 6).

We first show that the sets $K_{a,b}$ are combs, or in our terminology, $(1/a)$-shadings.

Henceforth, unless we explicitly state otherwise, $h \in \mathbf{H}$ and $\Gamma \in \mathcal{E}(h)$ are fixed.

Theorem 3.6 Let $\mu \in \mathcal{M}$, and let $a \in \mathbf{N}$. Then for any interval $J \subset \mathbf{R}$,

$$\mu(K_{a,b} \cap J) = \frac{1}{a}\lambda(J), \quad \forall\, b \in \mathbf{Z}.$$

Proof. Let J be any nonempty, bounded interval in $\mathbf{R}$, and let $0 < \varepsilon < \mu(J)$. Now, for each $c = 1, 2, \ldots, a-1$, we have that $ah\mathbf{Z} + \mathbf{Z} + ch$ is dense in $\mathbf{R}$, so we can choose $r_c \in (0, \varepsilon) \cap (ah\mathbf{Z} + \mathbf{Z} + ch)$. Then $K_{a,b} + r_c = K_{a,b+c}$, by property 6) of Theorem 3.5. Letting $r_0 = 0$, and applying properties 1), 2) and 4) of Theorem 3.5, we have

$$\begin{aligned} \biguplus_{c=0}^{a-1} (K_{a,b} + r_c) &= \biguplus_{c=0}^{a-1} K_{a,b+c} \\ &= \biguplus_{c=0}^{a-1} K(M_{a,b+c}) \\ &= K(\biguplus_{c=0}^{a-1} M_{a,b+c} \\ &= K(\mathbf{Z}) = \mathbf{R}. \end{aligned}$$

Let $J^+ = J \cup (J + \varepsilon)$ and $J^- = J \cap (J + \varepsilon)$. Then

$$J^- \subset J + r_c \subset J^+, \quad \forall\, c = 0, 1, 2, \ldots a-1,$$

and so for each such c,

$$\begin{aligned}
\mu(K_{a,b} \cap J) &= \mu((K_{a,b} \cap J) + r_c) \\
&= \mu((K_{a,b} + r_c) \cap (J + r_c)) \\
&= \mu(K_{a,b+c} \cap (J + r_c)),
\end{aligned}$$

hence,

$$\begin{aligned}
a\ \mu(K_{a,b} \cap J) &= \sum_{c=0}^{a-1} \mu(K_{a,b+c} \cap (J + r_c)) \\
&\leq \sum_{c=0}^{a-1} \mu(K_{a,b+c} \cap J^+) \\
&= \mu(\biguplus_{c=0}^{a-1} (K_{a,b+c} \cap J^+)) \\
&= \mu((\biguplus_{c=0}^{a-1} K_{a,b+c}) \cap J^+) \\
&= \mu(\mathbf{R} \cap J^+) \\
&= \lambda(J) + \varepsilon.
\end{aligned}$$

Similarly, using J^- in place of J^+,

$$a\ \mu(K_{a,b} \cap J) \geq \lambda(J) - \varepsilon.$$

Since ε is arbitrary, it follows that

$$\mu(K_{a,b} \cap J) = (1/a)\lambda(J).$$

□

We note that the only isometry invariance of μ required in the above proof was the *translation* i.e., that $\mu(A) = \mu(A + t)$ for all $t \in \mathbf{R}$.

The above theorem motivates the following definition.

Definition 3.7 If $\alpha \in [0, 1]$ and $\mu(A \cap J) = \alpha\ \lambda(J)$ for all $\mu \in \mathcal{M}$ and all bounded intervals $J \subset \mathbf{R}$, then we call A an *α-shading* of $\mathbf{R}$, and write $\mathrm{sh}(A) = \alpha$.

Using the finite-additivity of μ, it is easy then to construct combs of any rational shade. If $p, q \in \mathbf{N}$ with $p < q$, then for

$$A = \biguplus_{i=1}^{p} K_{q,b_i},$$

where $\{b_1, b_2, \ldots, b_p\}$ is any set of distinct numbers in $\{0, 1, \ldots q-1\}$, we have that $\mathrm{sh}(A) = p/q$. The obvious question is then, can combs be constructed having *irrational* shade? The affirmative answer to this question follows as a corollary to the next simple, yet fundamental theorem, which asserts that despite the fact that μ is only *finitely*-additive, the shades of our combs are *countably*-additive under μ.

Theorem 3.8 If $\{x_i\}_{i=1}^{\infty}$ is a sequence in $(0,1)$ such that

$$\sum_{i=1}^{\infty} x_i = 1,$$

and $\{A_i\}_{i=1}^{\infty}$ is a pairwise disjoint sequence of combs such that

$$\mathrm{sh}(A_i) = x_i, \quad \text{for each } i \in \mathbf{N},$$

then for any $M \subset \mathbf{N}$, the set

$$A_M := \biguplus_{i \in M} A_i$$

is a comb, and

$$\mathrm{sh}(A_M) = \sum_{i \in M} x_i.$$

Proof. Without loss of generality, we may assume that $\biguplus_{i \in N} A_i = \mathbf{R}$. We let

$$x_M = \sum_{i \in M} x_i,$$

and let $A = A_M$ and $B = \mathbf{R} \setminus A_M$. Let J be any bounded nonempty interval in $\mathbf{R}$, let $\varepsilon > 0$, and choose $n \in \mathbf{N}$ such that

$$\sum_{i \in M, i \leq n} x_i > x_M - \varepsilon.$$

Then we have

$$\begin{aligned}\mu(A\cap J) &\geq \mu(\biguplus_{i\in M, i\leq n} A_i \cap J)\\ &= \sum_{i\in M, i\leq n} x_i\,\lambda(J)\\ &> (x_M-\varepsilon)\lambda(J),\end{aligned}$$

and since ε was arbitrary,

$$\mu(A\cap J)\geq x_M\,\lambda(J). \tag{2}$$

Similarly,

$$\mu(B\cap J)\geq (1-x_M)\lambda(J). \tag{3}$$

But if equality does not hold in either (2) or (3), then

$$\begin{aligned}\lambda(J) &= \mu(A\cap J)+\mu(B\cap J)\\ &> x_M\,\lambda(J)+(1-x_M)\lambda(J)\\ &= \lambda(J),\end{aligned}$$

which is a contradiction. Hence

$$\mu(A_M\cap J)=x_M\,\lambda(J),$$

which proves the theorem. □

Corollary 3.9 For each $x\in(0,1)$, there exists a comb with shade x.

Proof. Let $\{A_i\}_{i=1}^{\infty}$ be a sequence of disjoint combs such that $\mathrm{sh}(A_i)=2^{-i}$ for each $i\in\mathbf{N}$ (for example, we can take $A_i=K_{2^i,2^{i-1}-1}$). We write x as a binary expansion,

$$x=\sum_{i=1}^{\infty} z_i 2^{-i}=(.z_1z_2z_3\cdots)_2,$$

let $M=\{i\in\mathbf{N}: z_i=1\}$,

$$x_i=2^{-i},$$

and apply Theorem 3.8. The comb $A=\biguplus_{i\in M}A_i$ has shade x. □

Remark 3.10 It is clear that the manipulation of the combs $K_{a,b}$ basically involves exploiting the properties of the underlying sets of integers $M_{a,b}$. Intuitively, we think of $M_{a,b}$ as being "every a'th integer, beginning with b," and splitting $\mathbf{Z}$ into a "copies", i.e. translates, of $M_{a,b}$ (see property (4) of Theorem 3.5). Thus each translate occupies a fraction $1/a$ of the entire set of integers. This is the simplest example of a set of integers with so-called *natural* or *asymptotic* density, which is usually defined (for positive integers) by

$$D(M) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \chi_M(i),$$

if the limit exists (see, for example, [11]). One might expect the shade of $K(M)$ to coincide with the density of M in $\mathbf{Z}$, if it exists, but this connection is by no means obvious. For now, we simply note that each $\mu \in \mathcal{M}$ induces a finitely-additive isometry invariant measure ν_μ on $2^{\mathbf{Z}}$ as follows:

$$\nu_\mu(M) := \mu(K(M) \cap I),$$

where I is any unit interval in $\mathbf{R}$. These measures coincide with the measure on the Carathéodory extension $\mathcal{D}_\mu$ of $\mathcal{D}_0$ in the well-known paper of R.C. Buck (cf. [1, p.562]), where $\mathcal{D}_0$ is the algebra of subsets of $\mathbf{Z}$ generated by all the finite subsets of $\mathbf{Z}$ and all subsets of $\mathbf{Z}$ that are arithmetic progressions. In that paper, it is shown that the quasi-progressions

$$\{[\alpha n + \beta] : n \in \mathbf{Z}\},$$

where $\alpha > 1$ is irrational and $[\ ]$ denotes the greatest integer function, are not in the class $\mathcal{D}\mu$, but have density $(1/\alpha)$. One can readily show that this number is also the ν_μ-measure of such a quasi-progression, and so these measures agree with the density on at least certain classes of subsets of $\mathbf{Z}$.

Although Buck's measures were defined on $\mathbf{N}$, all of the basic results go through for integers. However, one problem that comes up in our context is the following: What is $\mathrm{sh}(K(\mathbf{N}))$? It is easy to verify that if $M \subset \mathbf{Z}$ is finite, then $\mathrm{sh}(K(M)) = 0$, and so we have

$$\mathrm{sh}(K(-\mathbf{N})) + \mathrm{sh}(K(\{0\})) + \mathrm{sh}(K(\mathbf{N})) = \mathrm{sh}(K(\mathbf{Z})) = 1,$$

so that

$$\mathrm{sh}(K(-\mathbf{N})) = 1 - \mathrm{sh}(K(\mathbf{N})).$$

Now, since we are dealing with isometry invariant measures, and since $-\mathbf{N}$ is isometric to $\mathbf{N}$, it is tempting to conclude that

$$\mathrm{sh}(K(-\mathbf{N})) = \mathrm{sh}(K(\mathbf{N})) = \frac{1}{2}. \tag{4}$$

But this is not at all clear, since $K(-\mathbf{N})$ need not be isometric to $K(\mathbf{N})$. However, it can be shown that $\Gamma \in \mathcal{E}(h)$ can be chosen such that Γ is "almost symmetric", in fact, such that

$$\{\gamma \in \Gamma : -\gamma \notin \Gamma\} = \left\{-\frac{1}{2}, \frac{h}{2}, \frac{1}{2} - \frac{h}{2}\right\}. \tag{5}$$

The details are rather tedious, but it is basically a matter of starting with any $\Gamma_0 \in \mathcal{E}(h)$, and then for each $\gamma \in \Gamma_0$ (except those equivalent to an element of the right hand side of (5)), replace with $-\gamma$ the element of Γ_0 which is equivalent to $-\gamma$, to form a new index set $\Gamma \in \mathcal{E}(h)$. For such Γ, one can then show that (4) is satisfied.

We conclude this section with an easy and obvious theorem, which allows us to pass from intervals to Lebesgue measurable sets. We include the proof, since we are dealing with countable collections of intervals, but an only finitely-additive measure.

Theorem 3.11 Let $\alpha \in (0,1)$ and let A be an α-shading of $\mathbf{R}$. Then for any Lebesgue measurable set $E \subset \mathbf{R}$,

$$\mu(A \cap E) = \alpha\ \lambda(E).$$

Thus, A "combs" not only intervals, but all Lebesgue measurable sets.

Proof. Let E be any Lebesgue measurable set having finite Lebesgue measure. Let $\varepsilon > 0$ and choose $\{J_i\}_{i=1}^{\infty}$ to be a sequence of intervals in $\mathbf{R}$ such that $E \subset \bigcup_{i=1}^{\infty} J_i$ and $\sum_{i=1}^{\infty} \lambda(J_i) < \lambda(E) + \varepsilon$. Let n be a positive integer such that $\sum_{i=n+1}^{\infty} \lambda(J_i) < \varepsilon$. Then we have

$$\begin{aligned} \mu(A \cap E) &\leq \mu\Big(\bigcup_{i=1}^{\infty}(A \cap J_i)\Big) \\ &\leq \sum_{i=1}^{n} \mu(A \cap J_i) + \mu\Big(\bigcup_{i=n+1}^{\infty}(A \cap J_i)\Big) \end{aligned}$$

$$
\begin{aligned}
&\leq \alpha \sum_{i=1}^{n} \lambda(J_i) + \sum_{i=n+1}^{\infty} \lambda(J_i) \\
&= \alpha \sum_{i=1}^{\infty} \lambda(J_i) + (1-\alpha) \sum_{i=n+1}^{\infty} \lambda(J_i) \\
&< \alpha(\lambda(E) + \varepsilon) + (1-\alpha)\varepsilon \\
&= \alpha\, \lambda(E) + \varepsilon.
\end{aligned}
$$

It follows that

$$\mu(A \cap E) \leq \alpha\, \lambda(E). \tag{6}$$

By considering the $(1-\alpha)$-shading, A^c, as in the previous theorem, we also have that

$$\mu(A^c \cap E) \leq (1-\alpha)\lambda(E). \tag{7}$$

Hence, if either equality in (6) or (7) fails, then as in Theorem 3.8, we get a contradiction. The theorem is thus proved for $\lambda(E) < \infty$. It follows easily for the infinite case, for if $\eta > 0$, we can choose $N > 0$ sufficiently large that $\lambda([-N, N] \cap E) > \eta/\alpha$, so that

$$\mu(A \cap E) \geq \mu(A \cap [-N, N] \cap E) = \alpha\, \lambda([-N, N] \cap E) > \eta,$$

that is, $\mu(A \cap E) = \infty$. □

4 Some Examples of Shadings

In section 3 we saw one type of a comb, which is based on disjoint unions of combs of the type $K_{a,b}$. The combs $K_{a,b}$ are, in turn, formed from countable unions of translates of the index set Γ. The set Γ is a classic example of a Lebesgue nonmeasurable set, as seen in [6, p.69], where it is then used to construct $K_{2,0}$, which is shown to have the property

$$\lambda^*(K_{2,0} \cap E) = \lambda(E),$$

for every Lebesgue measurable set E.

It turns out that other well-known examples of Lebesgue nonmeasurable sets are, in fact, combs.

Example 4.1 In [16], Sierpiński constructs a set C of irrationals, such that, if $x+y$ is rational, then exactly one of $\{x,y\}$ is in C. In [8], it is shown that if $J \subset \mathbf{R}$ is an interval, then

$$\lambda^*(C \cap J) \geq \frac{1}{2}\lambda(J). \tag{8}$$

Using essentially the same method as implemented in [8], it can easily be shown that

$$\mu(C \cap J) = \frac{1}{2}\lambda(J), \tag{9}$$

for any $\mu \in \mathcal{M}$. We omit the details here, merely noting that the inequality in (8) comes from the *subadditivity* of λ^*, which is replaced by equality in (9), due to the finite-additivity of μ. Also, in this example, it is not enough for μ to be merely translation invariant.

The above example is also found in [19] as an example of an Archimedean set (a dense set of translators is $\mathbf{Q}$). The following two examples are also found in [19].

Example 4.2 Let $\mathcal{V}$ denote a Hamel basis for $\mathbf{R}$ over the rationals $\mathbf{Q}$. Fix $v_0 \in \mathcal{V}$ (we will assume $v_0 > 0$) and let

$$W = \left\{ \sum_{i=1}^{n} r_i v_i : r_i \in \mathbf{Q}, v_i \in \mathcal{V}, v_i \neq v_0, n \in \mathbf{N} \right\}.$$

Then

$$\mathbf{R} = \biguplus_{r \in \mathbf{Q}} (W + r v_0),$$

and W is Archimedean, since

$$W = W + rv, \quad \forall\, r \in \mathbf{Q}, v \in \mathcal{V} \setminus \{v_0\}.$$

If $\mu \in \mathcal{M}$, we easily have (using only translation invariance) that

$$\mu(W \cap J) = 0,$$

for any bounded interval J. To see this, assume that J is a bounded nonempty interval and that

$$\mu(W \cap J) = t\lambda(J), \quad \text{for some } \ t \in (0,1].$$

Choose $k \in \mathbf{N}$ such that $kt > 1$, and let $0 < \varepsilon < (kt-1)\lambda(J)/v_0$. Choose $\{r_1, r_2, \ldots, r_k\}$ to be distinct elements of $\mathbf{Q} \cap (0, \varepsilon)$, and let $J^+ = \bigcup_{i=1}^{k}(J + r_i v_0)$. Then

$$\begin{aligned}
kt\lambda(J) &= k\mu(W \cap J) \\
&= \sum_{i=1}^{k} \mu((W + r_i v_0) \cap (J + r_i v_0)) \\
&\leq \sum_{i=1}^{k} \mu((W + r_i v_0) \cap J^+) \\
&= \mu(\biguplus_{i=1}^{k} (W + r_i v_0) \cap J^+) \\
&\leq \lambda(J^+) < \lambda(J) + \varepsilon v_0 < kt\lambda(J).
\end{aligned}$$

This contradiction establishes the claim. We note that this example shows that a set of positive λ^*-measure can have shade zero.

Problem 1 What can be said about $\mu(W)$, where W is as in Example 4.2 ?

Example 4.3 Let $\mathbf{Q}_{\text{odd}}$ denote the set of rationals having odd denominator when expressed in lowest terms. We define an equivalence relation $\sim$ on $\mathbf{R}$ by

$$x \sim y \Leftrightarrow x - y \in \mathbf{Q}_{\text{odd}}.$$

From each equivalence class, choose an α and let B_α denote those members x from this equivalence class for which $x - \alpha$ is of the form p/q, where p/q is in lowest terms, and both p and q are odd. Let B be the union of all such B_α. Then B is a $\frac{1}{2}$-shading of $\mathbf{R}$. To verify this claim, observe that for any odd p and q, we have

$$B^c + p/q = B.$$

Let J be any bounded interval, choose $\varepsilon \in (0, \lambda(J))$, let $J^+ = J \cup (J + \varepsilon)$, and choose p and q odd such that $p/q \in (0, \varepsilon)$. Then using only the translation invariance of μ, we have that

$$\begin{aligned}
\lambda(J) &= \mu(B \cap J) + \mu(B^c \cap J) \\
&= \mu(B \cap J) + \mu(B \cap (J + p/q))
\end{aligned}$$

$$\begin{aligned} &\leq 2\,\mu(B \cap J^{+}) \\ &< 2\,(\mu(B \cap J) + \varepsilon) \end{aligned}$$

and similarly,

$$\lambda(J) < 2\,(\mu(B^c \cap J) + \varepsilon).$$

The claim follows. Also, this example can easily be generalized to produce similar combs of any rational shade.

The following easy lemmas will facilitate the next examples.

Lemma 4.4 Let W be any set of reals such that card $W < 2^{\aleph_0}$. Then for any $k \in \mathbf{N}$, there exist reals $r_1, r_2, \ldots r_k$ such that the sets W, $W + r_1$, $W + r_2$, $\ldots$ $W + r_k$ are pairwise disjoint. Moreover, if S is any subset of $\mathbf{R}$ with card $S = 2^{\aleph_0}$, the translators $r_1, \ldots r_k$ can be chosen from S.

Proof. Let card $S = 2^{\aleph_0}$. We observe that

$$\text{card}\,(W - W) \leq (\text{card}\,W)(\text{card}\,W) < 2^{\aleph_0},$$

and choose $r_1 \in S \setminus (W - W)$. Clearly then, $W \cap (W + r_1) = \emptyset$. Next let $W_1 = W \cup (W + r_1)$ and choose $r_2 \in S \setminus (W_1 - W_1)$ and we have that the sets W, $W + r_1$ and $W + r_2$ are pairwise disjoint. This process can be continued indefinitely, proving the lemma. □

Lemma 4.5 Let W be any set of reals such that card $W < 2^{\aleph_0}$. Then for any $\mu \in \mathcal{M}$ and any bounded interval J, we have

$$\mu(W \cap J) = 0.$$

As a corollary, it follows that if $A \doteq B$, then $\mu(A \cap J) = \mu(B \cap J)$.

Proof. Assume that $\mu(W \cap J) = t\lambda(J)$ for some $t \in (0, 1]$, choose $k \in \mathbf{N}$ such that $kt > 1$, let $0 < \varepsilon < (kt - 1)\lambda(J)$, choose $r_1, r_2, \ldots r_k$ as in the previous lemma and proceed as in Example 4.2. □

In [17], Sierpiński constructs a set A of Lebesgue measure zero with the property that each translate of A is equal to A, except at countably many points (assuming the continuum hypothesis). Sets of this type are also discussed by Erdös in [3]. In [7], Harazishvili constructs a set with similar properties, and this set serves as our next example.

Example 4.6 There exists a subset A of the line with the following properties:

a) card $(A \cap F) = $ card $(A^c \cap F) = 2^{\aleph_0}$ for every closed set F with positive Lebesgue measure,

b) $(A + t) \doteq A$, for each $t \in \mathbf{R}$, and

c) $f_s(A) \doteq A^c$, for each $s \in \mathbf{R}$,

where $f_s(x) := 2s - x$ is the reflection of the point x relative to s. We claim that A is a comb with shade $\frac{1}{2}$. To see this, let J be any nonempty bounded interval, and choose the unique $s \in \mathbf{R}$ such that $f_s(J) \doteq J$. In view of property c), we must have

$$f_s(A \cap J) = f_s(A) \cap f_s(J) \doteq A^c \cap J,$$

and so,

$$\begin{aligned} 2\,\mu(A \cap J) &= \mu(A \cap J) + \mu(f_s(A \cap J)) \\ &= \mu(A \cap J) + \mu(A^c \cap J) \qquad \text{(by Lemma 4.5.)} \\ &= \mu(J), \end{aligned}$$

and our claim follows.

Incidentally, it might be noticed that we did not use properties a) or b). In fact, all that is needed is a weakened form of c), wherein $f_s(A) \doteq A^c$ for densely many $s \in \mathbf{R}$.

The set A from the previous example could be called "almost-Archimedean," in that $A \doteq A + t$ for densely many t. The fact that this set of translators is not only dense in $\mathbf{R}$, but is *all* of $\mathbf{R}$, seems to be a trade-off — if we weaken the requirement that $A = A + t$ for densely many t, and instead require only that $A \doteq A + t$, for densely many t , then we can have *uncountably* many such t. On the other hand, as was pointed out in Example 4.2, one can have uncountably many t such that $A = A+t$, if A has zero shade. The question of whether such restrictions are necessary we answer in the negative by means of the following examples.

Example 4.7 Let $\alpha \in (0, 1)$. We shall construct a set A with the following properties:

a) A is Archimedean,

b) $(A+t)=A$ for $2^{\aleph_0}$ many $t \in \mathbf{R}$, and

c) A has shade α.

Using the notations of Example 4.2 it seems plausible that a subset Q_A of the rationals might be chosen such that $A := W + Q_A v_0$ is a comb of positive shade, since $\mathbf{R} = W + \mathbf{Q} v_0$. This turns out to be the case. In fact, let

$$Q_A = \mathbf{Q} \cap \bigcup_{m \in \mathbf{Z}} [m, m+\alpha).$$

It is clear that $A + w = A$ for each $w \in W$, and that card $W = 2^{\aleph_0}$. If α is equal to $1/q$ for some $q \in \mathbf{N}$, then it is easy to see, using methods previously employed, that

$$\biguplus_{k=0}^{q-1} (A + \frac{k}{q} v_0) = \mathbf{R},$$

whence $\operatorname{sh}(A) = 1/q$. We can then easily pass to α of the form p/q, where $p, q \in \mathbf{N}$. Finally, each $\alpha \in (0,1)$ is contained in an arbitrarily small interval with rational endpoints, say $\alpha \in (p/q, (p+1)/q)$, from which it is easily shown that $\operatorname{sh}(A) \in (p/q, (p+1)/q)$, and the claim follows. We note in passing that this also shows that an Archimedean set can have irrational shade. Also, this set has the interesting property of being a nontrivial set such that $\{A + t : t \in \mathbf{R}\}$ is only a countable family of sets.

Example 4.8 Let $A = W + Q_A v_0$, where $Q_A = \mathbf{Q} \cap (0, \infty)$, and the notation is as in the previous example. Then

a) A is Archimedean,

b) $(A+t)=A$ for $2^{\aleph_0}$ many $t \in \mathbf{R}$,

c) $\operatorname{sh}(A) = \frac{1}{2}$, and

d) $\operatorname{sh}(A \,\triangle\, (A+t)) = 0, \quad \forall\, t \in \mathbf{R}$.

The proof is straightforward and we omit it.

Remark 4.9 We note that a set A having the property

$$A + t \doteq A, \quad \forall\, t \in \mathbf{R} \tag{10}$$

can also be Archimedean. In fact, let A be any set satisfying (10). We define the set B by

$$B = \bigcup_{q \in \mathbf{Q}} (A + q) = A + \mathbf{Q}.$$

It is easy to see that B satisfies (10), and that $B + q = B, \quad \forall\, q \in \mathbf{Q}$, so B is Archimedean. Also, if A is comb with shade α, then so is B, since $A \doteq B$ (by Lemma 4.5).

5 Some Properties of Shadings

From the construction of the combs $K_{a,b}$, we can see that if $r_1, r_2, \ldots, r_n$ are rationals in $(0,1)$ such that $r_1 + r_2 + \cdots + r_n = 1$, then there exist disjoint combs $C_1, C_2, \ldots, C_n$ such that $\biguplus_{i=1}^{n} C_i = \mathbf{R}$ and $\mathrm{sh}(C_i) = r_i$ for each i. We simply write $r_i = p_i/q$, where $1 \le p_i < q$ for each i, and q is a common denominator, and then consider disjoint unions of p_i combs having shade $1/q$. The question immediately arises as to whether the r_i can be irrational, and we provide an affirmative answer with the following.

Theorem 5.1 Given $\{x_i\}_{i=1}^{\infty} \subset (0,1)$ such that $\sum_{i=1}^{\infty} x_i = 1$, there exist disjoint combs $\{C_i\}_{i=1}^{\infty}$ such that $\mathrm{sh}(C_i) = x_i$ for each $i \in \mathbf{N}$.

Proof. We again make use of the binary expansion of each x_i, with the condition that each expansion is non-terminating, e.g., $\frac{1}{2} = (.011111\ldots)_2$. We write

$$x_1 = (.x_{11}x_{12}x_{13}\cdots)_2 = \sum_{j=1}^{\infty} x_{1j}2^{-j}$$
$$x_2 = (.x_{21}x_{22}x_{23}\cdots)_2 = \sum_{j=1}^{\infty} x_{2j}2^{-j}$$
$$\vdots$$
$$x_i = (.x_{i1}x_{i2}x_{i3}\cdots)_2 = \sum_{j=1}^{\infty} x_{ij}2^{-j}$$

where each $x_{ij} = 0$ or 1. For each $j \in \mathbf{N}$, let $L_j = \{(i,j) : x_{ij} = 1\}$. Since each L_j is a finite set, we can order $\bigcup_{j=1}^{\infty} L_j$ by listing the elements of L_1, followed by those of L_2, etc. Let $L = \{L_1; L_2; \cdots\}$ denote this concatenation.

We now construct a disjoint family of sets of integers corresponding to the elements of L. For each $k \in \mathbf{N}$, let $\alpha(k)$ denote the k'th element of L, and let p_k be the power of 2 corresponding to $\alpha(k)$, i.e., $p_k = 2^j$ if and only if $\alpha(k) \in L_j$. Then $p_1 \le p_2 \le p_3 \le \cdots$ and we let $N_1 = M_{p_1,0}$ (see def. 3.4). For $k > 1$, define N_k recursively by letting n_k be the first positive integer not contained in $N_1 \cup N_2 \cup \cdots \cup N_{k-1}$, and letting $N_k = M_{p_k,n_k}$. It is easily verified that the N_k are well-defined and that they are pairwise disjoint, with $\biguplus_{k=1}^{\infty} N_k = \mathbf{Z}$. Then the combs $K(N_k)$ are pairwise disjoint, with $\biguplus_{k=1}^{\infty} K(N_k) = \mathbf{R}$, and $\text{sh}(K(N_k)) = 1/p_k$. Finally, for each fixed $i \in \mathbf{N}$, let

$$C_i = \biguplus \{K(N_{\alpha^{-1}(i,j)}) : j \in \mathbf{N}, x_{ij} = 1\}.$$

By Theorem 3.8, it is clear that $\text{sh}(C_i) = x_i$, and our claim is proved.

We note that the theorem is obviously true for finite sums $\sum_{i=1}^{N} x_i = 1$, as well as sums less than one. □

In taking the union of disjoint combs to form new combs, we see that these unions can be thought of as having the original disjoint combs as *subcombs.* With the exception of Example 4.1, the author has found fairly easy means of forming subcombs of the examples found in this paper, but it is not at all clear that an arbitrary comb has nontrivial subcombs, and we will have to leave this question unanswered for now:

Problem 2 If A is a comb with $\text{sh}(A) = a$, and b is a real number in the open interval $(0, a)$, does there exist a comb $B \subset A$ such that $\text{sh}(B) = b$? (It suffices, in view of the proof of Corollary 3.9, to find B with $\text{sh}(B) = \frac{1}{2}a$.)

The previous theorem does give us an easy way to construct *examples* of sets A and B having the property described in Problem 2. For $0 < b < a < 1$, we need only construct disjoint combs B and C of shades b and $a - b$, respectively, and let $A = B \cup C$. We can also construct systems of combs having other inclusion properties. For instance, for $0 < x < \frac{1}{2}$, we can construct combs A and B each with shade $\frac{1}{2}$ and such that $A \cap B$ is a comb of shade x. We simply let D_1, D_2, and D_3 be disjoint combs of shades $\frac{1}{2} - x, \frac{1}{2} - x$, and x, respectively, and then let $A = D_1 \cup D_3$ and $B = D_2 \cup D_3$. We can easily generalize this to systems of combs having any "admissible" intersection properties, by means of the following corollary to the previous theorem.

Corollary 5.2 Let $n \in \mathbf{N}$ and let $\{0,1\}^n$ denote the set of 2^n n-tuples with coordinates either 0 or 1. Let

$$v : \{0,1\}^n \to [0,1]$$

be such that

$$\sum_{x \in \{0,1\}^n} v(x) = 1.$$

Then there exist combs $\{C_i\}_{i=1}^n$ such that the following holds. If

$$x = (x_1, x_2, \ldots, x_n) \in \{0,1\}^n,$$

let

$$L(x) = \bigcap_{i=1}^{n} L_i(x),$$

where

$$L_i(x) = \begin{cases} C_i & \text{if } x_i = 1 \\ \mathbf{R} \setminus C_i & \text{if } x_i = 0. \end{cases}$$

Then $L(x)$ has shade $v(x)$.

The proof is easy, and we omit it.

We illustrate the above corollary with the following two examples.

Example 5.3 We construct combs C_1, C_2, C_3 with shades 2/5, 2/5, 4/5, respectively, such that the sets $C_1 \cap C_2$, $C_1 \cap C_3$, $C_2 \cap C_3$, $C_1 \cap C_2 \cap C_3$ are combs with shades 1/10, 2/5, 3/10, and 1/10, respectively.

In the notation of the above corollary, we let $n = 3$ and

$$v(0,0,0) = 1/10 \quad v(0,0,1) = 1/5 \quad v(0,1,0) = 1/10 \quad v(0,1,1) = 1/5$$
$$v(1,0,0) = 0 \quad v(1,0,1) = 3/10 \quad v(1,1,0) = 0 \quad v(1,1,1) = 1/10.$$

Example 5.4 Let $v_1, v_2, \ldots, v_n$ be reals in $(0,1)$. Then there exist combs $C_1, C_2, \ldots, C_n$ with the following "probabalistic independence" property: For any set M of distinct integers from $\{1,2,\ldots,n\}$,

$$\mathrm{sh}(\bigcap_{j\in M} C_j) = \prod_{j\in M} v_j.$$

This follows from the previous corollary with $v : \{0,1\}^n \to [0,1]$ given by

$$v(x) = \prod_{i=1}^{n} y_i, \text{ where } y_i = \begin{cases} v_i & \text{if } x_i = 1 \\ 1 - v_i & \text{if } x_i = 0. \end{cases}$$

We should also mention that one need not be restricted to finite collections of combs. Here is a denumerable collection with an independence property:

Example 5.5 We construct a sequence of combs $\{C_i\}_{i=1}^{\infty}$ such that $\mathrm{sh}(C_i) = \frac{1}{2}$ for each i, and such that the intersection of any n of these sets or their complements has shade 2^{-n}.

For each i, define $N_i \subset \mathbf{Z}$ by

$$N_i = \biguplus_{j=0}^{2^{i-1}-1} M_{2^i,j}.$$

Then N_i is merely a block of 2^{i-1} integers (starting at zero), followed by a gap of equal size, then a block, and so on. Let $C_i = K(N_i)$. It is easy to verify that these C_i have the above stated properties.

The previous corollary and examples illustrate that in special cases, one can have very nice intersection properties of combs. But the general situation is much more complicated, in that the intersection of two arbitrary combs can result in most unusual sets, even if the two combs to be intersected have identical shade. In some cases, we are guaranteed that the intersection will be a comb of positive Lebesgue outer measure. In fact, if A and B are combs with

$$\mathrm{sh}(A) = a \in (0,1), \ \mathrm{sh}(B) = b \in (0,1), \text{ and } a + b > 1,$$

then for any bounded nonempty interval J,

$$\mu(A \cap B \cap J) \leq \min\,(a,b)\mu(J) < \mu(J),$$

and

$$\begin{aligned}\mu(A\cap B\cap J) &= \mu(J)-\mu((A^c\cup B^c)\cap J)\\ &> \mu(J)(a+b-1)\\ &> 0,\end{aligned}$$

so that

$$0<\mu(A\cap B\cap J)<\mu(J).$$

Hence, $A\cap B$ is a comb with positive shade, *relative to* μ, and hence has positive Lebesgue outer measure. But we cannot guarantee that the measure of $A\cap B\cap J$ is independent of μ. And it is clear that we cannot expect that $A\cap B$ is a comb with *constant shade*, as we illustrate in the next example.

Example 5.6 Let C_1 and D_1 be combs with

$$\text{sh}(C_1)=\text{sh}(D_1)=3/4 \text{ and } \text{sh}(C_1\cap D_1)=1/4,$$

and let C_2 and D_2 be combs with

$$\text{sh}(C_2)=\text{sh}(D_2)=3/4 \text{ and } \text{sh}(C_2\cap D_2)=1/2$$

(we have made use of the Corollary to Theorem 5.1). Then letting

$$\begin{aligned}A &= ((-\infty,0]\cap C_1)\cup((0,\infty)\cap C_2),\\ B &= ((-\infty,0]\cap D_1)\cup((0,\infty)\cap D_2),\end{aligned}$$

we see that A and B are combs with shade $3/4$, but that

$$\mu(A\cap B\cap J)=\begin{cases}\mu(C_1\cap D_1\cap J)=(1/4)\mu(J) & \text{if } J\subset(-\infty,0]\\ \mu(C_2\cap D_2\cap J)=(1/2)\mu(J) & \text{if } J\subset(0,\infty).\end{cases}$$

Thus $A\cap B$ is a comb whose shade is not constant on $\mathbf{R}$.

We can see that by using Theorem 5.1 and taking intersections with intervals, as in the above example, we can build combs whose shades are given by step functions. The next question is then, can we construct combs whose shade varies continuously, by passing to smaller and smaller intervals? The affirmative answer to this question is the result of the next theorem.

Theorem 5.7 Let $f : \mathbf{R} \to [0,1]$ be continuous. Then there exists a set $F \subset \mathbf{R}$ such that

$$\lim_{\mu(J(x))\to 0} \frac{\mu(F \cap J(x))}{\mu(J(x))} = f(x), \quad \forall\, x \in \mathbf{R},$$

where $J(x)$ denotes an interval containing x.

Proof. For each $n \in \mathbf{N}$, we define a two-valued simple function

$$f_n : \mathbf{R} \to \{0, 2^{-n}\}$$

as follows. Let $f_0 \equiv 0$ on $\mathbf{R}$ and for $n \geq 1$, let

$$S_n = \{x \in \mathbf{R} : f(x) - \sum_{i=0}^{n-1} f_i(x) > 2^{-n}\},$$

and let

$$f_n(x) = \begin{cases} 2^{-n} & \text{if } x \in S_n \\ 0 & \text{if } x \notin S_n. \end{cases}$$

It is clear that for each n,

$$S_n = \biguplus_{i=1}^{2^{n-1}} f^{-1}\left(\frac{2i-1}{2^n}, \frac{2i}{2^n}\right]$$

and that $\sum_{i=1}^{n} f_i(x) \to f(x)$ uniformly on $\mathbf{R}$ as $n \to \infty$. We let

$$F = \biguplus_{n=1}^{\infty} C_n \cap S_n,$$

where $\{C_n\}_{n=1}^{\infty}$ is any pairwise disjoint family of combs such that $\mathrm{sh}(C_n) = 2^{-n}$ for each $n \in \mathbf{N}$. It can readily be shown that F has the desired properties, and we omit the details. □

The continuity of f ensures that the limit need only involve intervals containing the point x. If f were only, say, piecewise continuous, this limit could differ to the left or right of a discontinuity. This motivates the following definition.

Definition 5.8 Let $F \subset \mathbf{R}$, and let $\mu \in \mathcal{M}$. For each $x \in \mathbf{R}$, we define the *upper right μ-shade* $\overline{\mathrm{sh}}_\mu^+(F)(x)$ of F at x, and the *lower right μ-shade* $\underline{\mathrm{sh}}_\mu^+(F)(x)$ of F at x, by

$$\overline{\mathrm{sh}}_\mu^+(F)(x) = \limsup_{h\to 0+} \frac{\mu(F \cap [x, x+h))}{h}$$

and

$$\underline{\mathrm{sh}}_\mu^+(F)(x) = \liminf_{h\to 0+} \frac{\mu(F \cap [x, x+h))}{h}.$$

Similarly, we define the *upper left μ-shade* and *lower left μ-shade* of F at x by

$$\overline{\mathrm{sh}}_\mu^-(F)(x) = \limsup_{h\to 0+} \frac{\mu(F \cap (x-h, x])}{h}$$

and

$$\underline{\mathrm{sh}}_\mu^-(F)(x) = \liminf_{h\to 0+} \frac{\mu(F \cap (x-h, x])}{h},$$

respectively. If these quantities do not depend on the particular $\mu \in \mathcal{M}$, we call them the *upper right shade* $\overline{\mathrm{sh}}^+(F)(x)$ of F at x, etc. If we define the function $F_0 : \mathbf{R} \to [0, \infty)$ by

$$F_0(x) = \begin{cases} \mu([0,x) \cap F) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ \mu((x,0] \cap F) & \text{if } x < 0, \end{cases}$$

then we recognize the above four defined quantities as the so-called *derivates* of F_0, that is,

$$\overline{\mathrm{sh}}_\mu^+(F)(x) = D^+F_0(x) := \limsup_{h\to 0+} \frac{F_0(x+h) - F_0(x)}{h},$$

$$\underline{\mathrm{sh}}_\mu^+(F)(x) = D_+F_0(x) := \liminf_{h\to 0+} \frac{F_0(x+h) - F_0(x)}{h},$$

$$\overline{\mathrm{sh}}_\mu^-(F)(x) = D^-F_0(x) := \limsup_{h\to 0+} \frac{F_0(x) - F_0(x-h)}{h},$$

$$\underline{\mathrm{sh}}_\mu^-(F)(x) = D_-F_0(x) := \liminf_{h\to 0+} \frac{F_0(x) - F_0(x-h)}{h}.$$

If $\overline{\mathrm{sh}}_\mu^+(F)(x) = \underline{\mathrm{sh}}_\mu^+(F)(x)$, then the common value may be called the *right μ-shade* $\mathrm{sh}_\mu^+(F)(x)$ of F at x, and similarly we may define the *left μ-shade*

$\mathrm{sh}_\mu^-(F)(x)$ of F at x. In turn, if these two quantities exist and are equal, we call the common value the *μ-shade* $\mathrm{sh}_\mu(F)(x)$ of F at x. Again, if these three quantities are independent of μ, then they may be called the *right shade* $\mathrm{sh}^+(F)(x)$, the *left shade* $\mathrm{sh}^-(F)(x)$, and the *shade* $\mathrm{sh}(F)(x)$ of F at x, respectively. These quantities are the right derivative, left derivative, and derivative, respectively, of F_0 at x. Well known facts from analysis guarantee us that if F is any subset of $\mathbf{R}$, then our function F_0 is nondecreasing, so F_0 is differentiable almost everywhere. Thus the shade of F is defined at almost all points of $\mathbf{R}$. If F is Lebesgue measurable, then the upper and lower shades of F can take only the values 0 or 1 (cf. [18]).

We make several remarks.

Remark 5.9 We note that our use of the term "shade of A" prior to the last definition has been consistent, since the shades of our combs have been constant functions, with the exception of Example 5.6. In that example, the shade of $A \cap B$ is not defined at $x = 0$, however, the left and right shades are:

$$\mathrm{sh}^+(A \cap B)(x) = \begin{cases} 1/4 & \text{if } x < 0 \\ 1/2 & \text{if } x \geq 0, \end{cases}$$

$$\mathrm{sh}^-(A \cap B)(x) = \begin{cases} 1/4 & \text{if } x \leq 0 \\ 1/2 & \text{if } x > 0. \end{cases}$$

Remark 5.10 We can restate Theorem 5.7 as follows:

Given a continuous function $f : \mathbf{R} \to [0,1]$, there exists $F \subset \mathbf{R}$ such that $\mathrm{sh}(F) = f$.

Remark 5.11 It is easy to see that the above is also true for piecewise continuous functions, so long as either $f(x^+) = f(x)$ or $f(x^-) = f(x)$ at each x.

Remark 5.12 It is not difficult to see that since $\mathrm{sh}_\mu(F)$ is always a Lebesgue measurable function, it follows that for any Lebesgue measurable set E, we have

$$\mu(E \cap F) = \int_E \mathrm{sh}_\mu(F)\, d\lambda.$$

This generalizes Theorem 3.11.

Remark 5.13 Results analagous to Theorem 5.1 and its corollary remain true with shades that are piecewise continuous functions. An affirmative answer to Problem 2 would imply its truth with shades a and b replaced by piecewise continuous functions.

6 Archimedean Sets

As was remarked earlier, all of the Archimedean sets mentioned thus far have been combs of constant shade. We shall see that if an Archimedean set is a shading, then its shade is constant. But is every Archimedean set a shading? We only partially answer this question.

Theorem 6.1 Let $\mu \in \mathcal{M}$ and let A be an Archimedean set. Then for any bounded nonempty interval J, the quantity

$$\frac{\mu(A \cap J)}{\mu(J)}$$

is a constant independent of J. Hence, $\mathrm{sh}_\mu(A)$ exists and is constant on $\mathbf{R}$.

Proof. We only sketch the proof, since it is similar to that of Theorem 3.6. Let $\tau(A)$ denote the set of Archimedean translators of A, that is,

$$\tau(A) = \{t \in \mathbf{R} : A + t = A\}.$$

First, assume that J_1 and J_2 are two nonempty bounded intervals of equal length. Then it is easy to see that $\mu(A \cap J_1) = \mu(A \cap J_2)$, using the fact that $\tau(A)$ is dense in $\mathbf{R}$, and the translation invariance of μ. Therefore the theorem is true for intervals of unit length. From here it is easy to pass to intervals of rational length, and then using a limiting argument, to intervals of any finite length. □

What we have not shown, of course, is that the constant referred to in the statement of the above theorem is independent of the choice of μ, and we must leave this unsolved:

Problem 3 Is every Archimedean set a shading?

It is easy to see that for an Archimedean set A, the set of Archimedean translators $\tau(A)$ is an additive group (in [15] Archimedean sets are mentioned as special cases of locally compact abelian groups which have a character whose kernel is dense and not Haar measurable). One can then define an equivalence relation on $\mathbf{R}$ by means of

$$x \sim y \Leftrightarrow x - y \in \tau(A).$$

We can write

$$\mathbf{R} = \biguplus_{\gamma \in \Gamma} A_\gamma,$$

for some index set $\Gamma \subset \mathbf{R}$, where $A_\gamma = \gamma + \tau(A)$ for each $\gamma \in \Gamma$. We see that each A_γ is Archimedean with $\tau(A_\gamma) = \tau(A)$, and so for any $\Gamma' \subset \Gamma$, the set

$$\Gamma' + \tau(A) = \biguplus_{\gamma \in \Gamma'} A_\gamma$$

is Archimedean. We also have that

$$A = \biguplus_{\gamma \in \Gamma \cap A} A_\gamma, \tag{11}$$

and

$$A^c = \biguplus_{\gamma \in \Gamma \cap A^c} A_\gamma.$$

An elementary group-theoretic argument shows that Γ cannot be finite, and we conclude that each A_γ also has zero shade (for any $\mu \in \mathcal{M}$).

Assume that A is Archimedean and that $\mathrm{sh}_\mu(A) = a \in (0,1)$. Given the facts of the preceeding paragraph, especially (11), it is not unreasonable to conjecture, at least for a fixed $\mu \in \mathcal{M}$, that for $b \in (0,a)$, some subset Γ_b of $\Gamma \cap A$ might be chosen such that

$$\mathrm{sh}_\mu\Big(\biguplus_{\gamma \in \Gamma_b} A_\gamma\Big) = b.$$

In other words, we have another problem:

Problem 4 For a fixed $\mu \in \mathcal{M}$, do Archimedean sets necessarily have every (or even any) μ-shade of subcomb?

7 Shadings in Other Extensions of λ

In [7, p.117], Harazishvili gives an example of a set $A \subset \mathbf{R}$ which has the property

$$\nu(A \cap E) = \frac{1}{2}\lambda(E) \tag{12}$$

for any Lebesgue measurable set $E \subset \mathbf{R}$, where ν is any isometry invariant extension of λ containing A in its domain of definition (see Example 4.6). But he points out that extensions ν which are only translation invariant need not satisfy (12). In [6, p.71], it is pointed out that one can extend λ to measures, though not necessarily translation invariant, on σ-algebras of the form

$$\{(E_1 \cap A) \cup (E_2 \cap A^c) : E_1, E_2 \text{ are Lebesgue measurable}\},$$

where A is as above, by defining ν by means of

$$\nu((E_1 \cap A) \cup (E_2 \cap A^c)) = a\ \lambda(E_1) + b\ \lambda(E_2), \tag{13}$$

where a and b are any two numbers in $[0,1]$ for which $a + b = 1$. It is clear, however, that such extensions cannot be further extended to finitely-additive isometry invariant measures on $2^{\mathbf{R}}$ unless $a = b = \frac{1}{2}$.

Many translation invariant extensions of λ are known, most notably those due to Kakutani and Oxtoby [5], who obtained extensions to very large σ-algebras. (See [2] for an extensive bibliography on this subject.)

While the methods used in [5] are fairly advanced, a relatively easy method, similar to the method used to obtain (13), can be used to create non-trivial translation invariant extensions of λ. Let us say that a set A has the *Bernstein property* if

$$A \cap F \neq \emptyset \neq A^c \cap F \text{ for every uncountable closed } F \subset \mathbf{R} \tag{14}$$

(cf. [12, Problem 2.4.5]). In Theorem 2.8 of [12], it is claimed that if A satisfies this property, then one can define a translation invariant extension ν of λ on the σ-algebra $\mathcal{S}_A$ generated by A and the Lebesgue measurable subsets of $\mathbf{R}$ by setting

$$\nu((E_1 \cap A) \cup (E_2 \cap A^c)) = \frac{1}{2}(\lambda(E_1) + \lambda(E_2)),$$

where E_1 and E_2 are Lebesgue measurable. The problem with this is that for a measure to be translation invariant, the σ-algebra on which it is defined must also be translation invariant, and this need not be the case with $\mathcal{S}_A$, as we shall see. In private communications, Professor Mukherjea has pointed out that the claim of the theorem is at least true for all translations of sets which do belong to $\mathcal{S}_A$. While this author agrees, it turns out that we can construct A to be such that *none* of the members of $\mathcal{S}_A$ (except the Lebesgue measurable ones) belong to $\mathcal{S}_A$ under non-trivial translation. Before proceeding to verify these claims, we point out that the theorem in [12] can be repaired by taking A to be the set in Example 4.6, since property b) in that example guarantees that the σ-algebra so induced is translation invariant (for all translators). It is not claimed that this A satisfies the Bernstein property, but only the weaker property that both A and A^c have nonempty intersection with every closed set of positive Lebesgue measure. But the stronger property is not necessary for constructing the extension of λ. In fact, if it were not for this restriction, our counterexample would be very easy indeed, since all combs of positive shade necessarily have uncountable intersection with every set of positive Lebesgue measure.

To verify the claims above, we take the trouble to include a very general and useful theorem, which is interesting in its own right. It was inspired by the construction of Example 4.6 in [7].

Theorem 7.1 Let $\sim$ denote an equivalence relation on a set E, where E has cardinality $2^{\aleph_0}$, and each equivalence class is countable. For $x \in E$, let S_x denote the equivalence class containing x. Let Ω denote the least ordinal number having cardinality $2^{\aleph_0}$, and let Φ denote any family of subsets of E such that

$$\bigcup_{F \in \Phi} F = E,$$

$$\text{card}\ (\Phi) = 2^{\aleph_0},$$

and where

$$\text{card}\ (F) = 2^{\aleph_0}, \quad \text{for each } F \in \Phi.$$

Finally, let

$$\{F_\alpha\}_{\alpha<\Omega}$$

denote a transfinite sequence of all the elements of Φ, where each element is indexed $2^{\aleph_0}$ many times, that is,

$$\text{card}\ \{\alpha < \Omega : F_\alpha = F\} = 2^{\aleph_0}, \quad \text{for each } F \in \Phi.$$

Then there exists a family

$$\{e_\alpha\}_{\alpha<\Omega}$$

such that

a) $e_\alpha \in F_\alpha, \quad \forall\, \alpha \in [0,\Omega)$,

b) $S_{e_\alpha} \cap S_{e_\beta} = \emptyset, \quad \forall\, \alpha,\beta \in [0,\Omega), \alpha \neq \beta$, and

c) $E = \bigcup_{\alpha<\Omega} S_{e_\alpha}$.

Proof. Let $\{x_\beta : \beta < \Omega\}$ be a well ordering of E and put $e_\alpha = x_\gamma$, where

$$\gamma = \min\{\xi : x_\xi \in F_\alpha \setminus \bigcup_{\beta<\alpha} S_{e_\beta}\}.$$

Then it is easy to verify, using transfinite induction, that these e_α satisfy the requirements of the theorem. □

Corollary 7.2 An index set $\Gamma \in \mathcal{E}(h)$ (see 3.3) can be chosen such that

$$\text{card } (\Gamma \cap F) = 2^{\aleph_0},$$

for each closed set F with card $(F) = 2^{\aleph_0}$.

Proof. Let $E = R$, and let

$$\Phi = \{F \subset R : F \text{ is closed and } \text{ card } F = 2^{\aleph_0}\}.$$

Then the preceeding theorem applies. □

We now construct the example which refutes the claim in [12].

Example 7.3 For any $h \in \mathbf{H}$, we use the above corollary to choose $\Gamma \in \mathcal{E}(h)$ such that Γ intersects every uncountable closed subset of $\mathbf{R}$. The set $\Gamma + t$ will also have this property, for any $t \in \mathbf{R}$, and it follows that $K_{a,b}(h,\Gamma)$ has the Bernstein property, for any $a, b \in \mathbf{N}, a > 1$. Let $f : \mathbf{R} \to (\frac{1}{2}, \frac{3}{4})$ be continuous and strictly increasing, and let A be an f-shading constructed using the disjoint combs

$$\{K_{2^i,2^{i-1}-1}(h,\Gamma)\}_{i=1}^{\infty}$$

as a "basis" (see Def. 3.4 and Theorem 5.7). Then A has the Bernstein property, since

$$\Gamma \subset K_{2,0}(h,\Gamma) \subset A,$$

and

$$\Gamma + 3h \subset K_{4,3}(h,\Gamma) \subset A^c.$$

Let $\mathcal{S}_A$ denote the σ-algebra

$$\{(E_1 \cap A) \cup (E_2 \cap A^c) : E_1, E_2 \text{ are Lebesgue measurable}\}.$$

We omit the details here, but it is not very difficult to verify that the assumption

$$((E_1 \cap A) \cup (E_2 \cap A^c)) + t = (E_3 \cap A) \cup (E_4 \cap A^c),$$

for some Lebesgue measurable E_1,E_2,E_3, E_4 and some $t \in \mathbf{R}$, leads to a contradiction.

Remark 7.4 If we let $A = K_{3,0}(h,\Gamma)$, where Γ is as above, then no contradiction arises in defining ν as in (12), if we observe the restriction that the translators are from the set $3h\mathbf{Z} + \mathbf{Z}$. However, all of the translators $h\mathbf{Z} + \mathbf{Z}$ are are also invariant, and if they are to be included, then the σ-algebra includes all of the invariant sets of the form

$$(A_0 \cap E_0) \cup (A_1 \cap E_1) \cup (A_2 \cap E_2),$$

where $A_0 = A, A_1 = A_0 + h, A_2 = A_0 + 2h$, and we must have

$$\nu((A_0 \cap E_0) \cup (A_1 \cap E_1) \cup (A_2 \cap E_2)) = \frac{1}{3}(\lambda(E_0) + \lambda(E_1) + \lambda(E_2)). \quad (15)$$

Invariance of this type, where the set of invariant isometries form a subgroup G of all isometries on the space, is referred to in [7] as *G-invariance* and measures such as ν in (15) are called *G-measures.*

Remark 7.5 The above construction shows how easy it is to obtain sets which have the Bernstein property. In fact, for any piecewise continuous function $f : \mathbf{R} \to [0,1]$, one can construct an f-shading with the Bernstein property. This might lead one to wonder whether Theorem 7.1 and its corollary are really necessary to achieve this property. That they are necessary can be seen[1] by noting that there exists a Γ containing a perfect set ([10]). Also, sets like the set A of Example 4.6 can fail to have the Bernstein property, since there exists a Hamel basis containing a perfect set ([4], [9, pp.220-221]).

Remark 7.6 Sets with the properties of the set A in Example 4.6 can certainly be constructed that do have the Bernstein property, and had this been required rather than property a), the author of [7] could have included it at no extra cost. We sketch the construction of this set as an application of Theorem 7.1, and to give the reader a better understanding of the structure of the set in Example 4.6 to which we have made so many references. We do this by means of the following corollary to Theorem 7.1.

Corollary 7.7 A Hamel basis $\mathcal{V}$ for $\mathbf{R}$ over $\mathbf{Q}$ exists for which

$$\text{card } (\mathcal{V} \cap F) = 2^{\aleph_0},$$

for each closed set F with card $(F) = 2^{\aleph_0}$.

Proof. Let $\{v_\alpha\}_{\alpha<\Omega}$ be any Hamel basis for $\mathbf{R}$ over $\mathbf{Q}$. For each $\alpha < \Omega$, let T_α denote the span of $\{v_\beta\}_{\beta<\alpha}$, that is,

$$T_\alpha = \left\{ \sum_{i=1}^{n} q_i v_{\beta_i} : q_i \in \mathbf{Q}, \beta_i < \alpha, n \in \mathbf{N} \right\},$$

and let $U_\alpha = T_{\alpha+1} \setminus T_\alpha$. We note that T_α is countable for each $\alpha < \Omega$, so that U_α is as well, and that $\{U_\alpha\}_{\alpha<\Omega}$ is a pairwise disjoint family of sets whose union is $\mathbf{R}$ and thereby defines an equivalence relation on $\mathbf{R}$:

$$x \sim y \Leftrightarrow \{x, y\} \subset U_\alpha, \text{ for some } \alpha < \Omega.$$

Let $E = \mathbf{R}$, let

$$\Phi = \{F \subset \mathbf{R} : F \text{ is closed and } \text{card } F = 2^{\aleph_0}\},$$

and apply the preceeding theorem. It is easy to verify that the set $\{e_\alpha : \alpha < \Omega\}$ thusly obtained forms a new Hamel basis $\mathcal{V}$ with the properties we require. □

[1]—thanks to the wisdom of the anonymous referees, to whom I am indebted for spotting many imperfections in the original manuscript.

We note that in [7], a Hamel basis $\mathcal{V}$ with the properties given above is obtained directly, without the intermediate Hamel basis, by methods similar to the proof of Theorem 7.1. The set A is now constructed as follows. Let the Hamel basis $\mathcal{V}$ obtained above have a well-ordering, say $\preceq$, which is order-isomorphic to Ω. For a given $x \in \mathbf{R}$, we have a unique expansion

$$x = \sum_{i=1}^{n(x)} q(x,i)v(x,i),$$

where $q(x,i) \in \mathbf{Q}$ and $v(x,i) \in \mathcal{V}$ for each i, with

$$v(x,1) \prec v(x,2) \prec \cdots \prec v(x,n(x)).$$

We then let

$$A = \{x \in \mathbf{R} : q(x,n(x)) > 0\}.$$

The details of verifying the alleged properties of A are given in [7] and we omit them.

Remark 7.8 Given that property (12) holds for every $\nu \in \mathcal{M}$, it would be of interest to know if there exists a μ which is a translation invariant extension of λ to $2^{\mathbf{R}}$, for which $\mu(A \cap E) \neq \frac{1}{2}\lambda(E)$, for some Lebesgue measurable set E. We generalize this question as follows:

Problem 5 Does there exist an α-shading A , a translation invariant extension μ of λ to $2^{\mathbf{R}}$, and a bounded nonempty interval $J \subset \mathbf{R}$ such that $\mu(J \cap A) \neq \alpha\ \lambda(J)$? (If so, the shade of A must rely on invariance with respect to reflection about a point, as does, Example 4.6.)

8 Conclusion

The facts and examples in this paper merely scratch the surface. The interested reader will no doubt see many problems that the author has omitted. For instance, we know that for a set A of constant shade,

$$\mathcal{T}(A) := \{\mathrm{sh}(A \cap (A+t)) : t \in \mathbf{R}\}$$

must be contained in the interval

$$U_A := [\max(0, 2\ \mathrm{sh}(A) - 1), \mathrm{sh}(A)].$$

Can A be such that $\mathcal{T}(A) = U_A$? In view of Remark 4.9, one can see that $\mathcal{T}(A)$ can be a singleton, while Example 4.7 shows that $\mathcal{T}(A)$ can be countable and dense in U_A. Can $\mathcal{T}(A)$ be finite but contain more than one element? The author can show that it is easy to construct examples of A which are countable unions of Archimedean sets, for which $\mathcal{T}(A)$ contains a sequence of shades $\{s_i\}_{i=1}^{\infty}$ which approaches the shade of A. Can such values s_i be prescribed? It is likely that many results from the theory of uniformly-distributed and well-distributed sequences of integers can be brought to bear on such questions, at least for certain Archimedean sets and their countable unions (in view of Remark 3.10).

We have not even mentioned shadings in $\mathbf{R}^n$ for $n > 1$. Clearly, the definition of shading would have to be changed for $n \geq 3$, but one can consider "$G-$shadings" for a group G of isometries on $\mathbf{R}^n$. But $\mathbf{R}^2$ is interesting enough. Let λ and μ now represent the Lebesgue measure in $\mathbf{R}^2$ and any isometry invariant total extension of λ to the power set of $\mathbf{R}^2$. It is obvious that if A and B are any of the constant shadings of the line presented in this paper, then

$$\mu(E \cap (A \times B)) = \mathrm{sh}(A)\mathrm{sh}(B)\lambda(E). \tag{16}$$

But it is only obvious because we know how such sets are constructed, and so the same basic manipulations (translations, etc.) can be performed in $\mathbf{R}^2$ to verify equation (16). But if A and B are any two arbitrary combs of constant shade in $\mathbf{R}$, it is not at all obvious that their product is a comb of constant shade in $\mathbf{R}^2$ (although it seems likely).

Also, can a subset $D \subset \mathbf{R}^2$ be constructed for which every line in $\mathbf{R}^2$ intersects D to form a constant shading of $\mathbf{R}$?

Another interesting problem would be to see if shadings A of $\mathbf{R}$ can be created with arbitrary constant shade, which satisfy the Sierpiǹski property

$$A \doteq A + t, \quad \forall\, t \in \mathbf{R}.$$

We could go on, but will stop here, hoping that this article might spark some interest in what are, in this author's opinion at least, very interesting and beautiful sets.

References

[1] R. Creighton Buck, The Measure Theoretic Approach to Density, *Am. J. Math.*, **68** (1946) 560-580.

[2] K. Ciesielski, How Good is Lebesgue Measure?, *Math. Intelligencer*, **11** (1989) 54-58.

[3] P. Erdos, Some Remarks on Set Theory, III, *Michigan Math. J.*, **2** (1953) 51-57.

[4] F.B. Jones, Measure and Other Properties of a Hamel Basis, *Bull. Amer. Math. Soc.*, **48** (1942) 472-481.

[5] S. Kakutani and J. Oxtoby, Construction of a Non-Separable Invariant Extension of the Lebesgue Measure Space, *Ann. of Math.*, **52** (1950) 580-590.

[6] P. Halmos, Measure Theory, Van Nostrand Reinhold Co., New York, 1950.

[7] A.B. Harazishvili, Invariant Extensions of Lebesgue Measure, Tbilis. Gos. Univ., Tbilisi, 1983 (Russian).

[8] E. Hewitt and K. Stromberg, Some Examples of Nonmeasurable Sets, *J. Austral. Math. Soc.*, **18** (1974) 236-238.

[9] John C. Morgan II, Point Set Theory, Marcel Dekker, Inc., New York, 1990.

[10] J. Silver, Counting the Number of Equivalence Classes of Borel and Coanalytic Equivalence Relations, *Ann. Math. Logic*, **18** (1980) 1-28.

[11] L. Kuipers and H. Neiderreiter, Uniform Distribution of Sequences, Wiley, New York, 1974.

[12] A. Mukherjea and K. Pothoven, Real and Functional Analysis, Plenum Press, New York, 1978.

[13] H.W. Pu, Concerning non-measurable subsets of a given measurable set, *J. Austral. Math. Soc.*, **13** (1972) 177-179.

[14] W. Rudin, Well-distributed measurable sets, *Am. Math. Monthly*, **90** (1983) 41-42.

[15] S. Saeki and K. Stromberg, Measurable subgroups and nonmeasurable characters, *Math. Scand.*, **57** (1985) 359-374.

[16] W. Sierpiǹski, Sur un problème conduisant à un ensemble non mesurable, *Fund. Math.*, **10** (1927) 177-179.

[17] W. Sierpiǹski, Sur les translations des ensembles linéaires, *Fund. Math.*, **19** (1932) 22-28.

[18] A. Simoson, An "Archimedean" Paradox, *Am. Math. Monthly*, **89** (1982) 114-116,125.

[19] A. Simoson, On Two Halves Being Two Wholes, *Am. Math. Monthly*, **91** (1984) 190-193.

Received August 23, 1990