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# ON SCORZA DRAGONI'S PROPERTY FOR THE DENSITY TOPOLOGY

**Abstract**

We show that the Scorza Dragoni theorem does not hold for the density topology and we prove some special forms of this theorem.

Denote by $\mathbb{R}$ the set of all reals, by $\mathbf{L}$ the $\sigma$-algebra of all Lebesgue measurable subsets of $\mathbb{R}$, and by $\mu$ Lebesgue measure on $\mathbb{R}$. A point $x \in \mathbb{R}$ is said to be a density point of a set $A \in \mathbf{L}$ if $\lim_{r\to 0} \mu(A \cap (x-r, x+r))/2r = 1$. The family

$$T_d = \{A \in \mathbf{L};\ \text{if } x \in A,\ \text{then } x \text{ is a density point of } A\}$$

is a topology called the density topology [2, 4, 10]. Similarly, the family

$$T_{ae} = \{A \in T_d; \mu(A \setminus int(A)) = 0\},$$

where $int(A)$ denotes the Euclidean interior of $A$, is a topology called the a.e. topology [8]. Moreover, let $I \subset \mathbb{R}$ be an interval and let $T_e$ denote the euclidean topology in $I$. Evidently, the topological space $(\mathbb{R}, T_{ae})$ is separable but it is not a second-countable topological space.

From the more general Theorem 2 of [6] the following form of Scorza Dragoni's theorem can be concluded.

**Theorem 1** *Suppose that $I \subset \mathbb{R}$ is an interval, $(X, T_X)$ is a second-countable Hausdorff space, and $(Y, \varrho)$ is a separable metric space. Denote by $B(X)$ the $\sigma$-algebra of all Borel subsets of $X$ and by $(\mathbf{L} \bigotimes B(X))$ the product $\sigma$-algebra on $\mathbb{R} \times X$. Let $f : I \times X \to Y$ be a $(\mathbf{L} \bigotimes B(X))$-measurable function, i.e. $f^{-1}(U) \in (\mathbf{L} \bigotimes B(X))$ for every open subset $U \subset Y$. If all sections $f_t(x) = f(t,x), t \in I, x \in X$, are $T_X$-continuous, then for every $\varepsilon > 0$ there is a closed set $K \subset I$ such that $\mu(I \setminus K) < \varepsilon$ and the restriction $f|(K \times X)$ is $(T_e|K \times T_X)$-continuous, where $T_e|K = \{K \cap U; U \in T_e\}$ and $T_e|K \times X$ denotes the product topology on $K \times X$.*

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This theorem does not hold if $X = \mathbb{R}$ and $T_X = T_{ae}$. Indeed the following theorem holds:

**Theorem 2** *There is a Baire 1 function $f : \mathbb{R}^2 \to \mathbb{R}$ which is $(\mu \times \mu)$-almost everywhere $(T_e \times T_e)$-continuous and such that all its sections $f_t$ and $f^x(t) = f(t, x), t, x \in \mathbb{R}$, are $T_d$-continuous everywhere and $T_e$-discontinuous at most at one point and for every closed set $K \subset \mathbb{R}$ with $\mu(K) > 0$ the restricted function $f|(K \times \mathbb{R})$ is not $(T_e|K \times T_d)$-continuous.*

PROOF. Let points $a_n, n = 1, 2, \ldots,$ be such that $a_n > a_{n+1}$ for $n = 1, 2, \ldots$, $\lim_{n\to\infty} a_n = 0$ and 0 is a density point of the set $(-\infty, 0) \cup \bigcup_n (a_{2n}, a_{2n-1})$. For $n \geq 1$, let $b_n$ be the center of the interval $[a_{2n+1}, a_{2n})$. Put $h(x) = 0$ for $x \leq 0$ for $x \geq a_1$ and for $x \in [a_{2n}, a_{2n-1}], n \geq 1$, $h(b_n) = 1$ for $n \geq 1$, and $h$ is linear on the intervals $[a_{2n+1}, b_n]$ and $[b_n, a_{2n}], n \geq 1$. Then the function $h : \mathbb{R} \to [0, 1]$ is $T_d$-continuous everywhere, $T_e$-discontinuous at 0, $T_e$-continuous at every point $t \neq 0$, and such that $h(0) = 0$. Let $f(t, x) = h(t - x)$ for $t, x \in \mathbb{R}$. Evidently, $f$ is a Baire 1 function $(T_e \times T_e)$-continuous at every point $(t, x)$ such that $t \neq x$ and all sections $f_t$, and $f^x, t, x \in \mathbb{R}$, are $T_d$-continuous everywhere and $T_e$-discontinuous only at one point. Let $K \subset \mathbb{R}$ be a closed set with $\mu(K) > 0$. There is a point $u \in K$ which is a density point of $K$. Evidently, $f(u, u) = h(u - u) = h(0) = 0$. Let $s = \limsup_{t\to 0} h(t)$. Then $1 = s > 0$. If the restriction $f|(K \times \mathbb{R})$ is $(T_e|K \times T_d)$-continuous at $(u, u)$, then there are an open interval $J$ and a set $B \in T_d$ such that $u \in J \cap B$ and

$$(t, x) \in (J \cap K) \times B \Longrightarrow f(t, x) < s/2. \tag{1}$$

Then $0 = u - u \in int((J \cap K) - B)$ ([7], p.71, Th.2 ) and there is a point $(t, x) \in (J \cap K) \times B$ such that $t - x \in int((J \cap K) - B), h(t - x) > s/2$, contrary to (1). □

**Remark 1** *All sections $f_t$ and $f^x$ , $t, x \in \mathbb{R}$, of the function $f$ from Theorem 2 are $T_{ae}$-continuous [8].*

However, the following form of Scorza Dragoni theorem is true:

**Theorem 3** *Let $(X, T_X)$ be a topological space such that*

(2) *for every family $\{U_s\}_{s\in S}$ of $T_X$-open sets such that $X = \bigcup_{s\in S} U_s$ there is a sequence $s_1, \ldots, s_n, \ldots$ of indices from $S$ for which $\bigcup_{n=1}^{\infty} U_{s_n}$ is dense in $X$.*

*Suppose that $I \subset \mathbb{R}$ is an interval, $(Y, \varrho)$ is a metric space and $f : I \times X \to Y$ is a function such that all its sections $f^x, x \in X$, are Lebesgue measurable and all its sections $f_t, t \in I$, are $T_X$-equicontinuous at every point $x \in X$ ,i.e. for* *every $x \in X$ and for every $\eta > 0$ there is a set $U \in T_X$ containing $x$ and such that $\varrho(f(t,y), f(t,x)) < \eta$ for all $y \in U$ and $t \in I$. Then for every $\varepsilon > 0$ there is a closed set $K \subset I$ such that $\mu(I \setminus K) < \varepsilon$ and the restriction $f|(K \times X)$ is $(T_e|K \times T_X)$-continuous.*

PROOF. For $n = 1, 2, \ldots$ there are sequences of sets $U_{n,k} \in T_X$ and points $x_{n,k} \in U_{n,k}$, $k = 1, 2, \ldots$, such that the sets $\bigcup_{k=1}^{\infty} U_{n,k}$ are dense in $X$ and

$$\varrho(f(t,x), f(t,x_{n,k})) < 1/4n \tag{3}$$

for all $x \in U_{n,k}$ and $t \in I$.

Since all sections $t \to f(t, x_{n,k})$, $t \in I$, $k, n = 1, 2, \ldots$, are Lebesgue measurable, there exist closed sets $A_{n,k} \subset I$ such that $\mu(I \setminus A_{n,k}) < \varepsilon/2^{n+k+1}$ and the restricted functions $t \to f(t, x_{n,k}), t \in A_{n,k}$, are $(T_e|A_{n,k})$-continuous ([3], Th. 2B ). Let $K = \bigcap_{n,k=1}^{\infty} A_{n,k}$. Then $K \subset I$ is a closed set and $\mu(I \setminus K) < \varepsilon$. Fix a point $(u, v) \in K \times X$ and a positive number $\eta$. There is a positive integer $n$ such that $2/n < \eta$. If there is $k$ such that $v \in U_{n,k}$, then by the continuity of the section $t \to f(t, x_{n,k}), t \in K$, there is an open interval $J$ such that $u \in J$ and

$$\varrho(f(t, x_{n,k}), f(u, x_{n,k})) < \eta/2 \tag{4}$$

for all $t \in J \cap K$. Thus, for $(t, x) \in (J \cap K) \times U_{n,k}$ by (3) we have,

$$\begin{gathered}\varrho(f(t,x), f(u,v)) \le \varrho(f(t,x), f(t,x_{n,k})) + \\ \varrho(f(t,x_{n,k}), f(u,x_{n,k})) + \varrho(f(u,x_{n,k}), f(u,v)) \\ < 1/4n + \eta/2 + 1/4n < \eta/8 + \eta/2 + \eta/8 = 3\eta/4 < \eta.\end{gathered}$$

In the other case $v \in X \setminus \bigcup_{k=1}^{\infty} U_{n,k}$ and then there is a set $U \in T_X$ such that $v \in U$ and $\varrho(f(t,x), f(t,v)) < \eta/12$ for all $t \in I$ and $x \in U$. Since $\bigcup_k U_{n,k}$ is dense in $X$, there is $k$ such that $U_{n,k} \cap U \neq \emptyset$. Let $y \in U_{n,k} \cap U$ be a point and let $J$ be an open interval containing $u$ such that the inequality (4) holds for all $t \in J \cap K$. Then the point $(t, y) \in (J \cap K) \times U_{n,k}$ and

$$\varrho(f(t,y), f(u,y)) < 3\eta/4.$$

Consequently, for $(t, x) \in (J \cap K) \times U$ we have

$$\begin{gathered}\varrho(f(t,x), f(u,v)) \le \varrho(f(t,x), f(t,v)) + \\ \varrho(f(t,v), f(t,y)) + \varrho(f(t,y), f(u,y)) + \\ + \varrho(f(u,y), f(u,v)) < \eta/12 + \eta/12 + 3\eta/4 + \eta/12 = \eta. \ \Box\end{gathered}$$

**Theorem 4** *Let the spaces $(X, T_X)$ and $I$ be the same as those in Theorem 3 and let $(Y, \varrho)$ be a separable metric space. Suppose that all sections $f_t, t \in I$, of a function $f : I \times X \to Y$ are $T_X$-equicontinuous and all its sections $f^x, x \in X$, have the Baire property. Then there is a set $K \subset I$ residual in $I$ and such that the restriction $f|(K \times X)$ is $(T_e|K \times T_X)$-continuous.*

PROOF. The proof is similar to the proof of Theorem 3. It suffices only to find sets $A_{n,k}$ which are residual in $I$. □

**Remark 2** *Obviously the topologies $T_d$ and $T_{ae}$ in $\mathbb{R}$ satisfy the condition (2) from Theorem 3. So Theorems 3 and 4 are true for the topologies $T_d$ and $T_{ae}$. Since the space $(\mathbb{R}, T_{ae})$ is separable, Theorem 3 for $T_{ae}$ was essentially already known ([1], Th.3).*

The next example shows that, under Martin's Axiom (see [9]), Theorem 3 doesn't hold if condition (2) is dropped or approximate continuity is replaced by approximate upper semicontinuity. The function $f$ of this Example 1 is the same as that from Th.1 in [5], in which the continuum hypothesis is used instead of Martin's Axiom.

**Example 1.** Assume Martin's Axiom (MA). There is a transfinite sequence $a_1, \ldots, a_\alpha, \ldots, \alpha < \Omega$, of all reals such that $a_\alpha \neq a_\beta, \alpha \neq \beta, \alpha, \beta < \Omega$, and all sets $A_\alpha = \{a_\beta; \beta < \alpha\}, \alpha < \Omega$, are of Lebesgue measure zero (of the first category). Let $A = \{(a_\alpha, a_\beta); \beta < \alpha < \Omega\}$ and let

$$f(x,y) = \begin{cases} 1 & \text{for } (x,y) \in A \\ 0 & \text{otherwise in } \mathbb{R}^2. \end{cases}$$

If $I = X = \mathbb{R}$ and $T_X$ is the discrete topology in $X$, then all sections $f_t, t \in \mathbb{R}$, are $(T_X)$-equicontinuous and all sections $f^x, x \in X$, are Lebesgue measurable (have the Baire property), but for every closed set $K \subset I$ of positive Lebesgue measure (for every set $K \subset I$ which is residual in $I$) the restriction $f|(K \times X)$ is not $(T_e|K \times T_X)$-continuous. If $I = X = \mathbb{R}$ and $T_X = T_d$, then all sections $f_t, t \in I$, are $(T_X)$-upper semi-equicontinuous (i.e. for every $x \in X$ and for every $\eta > 0$ there is a set $U \in T_X$ containing $x$ and such that $f(t,y) - f(t,x) < \eta$ for all $y \in U, t \in I$), but for every closed set $K \subset I$ of positive measure (for every $K \subset I$ residual in $I$) the restriction $f|(K \times X)$ is not $(T_e|K \times T_X)$-upper- semicontinuous.

## References

[1] D. Averna, A. Fiacca, *Alcuni risultati sui teoremi di G. Scorza Dragoni e Tibaldo in spazi astratti*, Riv. Mat. Univ. Parma, **12** (1986), 217–225.

[2] A. M. Bruckner, *Differentiation of real functions*, Lecture Notes in Math. 659, Springer-Verlag, Berlin 1978.

[3] D. H. Fremlin, *Measurable functions and almost continuous functions*, Manuscripta Math. **33** (1981), 387–405.

[4] C. Goffman, C. J. Neugebauer, T. Nishiura, *Density topology and approximate continuity*, Duke Math. J. **28** (1961), 497–505.

[5] Z. Grande, *Semiéquicontinuité approximative et la mesurabilité* , Coll. Math. 45 (1981), 133–135.

[6] A. Kucia, *Scorza Dragoni type theorems*, Fund. Math. **138** (1991), 197–203.

[7] M. Kuczma, *An introduction to the theory of functional equations and inequalities. Cauchy's equation and Jensen's inequality*, PWN Warszawa-Kraków-Katowice 1985.

[8] R. J. O'Malley, *Approximately differentiable functions. The r topology*, Pacific J. Math. **72** (1977), 207–222.

[9] D. Martin, R. M. Solovay, *Internal Cohen extensions*, Ann. Math. Logic. **2** (1970), 143–178.

[10] F. D. Tall, *The density topology*, Pacific J. Math. **62** (1976), 275–284.