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UNIFORMLY ANTISYMMETRIC FUNCTIONS, UNIFORMLY ANTI-SCHWARZ FUNCTIONS

A function $f\colon \mathbb{R} \to \mathbb{R}$ is *symmetrically continuous* if for every $x \in \mathbb{R}$

$$\lim_{h\to 0+} |f(x-h) - f(x+h)| = 0.$$

As an opposite to this notion we say that a function $f\colon \mathbb{R} \to \mathbb{R}$ is *uniformly antisymmetric* (or *nowhere weakly symmetrically continuous*) provided for every $x \in \mathbb{R}$

$$(1) \qquad (\exists g > 0)(\forall 0 < h < g)\ |f(x-h) - f(x+h)| \geq g.$$

It is easy to see that that for $f\colon \mathbb{R} \to \mathbb{N}$ condition (1) is equivalent to

$$(2) \qquad (\exists g > 0)(\forall 0 < h < g)\ f(x-h) \neq f(x+h).$$

Moreover, if

$$S_x = \{h > 0\colon f(x-h) = f(x+h)\}$$

then (2) is equivalent to

$$\inf S_x > 0.$$

Theorem 1 [3] *There exists a function $f\colon \mathbb{R} \to \mathbb{N}$ for which S_x is finite for every $x \in \mathbb{R}$. In particular, f is uniformly antisymmetric.*

Open Problem 1 [3, 6] *Does there exist a uniformly antisymmetric function $f\colon \mathbb{R} \to \mathbb{R}$ with bounded range?*

Open Problem 2 [3] *Does there exist a uniformly antisymmetric function $f\colon \mathbb{R} \to \mathbb{N}$ with finite range?*

Known facts concerning Problem 2.

Theorem 2 [5] *There is no function $f\colon \mathbb{R} \to \mathbb{N}$ with finite range for which every set S_x is finite.*

For $g\colon \mathbb{R} \to (0,\infty)$ let G_g be an infinite graph on $\mathbb{R}$ such that vertices $x+h$ and $x-h$ are connected in G_g if and only if $0<|h|<g(x)$.

Theorem 3 [1] *There exists a uniformly antisymmetric function $f\colon \mathbb{R} \to \{1,2,\ldots,n\}$ if and only if there exists $g\colon \mathbb{R} \to (0,\infty)$ such that graph G_g is n-colorable.*

Recall that for a natural number $n>1$ symbol K_n stands for the graph with n vertices and all possible edges. Clearly K_{n+1} is not n-colorable.

Theorem 4 [1] *For every $g\colon \mathbb{R} \to (0,\infty)$ graph G_g contains K_4 as a subgraph. In particular, there is no uniformly antisymmetric function with three element range.*

Theorem 5 [2] *If the Continuum Hypothesis holds then there exists $g\colon \mathbb{R} \to (0,\infty)$ such that K_5 cannot be embedded into G_g.*

A notion dual to that of symmetric continuity is Schwarz continuity [6]. More precisely, a function $f\colon \mathbb{R} \to \mathbb{R}$ is *Schwarz continuous* if for every $x \in \mathbb{R}$

$$\lim_{h\to 0^+} |f(x-h)+f(x+h)-2f(x)| = 0.$$

In opposite to it we say that a function $f\colon \mathbb{R} \to \mathbb{R}$ is *uniformly anti-Schwarz* if for every $x \in \mathbb{R}$

$$(\exists g>0)(\forall 0<h<g)\ |f(x-h)+f(x+h)-2f(x)| \geq g.$$

Theorem 6 [2] *There exists an anti-Schwarz function $f\colon \mathbb{R} \to \mathbb{R}$ with bounded countable range.*

Open Problem 3 [2] *Does there exist an anti-Schwarz function $f\colon \mathbb{R} \to \mathbb{R}$ with two element (or finite) range?*

The existence of uniformly antisymmetric function $f\colon \mathbb{R}^2 \to \mathbb{N}$ seems to be also related to the following problem of P. Erdös.

Open Problem 4 [4, p. 314] *Does there exist a decomposition of $\mathbb{R}^2$ into countable many sets such that none of the sets spans an isosceles triangle?*

Such a decomposition is known to exist under the Continuum Hypothesis.

References

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