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A PROOF OF ABEL'S CONTINUITY THEOREM

Let S be the spaces of sequences $s = (s_n)_{n=0}^{\infty}$ of complex terms of convergent series with norm defined by $\|s\|_S = \sup_{n\geq 0} \left|\sum_{j=n}^{\infty} s_j\right|$ and J be the space of all sequences $\beta = (\beta_n)_{n=0}^{\infty}$ of bounded variation with norm $\|\beta\|_J = |\beta_0| + \sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}|$. The following Hölder's type of inequality holds:

If $s = (s_n) \varepsilon S$ and $\beta = (\beta_n) \varepsilon J$, then $\left|\sum_{n=0}^{\infty} s_n\beta_n\right| \leq \|s\|_S \cdot \|\beta\|_J$.

This may be seen easily by an application of summation by parts.

An interesting application of this inequality is a simple proof of the Stolz form of the Abel Continuity Theorem:

THEOREM: (Abel's Continuity Theorem). If $\sum_{n=0}^{\infty} \alpha_n$ converges and $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$, then $\lim_{z\to 1} f(z) = f(1)$, where z is restricted to approach the point 1 in such a way that $|z| < 1$ and $\dfrac{|1-z|}{1-|z|}$ remains bounded.

Proof: First of all let C be a positive absolute constant such that $\dfrac{|1-z|}{1-|z|} \leq C$ and $|z| < 1$. Notice that $\left|\sum_{p=N}^{\infty} \alpha_p z^p\right| \leq \|(z^p)_{p=N}^{\infty}\|_J \cdot \|(\alpha_p)_{p=N}^{\infty}\|_S$ for $N \geq 1$ and by the above inequality applied to the sequences (α_p) and (z^p), since $(z^p)_{p=N}^{\infty} \varepsilon J$; in fact, since $|z| < 1$

$$\|(z^p)_{p=N}^{\infty}\|_J = |z|^N + |z^N - z^{N+1}| + |z^{N+1} - z^{N+2}| + \ldots. =$$

$$= |z|^N + |1-z|(|z|^N + |z|^{N+1} + \ldots..) = |z|^N(1 + \frac{|1-z|}{1-|z|}).$$

Now using the hypothesis we have $\|(z^p)_{p=N}^{\infty}\|_J \leq 1 + C$. Consequently, $\left|\sum_{p=N}^{\infty} \alpha_p z^p\right| \leq (1+C)\|(\alpha_p)_{p=N}^{\infty}\|_S \to 0$ as $N \to \infty$ since $(\alpha_n) \in S$. Then $\left|\sum_{p=0}^{\infty} \alpha_p z^p - \sum_{p=0}^{\infty} \alpha_p\right| \leq \left|\sum_{p=0}^{N-1} \alpha_p z^p - \sum_{p=0}^{N-1} \alpha_p\right| + \left|\sum_{p=N}^{\infty} \alpha_p z^p - \sum_{p=N}^{\infty} \alpha_p\right| = A+B$. For $\varepsilon > 0$, fix N so that $B \leq (2+C)\|(\alpha_p)_{p=N}^{\infty}\|_S \leq \varepsilon/2$. For $|z-1|$ sufficiently small, $A < \varepsilon/2$. Hence $\lim_{z\to 1} \sum_{p=0}^{\infty} \alpha_p z^p = \sum_{p=0}^{\infty} \alpha_p$, so that the theorem is proved.

For more information about the space S the interested reader is referred to [1], however we would like to point out that J is the dual space of S.

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References

[1] Geraldo Soares de Souza and G.O. Golightly - On Some Spaces of Summable Sequences and Their Duals. Preprint.

[2] G.O. Golightly - Sine Series on $[0,\ell]$ for Certain Entire Functions and Lidstone Series. Journal of Mathematical Analysis and Applications, to appear.

[3] A. Zygmund, Trigonometric Series (Cambridge University Press, 1959).

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