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Fourier Series of Functions of Generalized Bounded Variation

The work described here appears in [4] and [5].

The ideas of Harmonic Bounded Variation (HBV) and Λ-Bounded Variation (ΛBV) began in the work of Goffman and Waterman ([1],[2],[3],[8]). Another notion of generalized bounded variation is described as follows. Let ϕ be a non-negative convex function defined on $[0,\infty)$, with $\phi(0)=0$ and $\phi(x)>0$ for $x>0$, and let $\Lambda=\{\lambda_n\}$ be a non-decreasing sequence of positive real numbers such that $\Sigma 1/\lambda_n=\infty$. We say that f is of $\phi\Lambda$-Bounded Variation ($\phi\Lambda$BV) on $[a,b]$ if there is a $c>0$ so that, for any collection, $\{[a_n,b_n]\}$, of non-overlapping subintervals of $[a,b]$, the sum $\Sigma\phi(c|f(b_n)-f(a_n)|)/\lambda_n<\infty$. When $\phi(x)=x^p$, $p\geq 1$, this class is called $\Lambda BV^{(p)}$, and has been studied by Shiba [6]. Both ΛBV and $\phi\Lambda$BV may be made into Banach spaces with suitable norms.

In [4], we obtain the following

Theorem

Let $f:R \to R$ be of period 2π;

(i) if $f\varepsilon\Lambda BV$, $\omega_1(f;\delta)=O(1/\sum_{1}^{[1/\delta]} 1/\lambda_j)$;

(ii) if $f\varepsilon\phi\Lambda BV$, $\omega_1(f;\delta)=O(\phi^{-1}(1/\sum_{1}^{[1/\delta]} 1/\lambda_j))$;

(iii) if $f\varepsilon\Lambda BV^{(p)}$, $\omega_p(f;\delta)=O(1/(\sum_{1}^{[1/\delta]} 1/\lambda_j)^{1/p})$.

A simple calculation shows that $|\hat{f}(n)|\leq(1/4\pi)\omega_1(f;\pi/n)$, so we have the following

<u>Corollary</u>

(i) If $f\varepsilon\Lambda BV$, $\hat{f}(n)=O(1/\sum_1^n 1/\lambda_j)$;

(ii) if $f\varepsilon\phi\Lambda BV$, $\hat{f}(n)=O(\phi^{-1}(1/\sum_1^n 1/\lambda_j))$.

This result is best possible in a sense given by:

<u>Theorem</u>

If $\Gamma BV\supsetneq\Lambda BV$, there is a function $f\varepsilon\Gamma BV$ with

$\hat{f}(n)\neq O(1/{}_1^n 1/\lambda_j)$.

These results, as they pertain to ΛBV, were also discovered by Wang [7].

We obtain in [5] the following improvement of a theorem of Shiba [6]:

<u>Theorem</u>

If $f\varepsilon\Lambda BV^{(p)}$, $1\leq p<2r$, $1\leq r<\infty$, $(1/r)+(1/s)=1$, and

$$\sum_{n=1}^{\infty} n^{-1/2}\left(\sum_{k=1}^{n} 1/\lambda_k\right)^{-1/2r}\omega_{p+(2-p)s}^{1-p/2r}(f;\pi/n)<\infty,$$

then the Fourier series of f converges absolutely.

In his proof, Shiba uses the estimate $\sum_1^n 1/\lambda_j\geq n/\lambda_n$. That this estimate may give up a lot is seen in the fact that it is possible to have $n/\lambda_n=o(\sum_1^n 1/\lambda_j)$, as is the case in HBV, where $n/\lambda_n\equiv 1$ and $\sum_1^n 1/\lambda_j=\sum_1^n 1/j\sim\log n$. In our proof we are able to dispense with this estimate.

We say that a convex function ϕ is Δ_2 if there is a constant $k\geq 2$ so that $\phi(2x)\leq k\phi(x)$ for $x>0$. For such ϕ we have:

Theorem

If $f \in \phi\Lambda BV$, ϕ is Δ_2, $1 \leq p < 2r$, $1 \leq r < \infty$, $(1/r)+(1/s)=1$, and

$$\sum_{n=1}^{\infty} n^{-1/2}\left(\phi^{-1}\left(\left(\sum_{k=1}^{n} 1/\lambda_k\right)^{-1}\omega^{2r-p}_{p+(2-p)s}(f;\pi/n)\right)\right)^{1/2r} < \infty,$$

then the Fourier series of f converges absolutely.

We note in both cases above that the theorems for $\phi\Lambda BV$ are not direct generalizations of those for $\Lambda BV^{(p)}$.

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