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NOTE ON POINT SET THEORY

A large number of analogies between Baire category and Lebesgue measure are unified and generalized in [3]. Here an additional analogy established in [5] is generalized to perfect category bases $(X,\mathcal{C})$, where X is a dense-in-itself complete metric space. For definitions and properties used below refer to [1]-[4].

Theorem. For any given sequence of Baire sets, there exists in each abundant Baire set a denumerable set which cannot be represented as the limit of any subsequence of the given sequence.

Proof. Let $\langle E_n \rangle_{n\in\mathbb{N}}$ be a given sequence of Baire sets and let S be an abundant Baire set. According to the Fundamental Theorem, there exists a region A in which S is abundant everywhere. By Theorem 1.III.2 of [3] we have

$$A - S = \bigcup_{i=1}^{\infty} T_i$$

where each set T_i is a singular set. We proceed to determine a dyadic schema of subregions A_σ of A, where σ varies over all finite sequences of elements of the set $\mathbb{B} = \{0,1\}$.

Define A_0 and A_1 to be two disjoint subregions of A each of which has diameter ≤ 1 and is disjoint from the set T_1. For fixed $\beta \in \mathbb{B}$ we denote by $R_{\beta,1}$ the first one of the sets E_1, $X - E_1$ which is abundant in A_β and choose a subregion C_β of A_β in which $R_{\beta,1}$ is abundant everywhere. Since $R_{\beta,1}$ is a Baire set we have

$$C_\beta - R_{\beta,1} = \bigcup_{i=1}^{\infty} T_{\beta,i}$$

where each set $T_{\beta,i}$ is singular. We then define $A_{\beta 0}$ and $A_{\beta 1}$ to be two disjoint subregions of C_β each of which has diameter $\leq \frac{1}{2}$ and is disjoint from T_1, T_2, $T_{\beta,1}$ and $T_{\beta,2}$.

Assume $m \in \mathbb{N}$ and that for $\sigma \in \mathbb{B}^m$ and all $i \in \mathbb{N}$ we have already determined the set $R_{\sigma,m}$, the singular sets $T_{\sigma,i}$, and regions $A_{\sigma 0}$, $A_{\sigma 1}$. Fix $\beta \in \mathbb{B}^{m+1}$. Let $R_{\beta,m+1}$ denote the first one of the sets E_{m+1}, $X - E_{m+1}$ which is abundant in A_β and let C_β be a subregion of A_β in which $R_{\beta,m+1}$ is abundant everywhere. Then

$$C_\beta - R_{\beta,m+1} = \bigcup_{i=1}^{\infty} T_{\beta,i}$$

where each set $T_{\beta,i}$ is singular. Define $A_{\beta 0}$ and $A_{\beta 1}$ to be two disjoint subregions of C_β each of which has diameter $\leq \frac{1}{m+2}$ and is disjoint from all previously defined sets T with index $i \leq m+2$.

Let

$$P = \bigcap_{m=1}^{\infty} \bigcup_{\sigma \in \mathbb{B}^m} A_\sigma$$

be the perfect set obtained from the dyadic schema thus determined and let D be a denumerable subset of P which is everywhere dense in P. It is clear that $D \subset A \cap S$.

For each $m \in \mathbb{N}$ define

$$P_m = \bigcup \{A_{\sigma 0} \cup A_{\sigma 1} : \sigma \in \mathbb{B}^m \text{ and } R_{\sigma,m} = E_m\}$$
$$Q_m = \bigcup \{A_{\sigma 0} \cup A_{\sigma 1} : \sigma \in \mathbb{B}^m \text{ and } R_{\sigma,m} = X - E_m\}$$

Then

$$P = \bigcap_{m=1}^{\infty} (P_m \cup Q_m)$$

Because P is disjoint from all the sets $T_{\sigma,i}$, for all $\sigma \in \mathbb{B}^m$ and all $i \in \mathbb{N}$, we have $P \subset R_{\sigma,m}$ so that

$$P \cap P_m \subset E_m \qquad \text{and} \qquad P \cap Q_m \subset X - E_m$$

This implies

$$\begin{aligned} P - E_m &= [P \cap (P_m \cup Q_m)] - E_m \\ &= [(P \cap P_m \cap E_m) \cup (P \cap Q_m)] - E_m \\ &= P \cap Q_m \end{aligned}$$

for every $m \in \mathbb{N}$.

Suppose now that there did exist a subsequence $\langle E_{n_k} \rangle_{k \in \mathbb{N}}$ of the sequence $\langle E_n \rangle_{n \in \mathbb{N}}$ such that

$$D = \lim_k E_{n_k} \quad .$$

Then it follows that

$$P—D = \lim_k (P—E_{n_k}) \quad .$$

As seen from the preceding paragraph, each of the sets $P—E_{n_k}$ is a closed set. Hence, the set

$$P—D = \bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} (P—E_{n_k})$$

is an $\mathcal{F}_\sigma$-set. But, D being a denumerable set everywhere dense in the perfect set P, this leads to the contradiction that P is of the first category in itself!

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