INROADS *Real Analysis Exchange Vol. 7 (1981-82)*

A. K. Layek, Multipurpose School - Mathematics Section,
Akbar Road, Durgapur - 713204, Burdwan, West Bengal, India.

On Strong Essential Cluster Sets

1. Let H, R and M^* stand for the open upper half plane, real line and Lebesgue outer measure, respectively. M^* is linear or planar; the choice will be clear from the context. Let $L(x)$ denote the ray in H emanating from $x \in R$ in the direction $\pi/2$ and let $L(x,r)$ be a segment of $L(x)$ with one end at x and of length r.

Let $\{I\}$ be the collection of closed rectangles of the form $[a,b]\times[0,k]$, $a<0<b$, a, b and k are rationals. For $I \in \{I\}$ let $I(x_0)$ denote the closed rectangle obtained by mapping (x,y) into $(x_0 + x, y)$. The strong outer upper density of a set $E \subset H$ at x is defined by

$$d^*_s(E,x) = \lim_{n\to\infty}\left[\sup_{D(I)<1/n}\left\{\frac{M^*(I(x)\ E)}{M^*(I(x))} : I \in \{I\}\right\}\right]$$

where $D(I)$ denotes the diameter of I.

The directional upper outer density of a set $E \subset H$ at x in the direction $\frac{\pi}{2}$ is defined by

$$\overline{d}^*(E,x) = \lim_{r\to 0} \sup \frac{M^*(E \cap L(x,r))}{r}$$

In particular, if the sets concerned are measurable then M^* and d^* will be replaced by M and d, respectively.

Let $f : H \to W$, where W is a topological space. The strong essential cluster set $C_s(f,x)$ of f at x is the set of all $w \in W$ such that for every open set U of W containing w, $\overline{d}^*_s(f^{-1}(U),x)>0$.

The definition of the directional essential cluster set $C_e(f,x,\pi/2)$ of f at x in the direction $\pi/2$ is similar, with $\overline{d}^*_s$ replaced by $\overline{d}^*$.

2. O'Malley [1] proved that if f: H→R is measurable, then for all but a measure zero set of points x in R,

$$C_s(f,x) = C_e(f,x,\Pi/2)$$

If further f is continuous, then for all but a first category set of points $x \in R$,

$$C_s(f,x) = C_e(f,x,\Pi/2) .$$

In this note we have studied the relationship between $C_e(f,x,\Pi/2)$ and $C_s(f,x)$ for arbitrary functions and for functions of Baire type 1.

3. Now we shall prove the auxiliary lemmas for our results.

<u>Lemma 1</u>. <u>Let</u> E H <u>be</u> <u>measurable</u>. <u>Then</u> <u>the</u> <u>set</u>

$$B(E) = \{x : x \in R, \overline{d}_s(E,x) < \overline{d}(E,x)\}$$

<u>is</u> <u>of</u> <u>measure</u> <u>zero</u>.

<u>Proof</u>: For a fixed positive integer n and positive rationals p, q and r with $p < q$ and $\sqrt{2}r < \frac{1}{n}$ let

$$B_{npqr} = \{x : \frac{M(I(x) \cap E)}{M(I(x))} \leq p < q \leq \frac{M(L(x,r) \cap E)}{r} \text{ for all } I \in \{I\},\ D(I) < \frac{1}{n}\}$$

Then B(E) is contained in the countable union of all the sets B_{npqr}.

If possible, let $\xi \in B_{npqr}$ be a point of point of density of the set B_{npqr}. Then for ε, $0 < \varepsilon < \frac{q-p}{q}$, there exists η, $0 < \eta < r/2$, such that

$$\frac{M(\xi-h, \xi+h) \cap B_{npqr})}{2h} > 1 - \varepsilon$$

for all $h < \eta$.

Let h be a rational such that $0 < h < \eta$, and set $[-h,h]\times[0,r] = I'$. Then $D(I') < \frac{1}{n}$ and

$$M(I'(\xi) E) > \int_{(\xi-h,\xi+h)\cap B_{npqr}} M(L(x,r) \cap E)\, dx$$

$$> 2qr(1-\varepsilon)h = q(1-\varepsilon)MI'(\xi) .$$

Since $q(1-\varepsilon) > p$, this is a contradiction to the fact that $\xi \in B_{npqr}$. Hence each of the sets B_{npqr} is of measure zero, and the proof is complete. (Lemma 1 is also proved in [1], but the proof above is more elementary).

Lemma 2. *Let* K H *be arbitrary. Then the set*

$$B^*(K) = \{x : \overline{d}^*_s(K,x) < \overline{d}^*(K,x)\}$$

is of measure zero.

Proof. Let $E \subset H$ be a measurable cover of K such that $M(E \cap Q) = M^*(K \cap Q)$ for each bounded measurable set $Q \subset H$. Then

$$B^*(K) \subset B(E) \cup T(E),$$

where B(E) is the set in Lemma 1 and T(E) is the measure zero set of all $x \in R$ at which $L(x) \cap E$ is non-measurable. By Lemma 1, B(E) is of measure zero, and the proof is complete.

Lemma 3. Let $E \subset H$ be an F_σ-set. Then the set

$$C(E) = \{x: \overline{d}_s(E,x) < \overline{d}(E,x)\}$$

is of the first category.

Proof. Let $E = \bigcup_{t=1}^{\infty} F_t$, where F_t is a closed set for each t. For fixed positive integers n, k and positive rationals p, q and r with $p < q$ and $\sqrt{2}r < \frac{1}{n}$ let

$$C_{nkpqr} = \{x: \frac{M(I(x) \cap E)}{M(I(x))} \leq p < q \leq \frac{M(L(x,r) \cap E_k)}{r} \text{ for all } I \in \{I\},\ D(I) < \frac{1}{n}\},$$

where $E_k = \bigcup_{t=1}^{k} F_t$. Then $C(E)$ is contained in the countable union of all the sets C_{nkpqr}.

If possible, let C_{nkpqr} be dense in an interval $(a,b) \subset R$. Then since E_k is a closed set and for $x \in C_{nkpqr}$ we have

$$\frac{M(L(x,r) \cap E_k)}{r} \geq q,$$

it follows that for all $x \in [a,b]$

$$\frac{M(L(x,r) \cap E_k)}{r} \geq q$$

Let $x' \in (a,b) \cap C_{nkpqr}$. Let y be a rational such that $0<y<\frac{r}{2}$ and $[x'-y, x'+y] \subset (a,b)$. Let $I' = [-y,y]x[0,r]$. Then $D(I') < \frac{I}{n}$ and

$$M(I'(x') \cap E_k) = \int_{(x'-y,\ x'+y)} M(L(x',r) \cap E_k)\, dx \geq 2qry = M(I'(x'))q$$

i.e.

$$\frac{M(I'(x')\cap E_k)}{M(I'(x'))} \geq q .$$

This is a contradiction to the fact that $x' \in C_{nkpqr}$. Hence each of the sets C_{nkpqr} is no-where dense and by (2) the set C(E) is of the first category.

Theorem. If $f : H \to W$ is arbitrary, where W is a second countable topological space, then except for a measure zero set of points x in R

$$C_e(f,x,\tfrac{\Pi}{2}) \subset C_s(f,x)$$

If further f is of Baire type 1 then the exceptional set is also of first category.

Proof. Let $B = \{V_n\}$ be a countable basis for the topology of W. Let $E_n = f^{-1}(V_n)$ and

$$P = \{x : x \in R,\ C_e(f,x,\tfrac{\Pi}{2}) \not\subset C_s(f,x)\}$$

Let $x' \in P$. Then there is a $w' \in C_e(f,x',\frac{\Pi}{2}) \setminus C_s(f,x')$. Since $w' \in C_e(f,x',\frac{\Pi}{2})$ and $w' \notin C_s(f,x')$ there is an n' such that $\overline{d}^* (E_{n'},x') > 0$ and $\overline{d}^*_s (E_{n'},x') = 0$. Hence $x' \in B(E_{n'})$, where

$$B(E_n) = \{x : \overline{d}^*(E_n,x) > \overline{d}^*_s(E_n,x)\}$$

Thus it is proved that

$$P \subset \bigcup_{n=1}^{\infty} B(E_n)$$

If f is arbitrary, then by Lemma 2 each of the sets $B(E_n)$

is of measure zero and hence P is of measure zero. Again if f is of Baire type 1 then each of the sets E_n is an F_σ set and $B(E_n) = C(E_n)$ of Lemma 3. Now by Lemma 3 each of the sets $C(E_n)$ is of first category and hence P is of the first category. This completes the proof.

Remark: O'Malley has constructed an arbitrary function in ([1], Example 4) for which the containment in the statement of the above theorem is proper for each x R. Example 3 in [1] also ensures that the exceptional set in the first part of the above theorem cannot be of the first category.

Question: Could the containment in the second part of the theorem be replaced by equality?

The author is thankful to the referee for his useful comments.

Reference

1. R.J. O'Malley, "Strong Essential Cluster Sets", Fund. Math. 78 (1973) 38-42.

Received November 10, 1980 and in revised form April 6, 1981