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MARTINGALE PROOF OF THE EXISTENCE OF LEBESGUE POINTS

The usual proof of the existence of Lebesgue points of a summable function is via Vitali's covering theorem or its modifications. We give here an alternative proof which reduces geometric considerations to a very simple lemma. Our proof is based on Lévy's martingale convergence theorem.

Let $\mathcal{A}$ be a family of sets and let X be any set. Then $\mathcal{A}|X = \{A \cap X : A \in \mathcal{A}\}$. By $\sigma\mathcal{A}$ we denote the σ-field generated by $\mathcal{A}$. Let $\mathbb{Z}$ be the set of all integers. The symbol χ_A will stand for the characteristic function of a set A. The n-dimensional Lebesgue measure in $\mathbb{R}^n$ will be denoted by λ (the same symbol λ will be used for each positive integer n). For any measure space $(\Omega, \mathcal{F}, \mu)$ we shall denote by $L_1(\Omega)$ the family of real functions f measurable with respect to $\mathcal{F}$ such that $\int_\Omega |f|\, d\mu < \infty$. If Ω is an open subset of $\mathbb{R}^n$, $\mathcal{F}$ will be the family of Lebesgue measurable sets and $\mu = \lambda$. For a subset X of $\mathbb{R}^n$ and a vector $x \in \mathbb{R}^n$ put $x + X = \{x + y : y \in X\}$.

Let $f \in L_1(U)$, where U is an open subset of $\mathbb{R}^n$. We say that $x \in U$ is a *Lebesgue point* of f if

$(1/\lambda(Q_m)) \int_{Q_m} |f(s) - f(x)|\, ds \to 0$ for each sequence of cubes Q_m such that $x \in Q_m \subseteq U$, $m = 1, 2, \ldots$, and $\lambda(Q_m) \to 0$ (without loss of generality we can assume that x is the center of Q_m, i.e., $Q_m = x + (-\delta_m, \delta_m)^n$ for some $\delta_m > 0$).

Let $(\Omega, \mathcal{B}, P)$ be a probability space. Let $f \in L_1(\Omega)$ and $\mathcal{F}$ be a σ-algebra (i.e. a σ-field containing Ω) contained in $\mathcal{B}$. Let $E(f|\mathcal{F})$ denote a function measurable with respect to $\mathcal{F}$ such that $\int_A f\, dP = \int_A E(f|\mathcal{F})\, dP$ for each $A \in \mathcal{F}$. Its existence is guaranteed by the Radon - Nikodym theorem.

Let us now recall the theorem in question.

THEOREM 1 (H. Lebesgue). *Let* $f \in L_1(\mathbb{R}^n)$. *Then almost every point of* $\mathbb{R}^n$ *is a Lebesgue point of* f.

An essential role in our proof of the above theorem will be played by the following theorem of P. Lévy ([3], Theorem 9.4.8, p.340; for an elementary proof see [2], Theorem 1.4; see also Remark 2 below).

THEOREM 2 (P. Lévy). *Let* $(\Omega, \mathcal{B}, P)$ *be a probability space. Let* $\{\mathcal{B}_m : m \geq 1\}$ *be an increasing sequence of* σ*-algebras contained in* $\mathcal{B}$ *and* $\mathcal{B}_\infty = \sigma \bigcup_{m=1}^{\infty} \mathcal{B}_m$.

Then for each function $f \in L_1(\Omega)$

$$\lim_{m\to\infty} E(f|\mathcal{B}_m)(\omega) = E(f|\mathcal{B}_\infty)(\omega)$$

for almost every $\omega \in \Omega$.

We need two lemmas.

LEMMA 1. *Let* $(\Omega, \mathcal{B}, P)$ *be a probability space and* $f \in L_1(\Omega)$. *Then* f *is the limit of a uniformly convergent sequence of functions* $f_m \in L_1(\Omega)$ *with* $f_m(\Omega)$ *countable,* $m = 1, 2, \ldots$.

Proof. Let $A_m^k = f^{-1}((k/m, (k+1)/m])$ and $f = \sum_{k\in\mathbb{Z}} (k/m)\chi_{A_m^k}$, for $k, m \in \mathbb{Z}$, $m > 0$. Then $\sup\{|f(\omega) - f_m(\omega)| : \omega \in \Omega\} \le 1/m$ and $f_m \in L_1(\Omega)$.■

Let $T = \{0, 1/3\}^n$. Let us define for $t \in T$ and $m \in \mathbb{Z}$ a covering of $\mathbb{R}^n$

$\mathcal{A}_m^t = \{ t + ((k_1\cdot 2^{-m}, (k_1+1)\cdot 2^{-m}] \times \ldots \times (k_n\cdot 2^{-m}, (k_n+1)\cdot 2^{-m}]) :$ $k_i \in \mathbb{Z}$ for $i \in \{1,\ldots, n\}\}$.

Let us notice that the elements of $\mathcal{A}_m^t$ are pairwise disjoint and their union covers $\mathbb{R}^n$.

LEMMA 2. *Let* $x = (x_1,\ldots, x_n) \in \mathbb{R}^n$ *and for* $t \in T$ *let* I^t *be the unique element of* $\mathcal{A}_m^t$ *such that* $x \in I^t$. *Let* δ *be a real number satisfying* $3^{-1}\cdot 2^{-m} > \delta > 0$. *Then* $x + (-\delta, \delta)^n \subseteq \bigcup\{I^t : t \in T\}$.

Proof. Let $x_i \in I_i \cap J_i$ where $I_i = (k_i\cdot 2^{-m}, (k_i+1)\cdot 2^{-m}]$ and $J_i = (p_i\cdot 2^{-m} + 3^{-1}, (p_i+1)\cdot 2^{-m} + 3^{-1}]$ for some (unique) $k_i, p_i \in \mathbb{Z}$. Since the distance from each end-point of I_i to the end-points of J_i is greater than δ, $(x_i - \delta, x_i + \delta) \subseteq I_i \cup J_i$.

As $\bigcup\{I^t : t \in T\}$ is the Cartesian product of the sets $I_i \cup J_i$, $i = 1,\dots n$, the result follows.∎

Proof of Theorem 1. It is enough to prove that almost every point of the cube $(0, 1)^n$ is a Lebesgue point of f. Thus we may assume that $f \in L_1((0, 1)^n)$.

At first we suppose that f has a countable range. For $m = 1, 2,\dots$ and $t \in T$ we define $\mathcal{B}^t_m = \sigma\, \mathcal{A}^t_m|(0, 1)^n$. Let us notice that for each $t \in T$ the σ-algebra $\mathcal{B}^t_\infty = \sigma \bigcup_{m=1}^{\infty} \mathcal{B}^t_m$ is the σ-algebra of all Borel subsets of $(0, 1)^n$. Let $a \in \mathbb{R}$. By Theorem 2 for almost every $x \in (0, 1)^n$ we have

$$E(|f - a||\mathcal{B}^t_m)(x) \to E(|f - a||\mathcal{B}^t_\infty)(x) = |f(x) - a|.$$

Thus for almost every $x \in f^{-1}(a)$

$$E(|f - a||\mathcal{B}^t_m)(x) \to 0.$$

Since f has a countable range we obtain

$$(*) \quad E(|f - f(x)||\mathcal{B}^t_m)(x) \to 0$$

for almost every $x \in (0, 1)^n$.

We shall show now that such x is a Lebesgue point of f. For $t \in T$ we define a sequence $\{I^t_m : m = 1, 2,\dots\}$, where I^t_m is the unique element of $\mathcal{A}^t_m$ such that $x \in I^t_m$. The inclusion $I^t_m \subset (0, 1)^n$ holds for each $t \in T$ and $m \geq K$ for some positive integer K. Then

$$E(|f - f(x)||\mathcal{B}^t_m)(x) = (1/\lambda(I^t_m))\int_{I^t_m} |f(s) - f(x)|\, ds.$$

Thus by $(*)$ we have

$$(**)\qquad (1/\lambda(I_m^t))\int_{I_m^t} |f(s) - f(x)|\, ds \to 0.$$

For any $\delta > 0$ we find the unique m such that $2^{-m}\cdot 3^{-1} > \delta \geq 2^{-m-1}\cdot 3^{-1}$. By Lemma 2 we obtain

$$0 \leq (1/(2\delta)^n) \int_{x+(-\delta,\delta)^n} |f(s) - f(x)|\, ds \leq$$

$$(1/(2\delta)^n)\Big(\sum_{t\in T} \int_{I_m^t} |f(s) - f(x)|\, ds \Big) \leq$$

$$3^n\Big(\sum_{t\in T} (1/\lambda(I_m^t))\int_{I_m^t} |f(s) - f(x)|\, ds \Big).$$

Hence by (**) x is a Lebesgue point of f.

Now let f be an arbitrary function from $L_1((0, 1)^n)$. By Lemma 1 f is the limit of a uniformly convergent sequence of functions $\{f_m : m = 1, 2,\dots\} \subseteq L_1((0, 1)^n)$, where each f_m has a countable range. Let $A = \{x \in (0, 1)^n : x$ is a Lebesgue point for each f_m, $m = 1, 2,\dots\}$. Then $\lambda(\mathbb{R}^n \setminus A) = 0$.

Let $x \in (0, 1)^n$ and let $Q \subseteq (0, 1)^n$ be a cube. Then

$$(1/\lambda(Q))\int_Q |f(s) - f(x)|\, ds \leq$$

$$\int_Q |f_m(s) - f_m(x)|\, ds + 2 \sup \{|f(s) - f_m(s)| : s \in (0, 1)^n\}.$$

Hence if $x \in A$ and Q is a cube of center x we have

$$\limsup_{\lambda(Q)\to 0} (1/\lambda(Q))\int_Q |f(s) - f(x)|\, ds \leq 2 \sup \{|f(s) - f_m(s)| : s \in (0, 1)^n\}$$

for $m = 1, 2,\dots$. Thus

$$\lim_{\lambda(Q)\to 0} (1/\lambda(Q))\int_Q |f(s) - f(x)|\, ds = 0. \blacksquare$$

Remark 1. Theorem 1 implies Lebesgue's density theorem. L. Zajíček [5] and F. Cater [1] have recently proved the one dimensional version of this theorem without using a covering lemma.

Remark 2. It was pointed out to us by Professor K. Krickeberg and Professor M. Laczkovich that Lévy's theorem in the case of σ-algebras generated by countable partitions, the only one we use, was already known to de la Vallée Poussin [4].

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