Real Analysis Exchange
Vol. 18(2), 1992/93, pp. 471–475

T. Natkaniec,*Instytut Matematyki WSP, ul. Chodkiewicza 30, 85-064 Bydgoszcz, Poland, email: wspb11@pltumk11.bitnet

ON A PROBLEM CONCERNING UNIVERSALLY BAD DARBOUX FUNCTIONS

Introduction. For a given family $\mathcal{F}$ of real functions let $c(\mathcal{F})$ denote the following condition

$c(\mathcal{F})$: there exists a Darboux function f such that $f+g$ is Darboux for no $g \in \mathcal{F}$.

Such a function f is called *a universally bad Darboux function* for $\mathcal{F}$. The problem for which families $\mathcal{F}$ the condition $c(\mathcal{F})$ is fulfilled was considered by many authors (see e.g. [8], [10], [3], [6], [5]). In particular, it is proved in [6] that if the additivity of the ideal of all first category subsets of $\mathbb{R}$ is equal to 2^ω (e.g. if Martin's Axiom or CH hold) then $c(C^*)$ holds for the family C^* of all nowhere constant, continuous functions.[1]

In [6] there was posed the following problem:

Problem 1 *Does the condition $c(C^{**})$ hold for the family C^{**} of all non-constant continuous functions?*

In the present note we answer this question in the negative.

Notation. We consider real functions defined on the real line. A functin f is said to be

- *a Darboux function* iff $f(J)$ is an interval for every interval J,

*Supported by KBN Research Grant 1992-94, No 2 1144 91 01.

Key Words: Darboux, Cantor type, Cesaro type, cliquish functions

Mathematical Reviews subject classification: Primary 26A15 Secondary 54C40.

Received by the editors February 17, 1992

[1]Note that this result is independent of ZFC. Indeed, during the last Winter School on Abstract Analysis in Poděbrady, Prof. Todorčević informed me that recently Prof. T. Steprans from Toronto constructed a model for $\mathbf{ZFC} + \neg c(C^*)$.

- *nowhere constant* iff for no $y \in \mathbb{R}$ the set $f^{-1}(y)$ contains a non-degenerate interval,
- of *the Cesaro type* iff there exist non-degenerate intervals $I, J \subset \mathbb{R}$ such that $f^{-1}(y)$ is dense in I for each $y \in J$,
- *cliquish* iff for each $\varepsilon > 0$, every non-empty open set $U \subset \mathbb{R}$ contains a non-empty open set W such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in W$,
- of *the Cantor type with respect to a Cantor set* F iff F is a nowhere dense perfect set and f is increasing, continuous, strictly increasing in the points of F and constant on every component of the complement of F.

Recall that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is cliquish if it is pointwise discontinuous (i.e. the set $C(f)$ of all points at which f is continuous is dense in $\mathbb{R}$) [1]. Moreover, if f is Darboux and not cliquish then it is of the Cesaro type [4] (cf [9]).

Theorem 1 *For each Darboux function f there exists a Cantor type function g such that $f + g$ is Darboux.*

Proof. First assume that f is cliquish. Then the set $C(f)$ is residual, so there exists a Cantor set C contained in $C(f)$ (see e.g. [7], Lemma 5.1). Let $g : \mathbb{R} \longrightarrow [0, 1]$ be a function of the Cantor type with respect to C. Then $f + g$ has the Darboux property on each component of the complement of C and it is continuous at each $x \in C$. Therefore $f + g$ has the Darboux property at each $x \in \mathbb{R}$ and consequently it is Darboux (see [2], p. 100 for the definition of a Darboux point of a real function).

Now assume that f is not cliquish. By Gibson's theorem [4] f is of the Cesaro type. Let K and J be compact intervals such that for each $y \in J$ the level set $f^{-1}(y)$ is dense in K. Let $J = [c, d]$. We shall choose a Cantor set $F \subset K$.

Lemma 1 *Assume that f is a Darboux function on I and $f^{-1}(y)$ is dense in I for each $y \in (c, d)$. Then for each $n \in N$ there exist an interval $I_0 \subset I$ and $c_0, d_0 \in [-\infty, \infty]$ such that*

(1) $c_0 \le c$, $d \le d_0$,

(2) $f^{-1}(y)$ *is dense in* I_0 *for each* $y \in (c_0, d_0)$,

(3) $f(x) \in (c_0 - 1/n, d_0 + 1/n)$ *for each* $x \in I_0$.

Proof. Fix $n \in N$. Put

$$c_0 = \inf\{z \le c : \ f^{-1}(y) \text{ is dense in } I \text{ for each } y \in (z, d)\}.$$

Note that if $c_0 > -\infty$ then there exist a non-degenerate interval $J \subset I$ and $z \in (c_0 - 1/n, c_0)$ such that $z \notin f(J)$. Therefore $f(x) > c_0 - 1/n$ for $x \in J$. If $c_0 = \infty$ then put $J = I$.

Now let

$$d_0 = \sup\{z \ge d : \ f^{-1}(y) \text{ is dense in } J \text{ for each } y \in (c_0, z)\}.$$

Observe that if $d_0 < \infty$ then there exist a non-degenerate interval $I_0 \subset J$ and $z \in (d_0, d_0 + 1/n)$ such that $z \notin f(I_0)$. Therefore $f(x) < d_0 + 1/n$ for $x \in I_0$. If $d_0 = \infty$ then $I_0 = J$.

It is easy to verify that I_0, c_0, d_0 satisfy all conditions (1) – (3).

□

Lemma 2 *Assume that f is a Darboux function on I and $f^{-1}(y)$ is dense in I for each $y \in (c, d)$. Then there exists a Cantor set $C \subset I$ such that for every $x \in C$, if x is not isolated in C from the right (left) then for each $n \in N$ there exist a component J of $I \setminus C$ and $c_J, d_J \in [-\infty, \infty]$ such that*

(1) $J \subset (x, x + 1/n)$ $(J \subset (x - 1/n, x))$,

(2) $c_J \le c$, $d \le d_J$,

(3) $f^{-1}(y)$ *is dense in* J *for each* $y \in (c_J, d_J)$,

(4) $f(x) \in (c_J - 1/n, d_J + 1/n)$.

Proof. Let Σ_n be the family of all finite binary sequences of the length equal to n and let $\Sigma = \bigcup_n \Sigma_n$. By Lemma 1 we can choose (inductively) a net $(I_\sigma)_{\sigma \in \Sigma}$ of closed subintervals of I such that

(*i*) if $\sigma \subset \delta$ then $I_\delta \subset I_\sigma$,

(*ii*) if σ and δ are inconsistent then $I_\sigma \cap I_\delta = \emptyset$,

(*iii*) for each $\sigma \in \Sigma$ there exist c_σ, d_σ such that

- $c_\sigma \le c$, $d \le d_\sigma$,
- $f^{-1}(y)$ is dense in I_σ for each $y \in (c_\sigma, d_\sigma)$,
- $f(I_\sigma) \subset (c_\sigma - 1/n, d_\sigma + 1/n)$ for each $\sigma \in \Sigma_n$.

Then $C = \bigcap_{n\in N} \bigcup_{\sigma\in\Sigma_n} I_\sigma$ has the required properties.

□

Let $g : I \longrightarrow [0, h]$ be a Cantor type function for C, where C is chosen as in Lemma 2 and $h = (d-c)/2$. We shall verify that $f+g$ is a Darboux function. Fix x_0, x_1 and y such that $x_0 < x_1$ and $y_0 < y < y_1$, where $y_i = (f+g)(x_i)$ for $i = 0, 1$. There are two possible cases: either $y \leq d - h$ or $y \geq c + h$. Assume that e.g. the first case holds (the proof is similar in the other case). Now we have two subcases.

First, assume that there exist a component $J = (a, b)$ of $I\setminus C$ and $t_J \in [0, h]$ such that $x_0 \in [a, b)$ and $g(x) = t_J$ for each $x \in [a, b]$. Then $f(x_0) = y_0 - t_J < y - t_J < d$. By the assumptions on f, there exists $x \in (x_0, x_1) \cap J$ such that $f(x) = y - t_J$, so $(f + g)(x) = y$.

Now assume that x_0 is an accumulation point of C from the right. Put $t = y - y_0$. Since g is continuous at x_0 and C fulfils the conditions (1) – (4) of Lemma 2, there exist a component J of $I \setminus C$ and $c_J, d_J \in [-\infty, \infty]$ such that

- $|g(x_0) - t_J| < t/2$, where $g(x) = t_J$ for $x \in J$,
- $J \subset (x_0, x_1)$,
- $f^{-1}(y)$ is dense in J for each $y \in (c_J, d_J)$, where $c_J \leq c$, $d \leq d_J$,
- $f(x_0) \in (c_J - t/2, d_J + t/2)$.

We shall verify that $y - t_J \in (c_J, d_J)$. Since $y < d$, $y - t_J < d \leq d_J$. On the other side, $y = y_0 + t = f(x_0) + g(x_0) + t > c_J + g(x_0) + t/2 > c_J + t_J$. Thus $y - t_J > c_J$, so $f(x) = y - t_J$ for some $x \in J$, and consequently $(f+g)(x) = y$.

□

Corollary 1 *There is no universally bad Darboux function for the class of all non-constant continuous functions.*

References

[1] W. W. Blendsoe, *Neighbourly functions*, Proc. Amer. Math. Soc. 3 (1952), 114–115.

[2] A. M. Bruckner and J. Ceder, *Darboux continuity*, Jber. Deutsch. Math. Ver. 67 (1965), 93–117.

[3] A. M. Bruckner and J. Ceder, *On the sum of Darboux functions*, Proc. Amer. Math. Soc. 51 (1975), 97–102.

[4] R. G. Gibson, *Concerning a characterization of continuity*, a manuscript.

[5] P. Komjáth, *A note on Darboux functions*, a manuscript.

[6] B. Kirchheim and T. Natkaniec, *On universally bad Darboux functions*, Real Analysis Exchange 16 (1990–1991), 481–486.

[7] J. C. Oxtoby, *Measure and category*, Springer-Verlag, 1971.

[8] T. Radaković, *Über Darbouxsche und stetige Funktionen*, Monat. Math. Phys. 38 (1931), 111–122.

[9] J. Smital and E. Stanova, *On almost continuous functions*, Acta Math. Univ. Comen. 37 (1980), 147–155.

[10] R. Švarc, *On the range of values of the sum of a continuous and a Darboux function*, Čas. Pest. Mat. 98 (1973), 178–180, 213.