Real Analysis Exchange Vol 15 (1989-90)

Li Baoling, Department of Mathematics, Northwest Normal University, Lanzhou, China

CONVERGENCE THEOREMS FOR THE VARIATIONAL INTEGRAL

1. Introduction

The variational integral is a kind of nonabsolute integrals originally defined by R. Henstock [1]. It is equivalent to the Riemann complete integral [2]. In [3], Yoto Kubota has shown some elementary properties of the integral, including the important Cauchy and Harnack extensions. In this paper, we shall establish some significant convergence theorems for the integral.

Definition 1.1 Let $[a,b]$ be a compact interval on the real line, and $\delta(\xi)$ a positive real function defined on $[a,b]$. The finite set

$$P = \{x_0, x_1, \dots, x_p; \xi_1, \dots, \xi_p\} \tag{1.1}$$

is said to be a δ-fine division over $[a,b]$ if

$$a = x_0 < x_1 < \dots < x_p = b \text{ and}$$
$$\xi_i \in [x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)) \text{ for } i = 1, 2, \dots, p.$$

Alternatively, we write

$$P = \{[u,v]; \xi\} \tag{1.2}$$

where $[u,v]$ denotes a typical subinterval in the division, and

$$\xi \in [u,v] \subset (\xi - \delta(\xi), \xi + \delta(\xi)). \tag{1.3}$$

Definition 1.2 An interval function S is said to be superadditive if, for any two adjacent non-overlapping intervals I_1 and I_2, we have

$$S(I_1 \cup I_2) \geq S(I_1) + S(I_2) \tag{1.4}$$

Definition 1.3 Let $f : [a,b] \longrightarrow R$ be a measurable function. Then f is said to be variationally integrable on $[a,b]$ if there is a function F such that, for every $\epsilon > 0$, there is a $\delta(\xi) : [a,b] \longrightarrow (0,\infty)$, and a superadditive interval function S such that

$$0 = S([a,a]) \leq S([a,b]) < \epsilon \tag{1.5}$$

and that whenever $\xi - \delta(\xi) < u \leq \xi \leq v < \xi + \delta(\xi)$ we have

$$|F(u,v) - f(\xi)(v - u)| \le S(u,v) \tag{1.6}$$

where $F(u,v) = F(v) - F(u)$. Here F is called the primitive of f. For convenience, when f is variationally integrable on $[a,b]$ we write $f \in (V)$, and

$$(V) \int_a^b f(x)dx = F(b) - F(a) \tag{1.7}$$

2. A basic convergence theorem and its simple corollaries

We state and prove the following basic convergence theorem.

Theorem 2.1 Let $f_n \in (V)$ with primitives F_n, $n = 1, 2, \ldots$, $f_n(x)$ tend to $f(x)$ everywhere in $[a,b]$ as $n \to \infty$, and $F_n(x)$ converge pointwise to a limit function $F(x)$. Then in order that $f \in (V)$ with primitive F, and

$$\lim_{n\to\infty} (V) \int_a^b f_n(x)dx = (V)\int_a^b f(x)dx \tag{2.1}$$

it is necessary and sufficient that, for every $\epsilon > 0$, there exists $M(\xi)$ taking integer values such that, for infinitely many $m(\xi) \ge M(\xi)$, there is a $\delta_m(\xi)$: $[a,b] \to (0, \infty)$ and a superadditive interval function S on $[a,b]$ such that

$$0 = S([a,a]) \le S([a,b]) < \epsilon$$

and whenever $\xi - \delta_m(\xi) < u \le \xi \le v < \xi + \delta_m(\xi)$ we have

$$|F_{m(\xi)}(u,v) - F(u,v)| \le S([u,v]) \tag{2.2}$$

where $F_m(u,v) = F_m(v) - F_m(u)$ and $F(u,v) = F(v) - F(u)$.

<u>Proof</u>. As $n \to \infty$, $f_n(x)$ tends to $f(x)$ everywhere in $[a,b]$. Thus, given $\epsilon > 0$, there is a $M(x)$ taking integer values such that

$$|f_n(x) - f(x)| < \epsilon \quad \text{for} \quad \text{all } n \ge M(x). \tag{2.3}$$

If $f \in (V)$ with primitive F, then, for the above $\epsilon > 0$, there is a $\delta_0(\xi)$: $[a,b] \to (0, \infty)$ and a superadditive interval function S_0 such that

$$0 = S_0([a,a]) \le S_0([a,b]) < \epsilon$$

and that whenever $\xi - \delta_0(\xi) < u \le \xi \le v < \xi + \delta_0(\xi)$ we have

$$|F(u,v) - f(\xi)(v - u)| \le S_0([u,v]). \tag{2.4}$$

Similarly, for each n, there is a $\delta_n^*(\xi) > 0$, and a superadditive interval function S_n such that

$$0 = S_n([a,a]) \le S_n([a,b]) < \epsilon 2^{-n}$$

and that whenever $\xi - \delta_n^*(\xi) < u \le \xi \le v < \xi + \delta_n^*(\xi)$ we have

$$|F_n(u,v) - f_n(\xi)(v - u)| \le S_n(u,v) \tag{2.5}$$

Now put $\delta_m(\xi) = \min\{\delta_0(\xi),\ \delta_m^*(\xi)\}$ and

$$S([u,v]) = S_0([u,v]) + \epsilon(v - u) + \sum_{n=1}^{\infty} S_n([u,v])$$

then S is a superadditive interval function, and

$$0 = S([a,a]) \le S([a,b]) < \epsilon(2 + b - a).$$

Hence it follows that whenever $\xi - \delta_m(\xi) < u \le \xi \le v < \xi + \delta_m(\xi)$ we have

$$\begin{aligned} &|F(u,v) - F_{m(\xi)}(u,v)| \\ &\le |F(u,v) - f(\xi)(v - u)| + |f(\xi)(v - u) - f_{m(\xi)}(\xi)(v - u)| \\ &+ |F_{m(\xi)}(u,v) - f_{m(\xi)}(\xi)(v - u)| \\ &\le S([u,v]) \end{aligned} \tag{2.6}$$

that is, the conditions are necessary.

Similarly, we prove that the conditions are sufficient.

Theorem 2.2 (Vitali's convergence theorem) If the following conditions are satisfied:

(i) $f_n(x)$ tends to $f(x)$ as $n \longrightarrow \infty$ almost everywhere in $[a,b]$ where $f_n \in (V)$, $n = 1, 2, \ldots$;

(ii) the primitives F_n of f_n, $n = 1, 2, \ldots$, are uniformly absolutely continuous uniformly in n, i.e., UAC on $[a,b]$,

then $F_n(x)$ converges pointwise to a limit function $F(x)$ an $n \longrightarrow \infty$, and for every $\epsilon > 0$, there is an integer N and there is a superadditive interval function S such that, for infinitely many $n \ge N$, and every $[u,v] \subset [a,b]$, we have

$$0 = S([a,a]) \le S([a,b]) < \epsilon$$

and

$$|F_n(u,v) - F(u,v)| \le S([u,v]) \tag{2.7}$$

where $F(u,v) = F(v) - F(u)$.

<u>Proof</u>. For simplicity, we may assume that $f_n(x)$ tends to $f(x)$ as $n \longrightarrow \infty$ everywhere in $[a,b]$. First, it is well-known that [4;p.37], for every $\epsilon > 0$, there is an integer N, for every partial division of $[a,b]$ given by

$$a \le a_1 < b_1 \le a_2 < b_2 \ldots \le a_k < b_k \le b$$

whenever n, $m \geq N$, we have

$$\sum_{i=1}^{k} |F_n(a_i, b_i) - F_m(a_i, b_i)| < \epsilon(4 + b - a) \tag{2.8}$$

Then $\lim F_n(u,v) = F(u,v)$ exists for any subinterval $[u,v]$ of $[a,b]$. It follows that we can find a subsequence $F_{n(j)}$ of F_n such that

$$\sum_{i=1}^{k} |F_{n(j)}(a_i, b_i) - F(a_i, b_i)| < \epsilon 2^{-j} \tag{2.9}$$

for any partial division as given above and for $j = 1, 2, \ldots$. Then put

$$S_j([u,v]) = \mathrm{Sup} \sum |F_{n(j)}(u',v') - F(u',v')| \tag{2.10}$$

where the supremum is over all divisions of $[u,v]$, and put

$$S([u,v]) = \sum_{j=1}^{\infty} S_j([u,v]). \tag{2.11}$$

Then S is a superadditive interval function and whenever $n(j) \geq N$, for every $[u,v] \subset [a,b]$, we have

$$0 = S([a,a]) \leq S([u,v]) < \epsilon$$

and

$$|F_{n(j)}(u,v) - F(u,v)| \leq S([u,v]) < \epsilon.$$

Hence the theorem is proved.

Corollary 2.3 Under the conditions of Theorem 2.2, we have $f \in (V)$, and

$$\lim_{n\to\infty} (V) \int_a^b f_n(x)dx = (V) \int_a^b f(x)dx \tag{2.12}$$

Corollary 2.4 Theorem 2.2 holds true with (2.7) replaced by

$$\omega(F_n - F;\ [u,v]) \leq S([u,v]) \tag{2.13}$$

where ω denotes the oscillation of $F_n - F$ over $[u,v]$.

Corollary 2.5 (Monotone convergence theorem) If the following conditions are satisfied:

(i) $f_n(x)$ tends to $f(x)$ almost everywhere in $[a,b]$ as $n \longrightarrow \infty$ where $f_n \in (V)$, $n = 1, 2, \ldots$;

(ii) the primitive $F_n(x)$ of $f_n(x)$ converge to a limit function $F(x)$ for all x;

(iii) $f_1(x) \le f_2(x) \le \ldots$, for x belonging to [a,b], then $f \in (V)$, and

$$\lim_{n\to\infty} (V) \int_a^b f_n(x)dx = (V) \int_a^b f(x)dx \tag{2.14}$$

Corollary 2.6 (Dominated convergence theorem) If the following conditions are satisfied:

(i) $f_n(x)$ tends to $f(x)$ almost everywhere in [a,b] as $n \to \infty$ where $f_n \in (V)$, $n = 1, 2, \ldots$;

(ii) $g(x) \le f_n(x) \le h(x)$ almost everywhere in [a,b], $n = 1, 2, \ldots$, where $g, h \in (V)$, then $f \in (V)$, and

$$\lim_{n\to\infty} (V) \int_a^b f_n(x)dx = (V) \int_a^b f(x)dx \tag{2.16}$$

3. The controlled convergence theorem

In this section, we shall establish a more general convergence theorem for the variational integral, namely, the controlled convergence theorem [4].

We state without proof a theorem [3] which we need later.

Theorem 3.1 If a function F is absolutely continuous on [a,b], and if its derivative $F'(x) = f(x)$ almost everywhere in [a,b], then $f \in (V)$.

Lemma 3.2 If the following conditions are satisfied:

(i) $f_n(x)$ tends to $f(x)$ almost everywhere in [a,b] as $n \to \infty$ where $f_n \in (V)$, $n = 1, 2, \ldots$;

(ii) the primitive F_n of f_n converges to a continuous function F on [a,b];

(iii) the primitive F_n of f_n is $AC_*(X)$ uniformly in n, i.e. $UAC_*(X)$, where X is a closed subset of [a,b],

then, for every $\epsilon > 0$, there are at most finitely many points $x_i \in X$ for $i = 1, 2, \ldots, K$, an integer N, and a $\delta(\xi) : X - \{x_i | i = 1, 2, \ldots, K\} \to (0,\infty)$ such that, for any δ-fine partial division $P = \{[u,v];\xi\}$ with $\xi \in X - \{x_i | i = 1, 2, \ldots, K\}$ we have

$$\sum |F_n(u,v) - F(u,v)| < \epsilon \quad \text{for infinitely many } n \ge N. \tag{3.1}$$

Proof By conditions (ii) and (iii), F is $AC_*(X)$. Define $G_n(a) = F_n(a)$, $G_n(b) = F_n(b)$, $G_n(x) = F_n(x)$ when $x \in X$ and linearly, on the complement of X. We shall prove that G_n is uniformly absolutely continuous on $[a,b]$.

Suppose $a, b \in X$ and $(a,b) - X = \bigcup_{j=1}^{\infty} (c_j, d_j)$. Since F_n is $UAC_*(X)$, then, for every $\epsilon > 0$, there is a $\eta > 0$, for every sequence of non-overlapping intervals $\{[a_i,b_i]\}$ whenever $\sum_i(b_i - a_i) < \eta$ and $a_i, b_i \in X$ we have

$$\sum_i \omega(F_n; [a_i, b_i]) < \epsilon \tag{3.2}$$

Take an integer K such that

$$\sum_{j\geq K+1} (d_j - c_j) < \eta. \tag{3.3}$$

Since $F_n(x)$ converges to $F(x)$ an $n \longrightarrow \infty$, then there exists an integer N^* such that, when $x = c_j$ or d_j, $j = 1, 2, \ldots, K$, for every $n \geq N^*$, we have

$$|F_n(x) - F(x)| < \frac{1}{2}$$

and consequently

$$|F_n(c_j,d_j) - F(c_j,d_j)| < 1$$

Put

$$\omega^* = \max \left\{\frac{|F(c_j,d_j)| + 1}{d_j - c_j}; j = 1, 2, \ldots, K\right\} \tag{3.4}$$

and $\quad \bar{\eta} = \min\{\eta, \epsilon/\omega^*\}$.

Then, for every sequence of non-overlapping intervals $\{[a_i, b_i]\}$, whenever $\sum_i (b_i - a_i) < \bar{\eta}$, and $n \geq N^*$, we have

$$\begin{aligned}\sum_i \omega(G_n; [a_i,b_i]) &\leq \sum_1 \omega(G_n; I_1) + \sum_2 \omega(G_n; I_2) + \sum_3 \omega(G_n; I_3) \\ &\leq 3\epsilon\end{aligned} \tag{3.5}$$

where each $[a_i,b_i]$ can be decomposed into at most three subintervals I_1, I_2 and I_3 such that I_1, I_2 and I_3 denote respectively the intervals for which (i) the endpoints of the intervals belong to X, (ii) the intervals are contained in $[c_j, d_j]$ where $j \leq K$, and (iii) the intervals are contained in $[c_j, d_j]$ where $j > K$, giving each term in inequality (3.5) less than ϵ.

In fact for the second term, assuming $I_2 \subset [c_j, d_j]$ for some j we have

$$\omega(G_n; I_2) = |G_n(I_2)| = \frac{|F_n(c_j, d_j)|}{d_j - c_j} |I_2|$$

$$\leq \frac{|F(c_j, d_j)| + 1}{d_j - c_j} |I_2| \leq \omega^* |I_2|. \tag{3.6}$$

Since G_n is absolutely continuous on $[a,b]$ for $n = 1, 2, \ldots, N^*$, then we can modify the $\bar{\eta}$ suitably such that, for every n, the inequality (3.5) still holds. Then G_n is uniformly absolutely continuous on $[a,b]$, and there exists $g_n(x)$ such that $G_n'(x) = g_n(x)$ almost everywhere in $[a,b]$. For $n \geq 1$, we can prove that $g_n(x)$ is convergent almost everywhere in $[a,b]$. Since $G_n(x)$ and $F_n(x)$ are differentiable almost everywhere in $[a,b]$, we have $G_n'(x) = F_n'(x) = f_n(x)$ almost everywhere in X. Then $f_n(x) = g_n(x)$ almost everywhere in X.

Since

$$g_n(x) = F_n(c_j, d_j)/(d_j - c_j) \qquad \text{for } x \in (c_j, d_j)$$

then $g_n(x)$ converges when $F_n(x)$ converges. It is easy to see that $G_n(x)$ converges to a limit function $G(x)$ as $n \to \infty$. Then, by Theorem 2.2, for every $\epsilon > 0$, there exists an integer N, and there is a superadditive interval function S such that, for infinitely many $n \geq N$ and every $[u,v] \subset [a,b]$, we have

$$0 = S([a,a]) \leq S([a,b]) < \epsilon$$

and

$$|G_n(u,v) - G(u,v)| \leq S([u,v]). \tag{3.7}$$

Now define $\delta(\xi)$ on $X - \{c_j, d_j | j = 1, 2, \ldots, K\}$ such that $0 < \delta(\xi) < d = \min\{|\xi - x|; x = c_j \text{ or } d_j \text{ for } j = 1, 2, \ldots, K\}$. Then, for any δ-fine partial division $P = \{[u,v]; \xi\}$ with $\xi \in X - \{c_j, d_j | j = 1, 2, \ldots, k\}$

$$\sum |F_n(u,v) - F(u,v)|$$

$$\leq \sum |F_n(u,\xi) - F(u,\xi)| + \sum |F_n(\xi,v) - F(\xi,v)| \tag{3.8}$$

For the first term on the right side of inequality (3.8), no change is needed to change when u belongs to X. When u belongs to (c_j, d_j), $j > K$, then

$$|F_n(u,\xi) - F(u,\xi)|$$
$$\le |F_n(u,d_j) - F(u,d_j)| + |F_n(d_j,\xi) - F(d_j,\xi)|$$
$$\le \omega(F_n;\ [c_j,\ d_j]) + \omega(F;[c_j,\ d_j]) + |G_n(d_j,\xi) - G(d_j,\xi)|$$

and similarly for the second term on the right side of inequality (3.8). Then we obtain

$$\sum|F_n(u,v) - F(u,v)| < 5\epsilon. \tag{3.9}$$

Theorem 3.3 (Controlled convergence theorem) If the following conditions are satisfied:

(i) $f_n(x)$ tends to $f(x)$ almost everywhere in $[a,b]$ as $n \longrightarrow \infty$ where $f_n \in (V)$, $n = 1, 2, \ldots$;

(ii) the primitives F_n of f_n converge to a continuous function F;

(iii) the primitives F_n of f_n are $UACG_*$, i.e., ACG_* uniformly in n, then $f \in (V)$, and

$$\lim_{n \longrightarrow \infty} (V) \int_a^b f_n(x)dx = (V) \int_a^b f(x)dx \tag{3.10}$$

Proof. Let $[a,b] = \bigcup_{i=1}^{\infty} X_i$ such that $F_n \in UAC_*(X_i)$ for each i. By Lemma 3.2, for each X_i and every $\epsilon > 0$, there is an integer N_i, and a positive function $\delta^i(\xi)$ defined on $X_i - G_i$ where G_i is a finite subset of X_i, for any δ^i-fine partial division $P = \{[u,v];\ \xi\}$ with $\xi \in X_i - G_i$, we have

$$\sum|F_n(u,v) - F(u,v)| < \epsilon 2^{-i}, \qquad n \ge N.$$

Put $$S_i([s,t]) = \text{Sup} \sum|F_n(u,v) - F(u,v)| \tag{3.11}$$

where $\sum$ is the sum over a δ^i-fine partial division $P = \{[u,v];\xi\}$ of $[s,t]$ with $\xi \in (X_i - G_i) \cap [s,t]$, and the supremum is over all the above divisions of $[s,t]$. Then S_i is a superadditive interval function, and when $n \ge N_i$, we have $0 = S_i([a,a]) \le S_i([a,b]) \le \epsilon 2^{-i}$. Since $G = \bigcup_{i=1}^{\infty} G_i$ is a denumerable set, put $G = \{x_k | k = 1, 2, \ldots\}$. Since the $F_n(x)$ and $F(x)$ are continuous then define $\delta_n^*(x_k)$ at each point x_k such that

$$\omega(F_n;\ [x_k - \delta_n^*(x_k),\ x_k + \delta_n^*(x_k)]) < \epsilon 2^{-k-n-1}$$
$$\omega(F;\ [x_k - \delta_n^*(x_k),\ x_k + \delta_n^*(x_k)]) < \epsilon 2^{-k-n-1}$$

whenever $x_k - \delta_n^*(x_k) < u \le x_k \le v < x_k + \delta_n^*(x_k)$. Put

$$S_0([u,v]) = \sum \epsilon 2^{-k} \tag{3.12}$$

where $\sum$ sums over x_k in $[u,v]$. Then S_0 is an additive interval function, and $0 = S_0([a,a]) \le S_0([a,b]) < \epsilon$. Put

$$S([s,t]) = \sum_{i}^{\infty} S_i([s,t]) + S_0([s,t]) \tag{3.13}$$

for $[s,t] \subset [a,b]$, then S is a superadditive interval function, and $n \ge N_i$ we have $0 = S([a,a]) \le S([a,b]) \le 2\epsilon$. Put

$$Y_1 = X_1, \quad Y_i = X_i - \cup_{k=1}^{i-1} X_k \qquad \text{for } i = 2, 3, \ldots$$

Define $M(\xi) = N_i$ when $\xi \in Y_i$, and define $\delta_m(\xi)$ for infinitely many $m(\xi) \ge M(\xi)$ as follows: $\delta_m(\xi) = \delta^i(\xi)$ when $\xi \in Y_i - G$, and $\delta_m(\xi) = \delta_m^*(x_k)$ when $\xi = x_k \in G$. Then

$$|F_{m(\xi)}(u,v) - F(u.v)| \le S([u,v]) \tag{3.14}$$

whenever $\xi - \delta_m(\xi) < u \le \xi \le v < \xi + \delta_m(\xi)$.

By the basic convergence theorem (Theorem 2.1), then $f \in (V)$, and the inequality (3.10) holds.

I am indebted to my supervisors Professor Lee Peng Yee and Professor Ding Chuan Song for sharing with me their knowledge and ideas. Their advice was invaluable in obtaining the results in this paper.

References

1. R. Henstock, Definitions of Riemann-type of the variational integrals, Proc. London Math. Soc. (3)11(1961) 402-418.

2. ___________, A Riemann-type integral of Lebesgue power, Canad. J. Math. 20(1968) 79-87.

3. Yoto Kubota, An elementary theory of the special Denjoy integral, Math. Japonica 24 (1980) 507-520.

4. Lee Peng Yee, Lanzhou lectures on Henstock integration, World Scientific 1989.

Received December 19, 1988