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WEIGHTED SYMMETRIC FUNCTIONS

In [11], I studied the classes of weighted symmetric functions, which are a generalization of the classes of symmetric and symmetrically continuous functions. By definition, we call a weight system of order n a set of real numbers

$$W_n = \{w_{-n}, \ldots, w_{-1}, w_0, w_1, \ldots, w_n\}$$

such that $\sum_{k=-n}^{n} w_k = 0$ and $|w_n|+|w_{-n}|>0$.

We say a weight system is even if

$$w_{-k} = w_k \qquad k = 0, 1, \ldots, n$$

with $\sum_{k=1}^{n} w_k \neq 0$; then $w_0 = -2\sum_{k=1}^{n} w_k \neq 0$

and a weight system is odd if

$$w_{-k} = -w_k \qquad k = 0, 1, \ldots, n$$

with $\sum_{k=1}^{n} w_k \neq 0$; then $w_0 = 0$.

We call symmetric difference with respect to a weight system W_n of order n for a finite real-valued function f(x) the following expression

$$\Delta f(x;W_n,h) = \sum_{k=-n}^{n} w_k f(x+kh/2).$$

A finite real-valued function f is said to be symmetric with respect to a weight system W_n if

$$\lim \Delta f(x;W_n,h) = 0, \qquad h\to 0.$$

In this paper, we generalize the properties of measurable symmetric functions proved by Mazurkiewicz [7], H. Auerbach [2], and C. J. Neugebauer [8] to functions symmetric with respect to an even weight system and we modify the proofs in those papers. It should be noted that Theorem 10 of this paper is a consequence of a more general result obtained via a different approach by Lee Larson [5,Theorem 1]. First we give a very useful proposition, which Zygmund has proved and used in many proofs [6], [10], and J. M. Ash has generalized [1] using the interval [u,2u]. We state it here in a slightly more general form with the interval [mu,(m+1)u], $0<m<=1$.

Proposition 1: Let 0 be a point of density for a measurable set E. Let m be a fixed number , $0<m<=1$ and let $\{a_i,b_i\}$, $i=1,2,...,n$ be a set of real numbers such that $b_i = 0$, $i=1,2,...n$. Then for all positive u sufficiently small, there is a v in [mu,(m+1)u] such that

$$a_i u + b_i v \in E \qquad i=1,2,...,n.$$

The proof is completely similar to that of Lemma 1 in [1]. Note thst this proposition is true for $u<0$ and v in [(m+1)u,mu]. It is also true if 0 is a point of outer density for a nonmeasurable set E. To prove this, we consider a measurable set which contains E and which in every interval has the same measure as E has outer measure and then the proof in [1] is valid. Zygmund has proved such a case in [10, Lemma 2].

C. J. Neugebauer has stated [8]:

Theorem 1: If f is measurable and symmetric in (a,b), then f is bounded in a neighborhood of almost every point of (a,b).

Marcinkiewicz and Zygmund have proved a similar lemma [6, Lemma 5, p. 13]. M. Ash has also stated a similar one [1, lemma 3].

We generalize this theorem to:

<u>Theorem 2: If f is finite measurable and symmetric with respect to a weight system of order n in (a,b), then f is bounded in a neighborhood of almost every point of (a,b).</u>

In fact, we prove the following stronger theorem:

<u>Theorem 3: If f is finite measurable in (a,b) and if for each x in (a,b)</u>

$$\Delta f(x;W_n,h) = O(1) \qquad h \to 0,$$

<u>then f is bounded in a neighborhood of almost every point of (a,b).</u>

<u>Proof</u>: Let

$E^*_m = \{x : |f(x)| < m$ and $|\Delta f(x;W_n,h)| < m$ for $0<|h|<1/m\}$.

Then we have

$$(a,b) \subset \bigcup_{m=1}^{\infty} E^*_m .$$

Since f is measurable, E^*_m is measurable. We show that f is bounded in a neighborhood of each point of density of E^*_m.

For the weight system W_n, we may assume $w_n \neq 0$ since

$$|w_n|+|w_{-n}| \neq 0.$$

If $w_n = 0$ and $w_{-n} \neq 0$, we have a similar argument.

Without loss of generality, we suppose x = 0 is a point of density for E^*_m and that u is small enough so that Prop. 1 applies.

That is $\qquad 0<|u|<n/2m.$

Then there is a v in (a,b) such that

for $u<0$, $2u <= v <= u$

for $u>0$ $u <= v <= 2u$

and $v+k(u-v)/n = [ku+(n-k)v]/n \in E^*_m$, $k=-n,\dots,n-1$.

Let $h=2(u-v)/n$, then $|h|=2|u-v|/n < 2|u|/n < 1/m$.

and
$$\Delta f(v;W_n,h) = \sum_{k=-n}^{n} w_k f(v+kh/2)$$
$$= \sum_{k=-n}^{n-1} w_k f(v+kh/2) + w_n f(u).$$

Thus
$$|w_n| * |f(u)| <= |\Delta f(v;W_n,h)| + \sum_{k=-n}^{n-1} |w_k| * |f(v+kh/2)|.$$

But $|\Delta f(v;W_n,h)| < m$

and $|f(v+kh/2)| < m$, $k = -n,\dots,n-1$.

Then
$$|w_n| * |f(u)| < m(1+ \sum_{k=-n}^{n-1} |w_k|)$$

and
$$|f(u)| < m(1+ \sum_{k=-n}^{n-1} |w_k|)/|w_n|.$$

That is, f is bounded in a neigborhood of $x = 0$.

It is well known [4, p. 174], [9, p. 129] that almost all the points of E^*_m are points of density for E^*_m. Since $(a,b) \subset \bigcup_{m=1}^{\infty} E^*_m$, f is bounded in a neighborhood of almost every point of (a,b).

Mazurkiewicz [7] and H. Auerbach [2] have shown:

Theorem 4: If f is bounded in [a,b] and symmetric in (a,b), then f is continuous a.e. in (a,b).

We generalize this theorem to:

Theorem 5: If f is bounded in [a,b] and symmetric with respect to an even weight system W_n in (a,b), then f is continuous a.e. in (a,b); hence, f is measurable in (a,b).

Proof: Since f is bounded, we set in the sense of Darboux

$$G(x) = \overline{\int_a^x} f(t)dt \ , \ g(x) = \underline{\int_a^x} f(t)dt.$$

We have

$$G(x+h)-G(x-h) = \overline{\int_0^h} f(x+t)dt + \overline{\int_0^h} f(x-t)dt.$$

$$G(x+kh)-G(x-kh) = \overline{\int_0^{kh}} f(x+t)dt + \overline{\int_0^{kh}} f(x-t)dt$$

$$= k\overline{\int_0^h} f(x+kt)dt + k\overline{\int_0^h} f(x-kt)dt.$$

Since W_n is even,

$$\Delta f(x;W_n,2h) = \sum_{k=-n}^{n} w_k f(x+kh)$$

$$= \sum_{k=1}^{n} w_k[f(x+kh)+f(x-kh)-2f(x)].$$

For $0 <= t <= h$, we have

$|\Delta f(x;W_n,2h)| < \varepsilon$ for h sufficiently small.

Then

$$\left| \sum_{k=1}^{n} w_k\{\overline{\int_0^h} f(x+kt)dt+\overline{\int_0^h} f(x-kt)dt-2f(x)h\}\right| < \varepsilon h$$

$$\left| \sum_{k=1}^{h} w_k \{[G(x+kh)-G(x-kh)]/2kh-f(x)\}\right| < \varepsilon/2. \qquad (1)$$

On the other hand, let

$M(x) = \lim \sup\{f(t): x-e < t < x+e\}$ e->0

$m(x) = \lim \inf\{f(t): x-e < t < x+e\}$ e->0.

According to Carathéodory [3, p.459], we have

$$G(x) = \int_a^x M(t)dt$$

$$g(x) = \int_a^x m(t)dt.$$

Thus G(x) and g(x) are absolutely continuous and almost everywhere differentiable [9, p. 105]. But the existence of the ordinary derivative implies the existence of the symmetric derivative and the two derivatives are equal. We have thus

$\lim [G(x+h)-G(x-h)]/2h = M(x)$ a.e. $h \to 0$,

$\lim [g(x+h)-g(x-h)]/2h = m(x)$ a.e. $h \to 0$.

If the symmetric derivative of G(x) exists at x, we have

$\lim [G(x+kh)-G(x-kh)]/2kh = \lim [G(x+h)-G(x-h)]/2h \quad h \to 0.$

(1) shows that $\lim [G(x+h)-G(x-h)]/2h = f(x)$ a.e. $h \to 0$.

Similarly,

$$\lim [g(x+h)-g(x-h)]/2h = f(x) \text{ a.e. } h \to 0.$$

That is $M(x) = m(x) = f(x)$ a.e.

Now since $M(x)$ and $m(x)$ are Riemann integrable, they are Lebesgue integrable. The condition $M(x)=m(x)$ a.e. shows that the two Lebesgue integrals of $M(x)$ and $m(x)$ are equal :

$$(L)\int_a^x M(t)dt = (L)\int_a^x m(t)dt.$$

But

$$\int_a^x M(t)dt = (L)\int_a^x M(t)dt \quad \text{and} \quad \int_a^x m(t)dt = (L)\int_a^x m(t)dt.$$

Thus $\int_a^x M(t)dt = \int_a^x m(t)dt$

and $G(x) = g(x)$.

The function $f(x)$ is therefore Riemann integrable and continuous a.e.

Neugebauer has proved [8] the following:

Theorem 6: If f is measurable and symmetric in (a,b), then f is continuous a.e. in (a,b).

We generalize this theorem to

Theorem 7: If f is measurable and symmetric with respect to an even weight system W_n in (a,b), then f is continuous a.e. in (a,b).

Proof: By theorem 2, f is bounded in a neighborhood of almost every point of (a,b). Hence, there is a set

$$E \subset (a,b)$$

with $|E| = b-a$

such that with each $x \in E$ there is associated a $\delta_x > 0$ so that f is bounded on

$$I_x = [x-\delta_x, x+\delta_x].$$

We may assume $I_x \subset [a,b]$.

Let {J} be a collection of closed intervals J such that for some $x \in E$, $J \subset I_x$

By the Vitali covering theorem, there is a sequence of disjoint intervals J_m in {J} such that

$$|E - \bigcup_{m=1}^{\infty} J_m| = 0.$$

Since $|E| = b-a$, we have $\sum_{m=1}^{\infty} |J_m| = b-a$.

But f is bounded on J_m, by Theorem 5, f is continuous a.e. in J_m, m =1, 2, ...

Hence, f is continuous a.e. in (a,b).

Mazurkiewicz has shown [7] the following theorem:

Theorem 8: If f is bounded in [a,b] and symmetric in (a,b), then f is Baire 1 in (a,b).

We generalize this theorem to

Theorem 9: If f is bounded in [a,b] and symmetric with respect to an even weight system W_n in (a,b), then f is Baire 1 in (a,b).

Proof: By Theorem 5, f is continuous a.e. in [a,b]. Since f is bounded, continuous a.e. in [a,b], f is Riemann integrable in [a,b]. Set

$$F(x) = \int_a^x f(t)dt.$$

We have by an argument similar to that in the proof of Theorem 5 :

$$\left| \sum_{k=1}^{n} w_k\{[F(x+kh)-F(x-kh)]/2kh - f(x)\}\right| < \varepsilon/2;$$ that is

$$\lim \sum_{k=1}^{n} w_k[F(x+kh)-F(x-kh)]/2kh = f(x)\sum_{k=1}^{n} w_k \quad h\to 0$$

with $\sum_{k=1}^{n} w_k \neq 0$. Since $F(x)$ is continuous, the function

$$\sum_{k=1}^{n} w_k[F(x+kh]-F(x-kh)]/2kh \text{ is continuous.}$$

Let $\{h_p\}$ be a sequence which converges to 0. Then

$$\lim \sum_{k=1}^{n} w_k[F(x+kh_p)-F(x-kh_p)]/2h_p = f(x)\sum_{k=1}^{n} w_k \quad h_p\to 0.$$

That is, f is Baire 1 in (a,b).

Neugebauer has extended Mazurkiewicz' and Auerbach's theorems to measurable symmetric functions.

To generalize Neugebauer's theorem, we need the following lemma, whose proof is similar to but more general than Neugebauer's [8].

Lemma: If f is measurable and symmetric with respect to an even weight system in (a,b), then the set $E = \{x: osc(f,x) = \infty\}$ is countable.

Proof: Let $E_m = \{x: osc(f,x) \geq m\}$. Then E_m is closed and $E = \bigcap_{m=1}^{\infty} E_m$ is also closed. Assume E is uncountable. Then E contains a perfect set P so that E-P is countable. Since f is continuous a.e., $|E| = 0$.

Thus, $G = (a,b) - P \neq \emptyset$.

Let (c,d) be a component of G, say $c \in P$ and $(c,d) \cap P = \emptyset$. For every $h > 0$, the set $(c-h,c) \cap E$ is uncountable. Since f is symmetric with respect to an even W_n, there exists p, $0 < p < (d-c)/(2n-1)$ such that

$|\Delta f(w;W_n,h)| < 1$ for $0 < h < p$, where w is the midpoint of (c-p,d). Let $E_p = E \cap (c-p,c)$ and $u \in E_p$. Let v be the reflection of u about w, $u+v=2w$. Then with $h=(v-u)/n$, all the 2n-1 points $w+kh/2$, $k=-(n-1),\dots,n$, $(k\neq 0)$, which we will call the multisymmetric points of u about w, are in (c,d). To see this, we study the positions of u, v, and the point $y=w-(n-1)h/2$ closest to u among the 2n-1 multisymmetric points of u. It is clear that $v<d$, $c-p<u<c$ with $p<(d-c)/(2n-1)$. Then

$$\begin{aligned} y &= w - (n-1)(v-u)/2n \\ &= (c-p+d)/2 + (n-1)u/2n - (n-1)v/2n \\ &> (c-p+d)/2 + (n-1)(c-p)/2n - (n-1)d/2n \\ &> [n(c-p+d) + (n-1)(c-p) - (n-1)d]/2n \\ &> [(2n-1)(c-p) + d]/2n \\ &> (2n-1)c/2n - (2n-1)p/2n + d/2n \\ &> (2n-1)c/2n - (d-c)/2n + d/2n > c. \end{aligned}$$

So y is in (c,d). But the other multisymmetric points of u are to the right of y; that is, for u in (c-p,c), all the 2n-1 multisymmetric points of u are in (c.d). Now since $osc(f,u) = \infty$, we can choose a sequence $\{u_i\}$ converging to u in (c-p,c) such that $|f(u_i)-f(u)| \to \infty$. For each u_i, let v_i be the reflection of u_i about w and $h_i = (v_i-u_i)/n$. Then

$$|w_n|*|f(u_i)-f(u)| \le |\Delta f(w;W_n,h)|+|\Delta f(w;W_n,h_i)|+ \sum_{k=-(n-1)}^{n} |w_k|*|f(w+kh_i/2)-f(w+kh/2)|.$$

Since $|w_n| \neq 0$ and $|f(u_i)-f(u)| \to \infty$, we have

$$\sum_{k=-(n-1)}^{n} |w_k|*|f(w+kh_i/2)-f(w+kh/2)| \to \infty.$$

We see then for each u in E_p, the oscillation of f is arbitrary large at one of the multisymmetric points of u about w; for if f has finite oscillations at all those 2n-1 points, the expression

$$\sum_{k=-(n-1)}^{n} |w_k|*|f(c+kh_i/2)-f(c+kh/2)| \text{ must be finite, } u_i \to u.$$

Moreover, let t be a multisymmetric point of u about w, then for some k, t=w+kh/2. Thus

$$u = w-n(t-w)/k, \quad k = -(n-1),\dots,n \quad k \neq 0.$$

That is, there are at most 2n-1 points u in E_p which have a common multisymmetric t about w. Each point u in E_p is the representative of a finite set of points that have a multisymmetric point in common. Thus for all the points u in E_p, the set of multisymmetric points at which f has arbitrary large oscillations is uncountable : contradiction with the fact that $(c,d) \cap E \subset E - P$, which is countable.

<u>Theorem 10: If f is measurable and symmetric with respect to an even weight system in (a,b), then f is Baire 1 in (a,b).</u>

<u>Proof</u>: Set $E = \{x : \operatorname{osc}(f,x) = \infty\}$. Then E is closed and countable. The set $G = (a,b) - E$ is open. Let $x \in G$. Let (c,d) be a component of G and $I_n = [c+1/n, d-1/n]$. Since $\operatorname{osc}(f,x) < \infty$ in I_n, f is bounded in I_n; hence, f is Baire 1 in I_n. Let $E_a = \{x : f(x)<a\}$, $E^a = \{x : f(x)>a\}$. Then $E_a \cap I_n$ and $E^a \cap I_n$ are F_σ sets. Hence the set

$$E_a \cap G = \bigcup_{(c,d)} \bigcup_{n} (E_a \cap I_n),$$

where $\bigcup_{(c,d)}$ is extended over all the components of G, is an F_σ set. Since E is countable, E_a is also an F_σ set. Similarly, E^a is an F_σ set. Therefore, f is Baire 1 in (a,b).

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