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MEASURABILITY OF PEANO DERIVATES AND APPROXIMATE PEANO DERIVATES

Abstract

We prove that Peano derivatives and approximate Peano derivatives of measurable functions are measurable.

The measurability of Peano derivates of order $k \geq 2$ of a measurable function does not seem to be covered in the literature, although some authors sometimes used this while proving related results. Since the measurability of the Peano derivates of measurable functions is not automatic it is desirable that it is proved somewhere. The purpose of the present note is to offer the proofs of the measurability of the Peano derivates and approximate Peano derivates of measurable functions. It is worthwhile to mention that this work was inspired by valuable communications from Professor C.E. Weil which are gratefully acknowledged.

2. Preliminaries

Let f be a real function defined in some neighborhood of x. Then f is said to have Peano derivative (resp. approximate Peano derivative) at x of order k if there exist real numbers α_i, $1 \leq i \leq k$ depending on x and f only such that

$$f(x+t) = f(x) + \sum_{i=1}^{k} \frac{t^i}{i!}\alpha_i + \frac{t^k}{k!}\,\varepsilon_k(x,t,f)$$

where

$$\lim_{t\to 0}\varepsilon_k(x,t,f) = 0 \ (\text{resp.} \lim_{t\to 0} \ \text{ap}\ \varepsilon_k(x,t,f) = 0).$$

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The number α_k is called the Peano derivative (resp. approximate Peano derivative) of f at x of order k and is denoted by $f_{(k)}(x)$ (resp. $f_{(k),a}(x)$). For convenience we take $\alpha_0 = f(x) = f_{(0)}(x) = f_{(0),a}(x)$.

Suppose that f has Peano derivative (resp. approximate Peano derivative) at x of order k. For $t \neq 0$ write

$$\omega_{k+1}(x,t;f) = \omega_{k+1}(x,t) = \frac{(k+1)!}{t^{k+1}}[f(x+t) - \sum_{i=0}^{k} \frac{t^i}{i!}\alpha_i].$$

The right-hand upper (resp. approximate upper) Peano derivate of f at x of order $k+1$ is defined by

$$\begin{aligned} \bar{f}^{+}_{(k+1)}(x) &= \limsup_{t \to 0^+} \omega_{k+1}(x,t) \text{ (resp. } \bar{f}^{+}_{(k+1),a}(x) \\ &= \limsup_{t \to 0^+} \text{ ap } \omega_{k+1}(x,t)). \end{aligned}$$

The other derivates are defined analogously. If all the four Peano derivates (resp. approximate Peano derivates) at a point x are equal then the common value is called the Peano derivative (resp. approximate Peano derivate) of f at x (possibly infinite) of order $k+1$.

The first order derivates (resp. approximate derivates) of f at x are denoted by $\bar{f}^{+}_{(1)}(x)$, $\underline{f}^{+}_{(1)}(x)$, $\bar{f}^{-}_{(1)}(x)$, $\underline{f}^{-}_{(1)}(x)$, (resp. $\bar{f}^{+}_{(1),a}(x)$, $\underline{f}^{+}_{(1),a}(x)$, $\bar{f}^{-}_{(1),a}(x)$, $\underline{f}^{-}_{(1),a}(x)$). The Lebesgue measure will be denoted by μ, the set of all positive integers by $\mathbb{N}$ and the set of all real numbers by $\mathbb{R}$.

3. Measurability of Peano derivates

Theorem 1 *Let $f : \mathbb{R} \to \mathbb{R}$ be measurable and let $k \in \mathbb{N}$. Then the set $E_k \subset \mathbb{R}$ of points x such that $f_{(k)}(x)$ exists finitely is measurable and $f_{(k)}$ is measurable on E_k. Further $\bar{f}^{+}_{(k+1)}$, $\underline{f}^{+}_{(k+1)}$, $\bar{f}^{-}_{(k+1)}$, $\underline{f}^{-}_{(k+1)}$ are all measurable on E_k.*

PROOF. For each $n \in \mathbb{N}$, let

$$F_n(x) = n^k \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(x + \frac{i}{n}).$$

Since f is measurable, F_n is measurable. Also

$$f_{(k)}(x) = \lim_{t \to 0} \frac{1}{t^k} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(x+it) = \lim_{n \to \infty} F_n(x)$$

for $x \in E_k$. Hence the first part is clear. We prove the second part. For $n \in \mathbb{N}$, define

$$g_n(x) = \sup_{0<t<1/n} \omega_{k+1}(x,t), \ x \in E_k.$$

Let η, ε be arbitrarily small positive numbers. Since by the above, f, $f_{(1)}, \ldots, f_{(k)}$ are measurable on E_k, there is a perfect set $P = P_\eta \subset E_k$ such that $\mu(E_k \sim P) < \eta$ and the functions $f, f_{(1)}, \ldots, f_{(k)}$ are continuous on P relative to P. Let $x_0 \in P$. Then there is ξ, $0 < \xi - x_0 < 1/n$ such that

$$g_n(x_0) - \varepsilon < \omega_{k+1}(x_0, \xi - x_0). \tag{3.1}$$

Since $\omega_{k+1}(x, \xi - x)$, as a function of x, is continuous at x_0 relative to P, there is δ, $0 < \delta, \min[\xi - x_0, 1/n - \xi + x_0]$, such that

$$|\omega_{k+1}(x, \xi - x) - \omega_{k+1}(x_0, \xi - x_o)| < \varepsilon \ \text{ for } \ x \in P \cap (x_0 - \delta, x_0 + \delta). \tag{3.2}$$

From (3.1) and (3.2)

$$g_n(x_0) - 2\varepsilon < \omega_{k+1}(x, \xi - x) \ \text{ for } \ x \in P \cap (x_0 - \delta, x_0 + \delta). \tag{3.3}$$

Now, if $x \in P \cap (x_0 - \delta, x_0 + \delta)$ then

$$\xi - x = \xi - x_0 + x_0 - x < \xi - x_0 + \delta < \xi - x_0 + 1/n - \xi + x_0 = 1/n$$

and

$$\xi - x = \xi - x_0 + x_0 - x > \delta + x_0 - x > 0.$$

Hence from (3.3)

$$g_n(x_0) - 2\varepsilon < g_n(x) \ \text{ for } x \in P \cap (x_0 - \delta, x_0 + \delta).$$

So g_n is lower semicontinuous at x_0 relative to P. Since x_0 is any point of P, g_n is lower semicontinuous on P relative to P. Thus g_n is measurable on P. Since η is arbitrary, it follows that for each $\nu \in \mathbb{N}$, there is a perfect set $P_\nu \subset E_k$ such that $\mu(E_k \sim P_\nu) < 1/\nu$ and g_n is measurable on P_ν. Hence g_n is measurable on $\bigcup_{\nu=1}^{\infty} P_\nu$. Since

$$\mu(E_k \sim \bigcup_{\nu=1}^{\infty} P_\nu) = \mu(\bigcap_{\nu=1}^{\infty}(E_k \sim P_\nu)) \leq \mu(E_k \sim P_\nu) < 1/\nu$$

for all $\nu \in \mathbb{N}$, $\mu(E_k \sim \bigcup_{\nu=1}^{\infty} P_\nu) = 0$ and so g_n is measurable on E_k. Since

$$\bar{f}^{+}_{(k+1)}(x) = \lim_{n\to\infty} g_n(x), \quad x \in E_k,$$

it follows that $\bar{f}^{+}_{(k+1)}$ is measurable on E_k. Similar arguments hold for $\bar{f}^{-}_{(k+1)}$, $\underline{f}^{+}_{(k+1)}$, and $\underline{f}^{-}_{(k+1)}$.

Corollary 1 *Let $f : \mathbb{R} \to \mathbb{R}$ be measurable. Then the set $E \subset \mathbb{R}$ of points x such that $f_{(k)}(x)$ exists (possibly infinite) is measurable and $f_{(k)}$ is measurable on E.*

PROOF. Since $f_{(k)} = \bar{f}^{+}_{(k)} = \underline{f}^{+}_{(k)} = \bar{f}^{-}_{(k)} = \underline{f}^{-}_{(k)}$ whenever $f_{(k)}$ exists (possibly infinite) the proof follows from the above theorem. □

4. Remarks

1. Since the Cesaro derivates of order k (for definition see [5]) are Peano derivates of order $k+1$ of continuous function [1], the Cesaro derivates are also measurable.

2. In [2, p. 54] the authors proved that a finite n-th Peano derivative of a measurable function f is measurable and remarked that similar argument would give the measurability of $\bar{f}_{(n+1)}$ and $\underline{f}_{(n+1)}$ which is not true [6, p. 20]. Now from Theorem 1 the measurability of $\bar{f}_{(n+1)}$ and $\underline{f}_{(n+1)}$ follows and the results obtained there remain valid. (We take this opportunity to mention that the defect mentioned in [6, pp. 19-20] has been corrected in two ways [3,7]).

3. It may be of some interest to note that the set

$$G_{mn} = \{x : x \in E_k;\ |\omega_{k+1}(x,t)| \leq m \quad \text{for} \quad 0 < |t| < 1/n\}$$

is measurable. In fact, from the second part of the above proof, the function

$$h_n(x) = \sup_{0<|t|<1/n} |\omega_{k+1}(x,t)|, \quad x \in E_k$$

is measurable. Since $G_{mn} = \{x : x \in E_k;\quad h_n(x) \leq m\}$ the result follows. Similarly for every $\varepsilon > 0$ the set

$$H_{\varepsilon n} = \{x : x \in E_k;\quad |\varepsilon_k(x,t,f)| \leq \varepsilon \quad \text{for} \quad 0 < |t| < 1/n\}$$

is measurable. The measurability of the sets G_{mn} and $H_{\varepsilon n}$ are used in various cases before.

5. Measurability of approximate Peano derivates

Lemma 1 *Let $Q \subset \mathbb{R}$ be a measurable set and let $f : Q \to \mathbb{R}$ be measurable. Let $k \in \mathbb{N}$. Let*

$$E_0 = Q$$

$$E_i = \{x : x \in E_{i-1};\ f_{(i),a}(x) \text{ exists finitely}\}, \quad 1 \leq i \leq k.$$

Suppose that E_i *is measurable and* $f_{(i),a}$ *is measurable on* E_i *for* $1 \leq i \leq k$. *Then* $\bar{f}^+_{(k+1),a}$, $\bar{f}^-_{(k+1),a}$, $\underline{f}^-_{(k+1),a}$, $\underline{f}^+_{(k+1),a}$ *are all measurable on* E_k.

PROOF. We first suppose, as a special case, that Q is bounded and closed, $Q = E_k$ and f, $f_{(1),a}, \ldots, f_{(k),a}$ are all continuous on Q relative to Q. Let

$$P = \{x : x \in Q;\ \bar{f}^+_{(k+1),a}(x) \leq a\}$$

where $a \in \mathbb{R}$. Let D be the set of all points of Q which are also points of density of Q. For each $x \in Q$ and $n \in \mathbb{N}$, let

$$G_n(x) = \{t : t \geq x,\ t \in Q;\ f(t) - \sum_{i=0}^{k} \frac{(t-x)^i}{i!} f_{(i),a}(x) \tag{5.1}$$
$$[.2cm] \leq (a + 1/n)\frac{(t-x)^{k+1}}{(k+1)!}\}.$$

Then $G_n(x)$ is measurable. For $\ell, m, n \in \mathbb{N}$, let

(5.2)
$$Q_{\ell,m,n} = \{x : x \in Q;\ \mu(G_n(x) \cap [x, x+h]) \geq (1 - 1/m)h \text{ for } 0 \leq h \leq \frac{1}{\ell}\}.$$

Then

$$P \cap D \subset \bigcap_n \bigcap_m \bigcup_\ell Q_{\ell,m,n} \subset P. \tag{5.3}$$

To see this let $x \in P \cap D$. Then $\bar{f}^+_{(k+1),a}(x) \leq a$ and x is a point of density of Q. So for each $n \in \mathbb{N}$, x is a point of right density of $G_n(x)$. Hence for any $m \in \mathbb{N}$ there is $\ell \in \mathbb{N}$ such that $x \in Q_{\ell,m,n}$. This proves the first inclusion in (5.3). Next, let $x \in Q_{\ell,m,n}$ for each $m, n \in \mathbb{N}$ and some $\ell \in \mathbb{N}$. Then x is a point of right density of $G_n(x)$ for each $n \in \mathbb{N}$. Hence $x \in P$, which proves the second inclusion in (5.3).

Let ℓ, m, n be fixed. We show that $Q_{\ell,m,n}$ is closed. Let $\{x_i\}$ be any sequence in $Q_{\ell,n,m}$ which converges to x_0. Then since Q is closed, $x_0 \in Q$. Let $0 \leq h \leq 1/\ell$. Let

$$H_i = G_n(x_i) \cap [x_i, x_i + h], \quad i \in \mathbb{N} \cup \{0\}. \tag{5.4}$$

Then from (5.2)

$$\mu(H_i) \geq (1 - 1/m)h \quad \text{for} \quad i \in \mathbb{N}. \tag{5.5}$$

Let $\xi \in H_i$ for infinite values of i. So there is a sequence $\{i_\nu\} \subset \mathbb{N}$ such that $\xi \in H_{i_\nu}$ for all $\nu \in \mathbb{N}$. Hence $\xi \in [x_0, x_0 + h]$. Since $\xi \in G_n(x_{i_\nu})$ for all $\nu \in \mathbb{N}$, we have from (5.1) $\xi \geq x_{i_\nu}$, $\xi \in Q$ and

$$f(\xi) - \sum_{j=0}^{k} \frac{(\xi - x_{i_\nu})^j}{j!} f_{(j),a}(x_{i_\nu}) \leq (a + 1/n) \frac{(\xi - x_{i_\nu})^{k+1}}{(k+1)!}$$

for all $\nu \in \mathbb{N}$. Since f, $f_{(1),a}, \ldots, f_{(k),a}$ are continuous on Q, $\xi \geq x_0$, $\xi \in Q$ and

$$f(\xi) - \sum_{j=0}^{k} \frac{(\xi - x_0)^j}{j!} f_{(j),a}(x_0) \leq (a + 1/n) \frac{(\xi - x_0)^{k+1}}{(k+1)!}$$

and hence $\xi \in G_n(x_0)$. So $\xi \in H_0$. Thus

$$\bigcap_{r=1}^{\infty} \bigcup_{i=r}^{\infty} H_i \subset H_0.$$

Since $F_r = \bigcup_{i=r}^{\infty} H_i$ is decreasing and F_1 is a bounded set,

$$\lim_{r \to \infty} \mu(F_r) = \mu(\bigcap_{r=1}^{\infty} F_r) \leq \mu(H_0).$$

Since $\mu(H_i) \leq \mu(F_r)$ for $i \geq r$ we have

$$\limsup_{i \to \infty} \mu(H_i) \leq \lim_{r \to \infty} \mu(F_r) \leq \mu(H_0).$$

Hence from (5.5) $(1 - 1/m)h \leq \mu(H_0)$. Since h is arbitrary, this gives, using (5.4) and (5.2), that $x_0 \in Q_{\ell,m,n}$. So, $Q_{\ell,m,n}$ is closed. Hence $\bigcap_n \bigcap_m \bigcup_\ell Q_{\ell,m,n}$ is measurable. Since P and $P \cap D$ differ by a set of measure zero, P is measurable.

Now we come to the proof of the general case. Let Q be bounded. Let $\varepsilon > 0$ be arbitrary. Since E_k is measurable and f, $f_{(1),a}, \ldots, f_{(k),a}$ are measurable on E_k there is a perfect set $Q_0 \subset E_k$, such that $\mu(E_k \sim Q_0) < \varepsilon$ and f, $f_{(1),a}, \ldots, f_{(k),a}$ are continuous on Q_0 relative to Q_0. Then, by the above special case, $\bar{f}^+_{(k+1),a}$ is measurable on Q_0. Since ε is arbitrary, $\bar{f}^+_{(k+1),a}$ is measurable on E_k. If Q is unbounded, we apply the argument on $Q_n = Q \cap [-n, n]$, $n \in \mathbb{N}$. Since $Q = \cup\, Q_n$, $\bar{f}^+_{(k+1),a}$ is measurable on E_k. Similarly $\underline{f}^+_{(k+1),a}$, $\bar{f}^-_{(k+1),a}$, $\underline{f}^-_{(k+1),a}$ are all measurable on E_k. This completes the proof.

Theorem 2 *Let Q be a measurable set and let $f : Q \to \mathbb{R}$ be measurable. Let $k \in \mathbb{N}$. Let E_k be the set of points $x \in Q$ such that $f_{(k),a}(x)$ exists and is finite. Then E_k is measurable and $f_{(k),a}$ is measurable on E_k. Further $\bar{f}^{+}_{(k+1),a}$, $\underline{f}^{+}_{(k+1),a}$, $\bar{f}^{-}_{(k+1),a}$, $\underline{f}^{-}_{(k+1),a}$ are all measurable on E_k.*

PROOF. Since f is measurable, $\bar{f}^{+}_{(1),a}$, etc. are all measurable [8, p. 299; 4]. Hence the set

$$E_1 = \{x : x \in G; -\infty < \bar{f}^{+}_{(1),a}(x) = \underline{f}^{+}_{(1),a}(x) = \bar{f}^{-}_{(1),a}(x) = \underline{f}^{-}_{(1),a}(x) < \infty\}$$

is measurable and $f_{(1),a}$ is measurable on E_1. Putting $k = 1$ in the lead we have that $\bar{f}^{+}_{(2),a}$, $\underline{f}^{+}_{(2),a}$, $\bar{f}^{-}_{(2),a}$, $\underline{f}^{-}_{(2),a}$ are all measurable on E_1. Hence the set

$$E_2 = \{x : x \in E_1; -\infty < \bar{f}^{+}_{(2),a}(x) = \underline{f}^{+}_{(2),a}(x) = \bar{f}^{-}_{(2),a}(x) = \underline{f}^{-}_{(2),a}(x) < \infty\}$$

is measurable and $f_{(2),a}$ is measurable on E_2. Putting $k = 2$ in the lemma $\bar{f}^{+}_{(3),a}$ etc., are all measurable on E_2. Proceeding inductively the set

$$E_k = \{x : x \in E_{k-1}; -\infty < \bar{f}^{+}_{(k),a}(x) = \underline{f}^{+}_{(k),a}(x), = \bar{f}^{-}_{(k),a}(x) = \underline{f}^{-}_{(k),a}(x) < \infty\}$$

is measurable and $f_{(k),a}$ is measurable on E_k. By the lemma $\bar{f}^{+}_{(k+1),a}$ are all measurable on E_k. This completes the proof.

Corollary 2 *Let $f : Q \to \mathbb{R}$ be measurable. Then the set $E \subset Q$ of points x such that $f_{(k),a}(x)$ exists (possibly infinite) is measurable and $f_{(k),a}$ is measurable on E.*

The proof is similar to that of Corollary 1.

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References

[1] J.A. Bergin, *A new characterization of Cesaro Perron Integrals using Peano derivatives*, Trans. Amer. Math. Soc., **228** (1977), 287-305.

[2] P.S. Bullen and S.N. Mukhopadhyay, *Peano derivatives and general integrals*, Pacific J. Math., **47** (1973), 43-58.

[3] P.S. Bullen and S.N. Mukhopadhyay, *The Peano derivative and the M_2 property of Zahorski*, Ind. J. Math. **28** (1986), 219-228.

[4] J.C. Burkill and U.S. Haslam-Jones, *The derivates and approximate derivates of measurable functions*, Proc. London Math. Soc. (2), **32** (1931), 346-355.

[5] J.C. Burkill, *The Cesaro-Perron scale of integration*, Proc. London Math. Soc. (2), **39** (1935), 541-552.

[6] M.J. Evans and C.E. Weil, *Peano Derivatives: A survey*, Real Analysis Exchange, **7** (1981-82), 5-23.

[7] M. Laczkovich, D. Preiss and C. Weil, *On unilateral and bilateral n-th Peano derivatives*, Proc. Amer. Math. Soc., **99** (1987), 129-134.

[8] S. Saks, *Theory of the integral*, Dover, 1964.