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Milan Matejdes, Katedra matematiky a fyziky VSLD, Marxova 24, 960 53 Zvolen, Czechoslovakia

Dedicated to Professor L. Mišík

THE SEMI-BOREL CLASSIFICATION OF THE EXTREME PATH DERIVATIVES

ABSTRACT. The goal of this paper is to investigate of the semi-Borel and Baire classification of the multifunction of all path derived numbers of a semi-Borel and Baire function of the class α. Consequently the classification of the extreme path derivatives is given. The results hold in the setting of ordinary, qualitative and approximate path differentiation, and the proofs are based on a classification of the collection of paths which is considered as a multifunction of the semi-Borel class α.

1. INTRODUCTION. In recent years various generalizations of the notion of the derivative have been studied. A development of an approach to differentiation which includes a number of known generalized derivatives was introduced in the excellent paper [3]. Namely a collection $E = \{E(x) : x \in R\}$ is a system of paths if each set $E(x)$ has x as a point of accumulation. For such a system E the extreme E-derivatives of f at a point x are:

$$\overline{f}'_E(x) = \limsup_{y\to x,\, y\in E(x)} (f(x)-f(y))/(x-y)$$

$$\underline{f}'_E(x) = \liminf_{y\to x,\, y\in E(x)} (f(x)-f(y))/(x-y) .$$

A number of familiar derivatives (for example ordinary, approximate, preponderant, congruent, qualitative, one-sided derivatives) can be expressed as path derivatives along a system of paths of an appropriate types [3].

The behavior of a path derivative is very closely linked to the geometry of the system E as can be seen from paper [3] where many proofs are based on a system of paths satisfying some of the intersection conditions that provide information related to the "thickness" of the paths. For example, for a system of paths E satisfying the external intersection condition, any E-derivative is in Baire class 1 [3,Corollary 6.3]. But there are cases, such as in the study of extreme approximate derivatives, where the path system of differentiation is not convenient. Namely, if $\bar{f}'_{ap}$ is the approximate upper derivative of a function f, then there is a system of paths E such that $\bar{f}'_{ap} = \bar{f}'_{E}$. But nothing can be said about intersection conditions for E . A similar situation arises in the setting of qualitative extreme derivatives. Oving to these facts, the idea of path differentiation will be generalized in our paper by Definition 2.1. This generalization was motivated by a very useful notion involving the concept of systems of sets (called simple systems) associated with each point [9],but the localness of the systems of sets is not convenient for classifying generalized derivatives. Definition 2.1 allows considerable flexibility for the systems of sets as well as the system of paths.

Another motivation for our concept came from paper [1] by Alikhani-Koopaei. His method of considering E as a multifunction seems to be a convenient tool for investigating various problems connected with path derivatives. The main result of [1] says that the extreme path derivatives of a continuous function relative to a continuous system of paths are in Baire class 2. One of the major goals of the present paper is to replace continuity by a generalized notion of continuity of E (see Theorems 3.9, 3.10 below) as well as some weaker assumptions on the graph of E (Theorems 3.3, 3.9, 4.4, 4.7, Corollaries 3.4, 3.6, 3.7, 4.5, 4.6).

Another aim is to study measurability of the extreme path derivatives even further and to investigate the Lebesgue and Baire measurability as well as the semi-Borel classification of the multifunction of all path derived numbers. The well-known results of Professor Mišík ([6],[7], see also [2],[4], [8]) concern only the semi-Borel classification of Dini and approximate unilateral extreme derivatives.

The paper is divided into five sections. In § 2 we introduce a generalized path differentiation of functions and the classification of derivation systems is given. The main lemma of § 2 has a purely topological character and its consequences for ordinary, qualitative and approximate differentiation are given in §§ 3, 4. In the final section we deal with the properties of E-primitives.

2. Basic definitions, notation and preliminary results

As was mentioned in § 1, the concept of path differentiation is not effective in the setting of some generalized derivatives. In order to obtain a convenient toll for investigating the semi-Borel classification of the multifunction of all path derived numbers, we introduce the following generalized idea of differentiation which unites the notion of path system and the concept of system of sets.

DEFINITION 2.1 (see also [5]). Let $(R, \mathcal{O})$, $(R^*, \mathcal{O}^*)$ be the real line with the ordinary topology and the extended real line with the topology of the two-point compactification of R, respectively. Let $\mathcal{T}$ be a topology on R. A quadruple $\mathcal{E} = (R, \mathcal{T}, E, \mathcal{C})$ is called a derivation system (briefly DS) where E is a multifunction from R into 2^R, $\emptyset \neq \mathcal{C} \subset 2^R$, $\emptyset \notin \mathcal{C}$.

DEFINITION 2.2. Let $\mathcal{E} = (R, \mathcal{T}, E, \mathcal{C})$ be a DS and let $f: R \longrightarrow R$ be a function. A point $z \in R^*$ is called an $\mathcal{E}$-derived number of f at point $x \in R$ if, for any $G \in \mathcal{O}^*$ with $z \in G$ and for any $U \in \mathcal{T}$ with $x \in U$, there exists a set $A \in \mathcal{C}$ such that $A \subset U \cap E(x) \setminus \{x\}$ and $(f(x)-f(y))/(x-y) \in G$ whenever $y \in A$. The set of all $\mathcal{E}$-derived numbers of f at a point x will be denoted by $D(f, \mathcal{E}, x)$. Define $D_{f,\mathcal{E}}: R \longrightarrow 2^{R^*}$ by $D_{f,\mathcal{E}}(x) = D(f, \mathcal{E}, x)$. If $D(f, \mathcal{E}, x) \neq \emptyset$, then the extreme $\mathcal{E}$-derivatives of f at the point x are:

$\bar{f}'_{\mathcal{E}}(x) = \sup D(f, \mathcal{E}, x)$ (the upper extreme $\mathcal{E}$-derivative),

$\underline{f}'_{\mathcal{E}}(x) = \inf D(f, \mathcal{E}, x)$ (the lower extreme $\mathcal{E}$-derivative).

If $D(f, \mathcal{E}, x)$ is a one point set, then that point is called the $\mathcal{E}$-derivative of f at x and it is denoted by $f'_{\mathcal{E}}(x)$. Note that $D(f, \mathcal{E}, x)$ is $\mathcal{O}^*$-closed.

We introduce a classification of derivation systems within which various generalized derivatives can be expressed.

DEFINITION 2.3. A derivation system $\mathcal{E} = (R, \mathcal{T}, E, \mathcal{C})$ will be said to be

-ordinary DS if $\mathcal{T} = \mathcal{O}$ and $\mathcal{C} = 2^R \setminus \{\emptyset\}$;

-essential DS if $\mathcal{C} = \{A : A$ is of the $\mathcal{T}$-second category$\}$;

-qualitative DS if $\mathcal{T} = \mathcal{O}$ and $\mathcal{E} = \{A : A$ is of the $\mathcal{O}$-second category$\}$;

-approximate DS if $\mathcal{T} = \mathcal{D}$ where $\mathcal{D}$ is the density topology and $\mathcal{E} = 2^R \setminus \{\emptyset\}$;

-congruent DS if $E(x) = E(0) + x$ for all $x \in R$;

-Baire DS if $E(x) \setminus \{x\}$ is of the $\mathcal{T}$-second category at x and $E(x)$ has the $\mathcal{T}$-Baire property for all $x \in R$;

-left (right)-sided DS if $E(x) \subset (-\infty, x]$ ($E(x) \subset [x, \infty)$) for all $x \in R$;

-unilateral DS if $\mathcal{E}$ is left or right-sided.

We shall also classify the extreme $\mathcal{E}$-derivatives and $\mathcal{E}$-derivatives according to the definitions above. E.g., $\bar{f}'_{\mathcal{E}}$ is called the ordinary (qualitative, approximate...) upper extreme $\mathcal{E}$-derivative of f if $\mathcal{E}$ is an ordinary (qualitative, approximate...) DS.

The approximate extreme derivatives $\bar{f}^+_{ap}$, $\underline{f}^+_{ap}$, $\bar{f}^-_{ap}$, $\underline{f}^-_{ap}$ (qualitative extreme derivatives $\bar{f}^+_q$, $\underline{f}^+_q$, $\bar{f}^-_q$, $\underline{f}^-_q$) are just the approximate (qualitative) extreme $\mathcal{E}$-derivatives, where $E(x) = [x, \infty)$, $E(x) = (-\infty, x]$ respectively. The Dini derivatives D^+f, D_+f, D^-f, D_-f correspond to the ordinary extreme $\mathcal{E}$-derivatives where $E(x) = [x, \infty)$, $E(x) = (-\infty, x]$ respectively.

The following notation and some facts about multifunctions will be needed below.

The set of all positive integers is denoted by N. If $a = \pm\infty$ and if $r \in R$, let $a \pm r = a$. Let $f : R \to R$ be a function, $F : R \to 2^R$ be a multifunction, $\mathcal{T}$ be a topology on R, $U \subset R \times R$,

$T \subset R$, $n \in N$, $\emptyset = \mathcal{E} \subset 2^R$, $\emptyset \notin \mathcal{E}$.

We set:

$f_o((x,y)) = (f(x)-f(y))/(x-y)$, $(x,y) \in R \times R$, $x \neq y$.

$F^-(T) = \{x \in R: F(x) \cap T \neq \emptyset\}$.

$F^+(T) = \{x \in R: R(x) \subset T\}$.

$Gr(F) = \{(x,y) \in R \times R: y \in F(x)\}$ (graph of F).

$P(F) = \{x \in R: x \in F(x)\}$ (set of fixed points of F).

$A(F,f,n,a,b) = Gr(F) \cap f_o^{-1}((a-1/n, b+1/n))$ where $a,b \in R^*$, $a<b$.

$\mathcal{E}_{\mathcal{T}}(T) = \{x \in R$: for all $U \in \mathcal{T}$ with $x \in U$, there is a set $V \in \mathcal{E}$ such that $V \subset U \cap T \setminus \{x\}\}$.

We define the multifunctions F_+, F_-, F_U, $\mathcal{E}_{\mathcal{T}}(F): R \to 2^R$ as follows:

$F_+(x) = F(x) \cap [x, \infty)$, $F_-(x) = F(x) \cap (-\infty, x]$,

$F_U(x) = U_x = \{y \in R: (x,y) \in U\}$, $\mathcal{E}_{\mathcal{T}}(F)(x) = \mathcal{E}_{\mathcal{T}}(F(x))$ for all $x \in R$.

Note that $F^-(T) = R \setminus F^+(R \setminus T)$ and if a single valued function $f: R \to R$ is given, then under the natural interpretation of $f(x)$ as a one point set we have $f^+(T) = f^-(T) = f^{-1}(T)$.

We state as a lemma the main result of this section which is the essence of §§ 3,4.

<u>LEMMA 2.4</u>. Let $\mathcal{F} = (R, \mathcal{T}, E, \mathcal{E})$ be a DS in which $\mathcal{E}$ has the following property: If $\{A_1, \dots, A_k\}$ is a finite collection of subsets of R such that $\bigcup_{n=1}^{k} A_n \in \mathcal{E}$, then $A_n \in \mathcal{E}$ for some $n \in$

$\{1,\dots,k\}$. If $f:R\to R$, $a,b\in R^*$, $a<b$, then

$$D^-_{f,\mathcal{E}}([a,b]) = \bigcap_{n=1}^{\infty} P(\mathcal{E}_{\mathcal{T}}(F_{A(E,f,n,a,b)})).$$

PROOF. Let $x\in R$ be an arbitrary point. Then $\mathcal{E}_{\mathcal{T}}(\{y\in E(x)\setminus\{x\}: (f(x)-f(y))/(x-y)\in(a-1/n,b+1/n)\}) = \mathcal{E}_{\mathcal{T}}(\{y: (x,y)\in f_o^{-1}(a-1/n,b+1/n))\cap Gr(E)\}) = \mathcal{E}_{\mathcal{T}}((f_o^{-1}((a-1/n,b+1/n))\cap Gr(E))_x) = \mathcal{E}_{\mathcal{T}}(A(E,f,n,a,b)_x) = \mathcal{E}_{\mathcal{T}}(F_{A(E,f,n,a,b)}(x))$. It is clear that if $D_{f,\mathcal{E}}(x)\cap[a,b]\neq\emptyset$, then for any $n\in N$ we have: $x\in\mathcal{E}_{\mathcal{T}}(\{y\in E(x)\setminus\{x\}: f_o((x,y))\in(a-1/n,b+1/n)\}) = \mathcal{E}_{\mathcal{T}}(F_{A(E,f,n,a,b)}(x))$.

Let $x\in\bigcap_{n=1}^{\infty} P(\mathcal{E}_{\mathcal{T}}(F_{A(E,f,n,a,b)}))$. Suppose $D_{f,\mathcal{E}}(x)\cap[a,b]=\emptyset$. That means for any $z\in[a,b]$ there are $G(z)\in\mathcal{O}^*$, $U_z(x)\in\mathcal{T}$ with $z\in G(z)$ and $x\in U_z(x)$, such that the set $\{y\in E(x)\setminus\{x\}: f_o((x,y))\in G(z)\}\cap U_z(x)$ does not contain any set from $\mathcal{E}$. Since $[a,b]$ is $\mathcal{O}^*$-compact, the open covering $\{G(z): z\in[a,b]\}$ of $[a,b]$ has a finit subcovering $\{G(z_1),\dots,G(z_k)\}$. Let $U_i = U_{z_i}(x)$ and $G_i = G(z_i)$ for $i=1,\dots,k$. Put $U(x)=U_1\cap\dots\cap U_k$. Then $S:=\bigcup_{i=1}^{k}(\{y\in E(x)\setminus\{x\}: f_o((x,y))\in G_i\}\cap U(x))$ does not contain any set from $\mathcal{E}$. Since $G_1,\dots,G_n$ covers $[a,b]$, there is an $n\in N$ such that $(a-1/n,b+1/n)\subset G_1\cup\dots\cup G_k$. Then $S = \{y\in E(x)\setminus\{x\}: f_o((x,y))\in G_1\cup\dots\cup G_k\}\cap U(x)\supset\{y\in E(x)\setminus\{x\}: f_o((x,y))\in(a-1/n,b+1/n)\}\cap U(x) =: S_o$. That means S_o does not contain any set from $\mathcal{E}$. Hence $x\notin\mathcal{E}_{\mathcal{T}}(S_o)$

and we obtain a contradiction to the assumption that x belongs to

$$\bigcap_{n=1}^{\infty} P(\mathscr{E}_{\mathcal{T}}(F_{A(E,f,n,a,b)})).$$

In connection with Lemma 2.4 a natural question arises in this setting: what information about F, $\mathcal{T}$ and $\mathscr{E}$ implies that $P(\mathscr{E}_{\mathcal{T}}(F))$ is a set of the Borel class α (Lebesque measurable)? For certain special cases this problem will be solved in the next two sections.

This section is concluded with two trivial lemmas which will be needed below.

<u>LEMMA 2.5</u>. Let $f: R \to R$ be a function. For $a \in R$ let

$$S_a = \{(x,y): f(x)-ax>f(y)-ay\},$$

$$T_a = \{(x,y): f(x)-ax<f(y)-ay\}.$$

Then

(a) $S_a = \bigcup_{r \in Q} \{x: f(x)-ax>r\} \times \{y: f(y)-ay<r\};$

(b) $T_a = \bigcup_{r \in Q} \{x: f(x)-ax<r\} \times \{y: f(y)-ay>r\}$

where $Q = \{r: r \text{ is a rational number}\}$.

<u>LEMMA 2.6</u>. Let $f: R \to R$, $E: R \to 2^R$. For any $a,b \in R$, $a<b$, we have

(a) $f_o^{-1}((a,\infty)) \cap Gr(E_-) = S_a \cap Gr(E_-);$

(b) $f_o^{-1}((-\infty,a)) \cap Gr(E_-) = T_a \cap Gr(E_-);$

(c) $f_o^{-1}((a,\infty)) \cap Gr(E_+) = T_a \cap Gr(E_+);$

(d) $f_o^{-1}((-\infty,a)) \cap Gr(E_+) = S_a \cap Gr(E_+);$

(e) $f_o^{-1}((a,b)) \cap Gr(E_-) = S_a \cap T_b \cap Gr(E_-);$

(f) $f_o^{-1}((a,b)) \cap Gr(E_+) = S_b \cap T_a \cap Gr(E_+)$

where S_a, T_a are as in Lemma 2.5.

The trivial proofs are omitted.

3. The classification of $D_{f,\mathcal{E}}$ and the extreme $\mathcal{E}$-derivatives for the ordinary and qualitative derivation systems

In this section we will investigate the semi-Borel and Baire classification and Borel, Lebesgue and Baire measurability of the multifunction of all ordinary and qualitative $\mathcal{E}$-derived numbers.

DEFINITION 3.1. Let $\mathcal{A} \subset 2^R$, $\mathcal{A} \neq \emptyset$. A multifunction $F: R \longrightarrow 2^{R^*}$ is lower (upper) semi-$\mathcal{A}$-measurable (briefly $F \in \mathcal{A}^-$ ($F \in \mathcal{A}^+$)) if $F^+((a,\infty]) \in \mathcal{A}$ ($F^+([-\infty,a)) \in \mathcal{A}$) for all $a \in R$. Let $\mathcal{A}_\alpha$ denote the family of all sets of the Borel additive class α. A multifunction $F: R \longrightarrow 2^{R^*}$ is a lower (upper) semi-Borel multifunction of class α, if $F \in \mathcal{A}_\alpha^-$ ($F \in \mathcal{A}_\alpha^+$). F is a Baire multifunction of class α, if $F \in \mathcal{A}_\alpha^- \cap \mathcal{A}_\alpha^+$. F is Borel (Lebesgue, Baire) measurable if $F^-(G) \in \mathcal{B}r$ ($F^-(G) \in \mathcal{L}$, $F^-(G) \in \mathcal{B}$) for all $G \in \mathcal{O}^*$ where $\mathcal{B}r$, $\mathcal{L}$, $\mathcal{B}$ is the family of all Borel, Lebesgue, Baire sets respectively. By $\sigma\mathcal{A}$ ($\delta\mathcal{A}$) we denote the smallest σ-additive (σ-multiplicative) family generated by $\mathcal{A}$. Let $c\mathcal{A} = \{A \subset R: R \setminus A \in \mathcal{A}\}$.

In order to achieve our goal, we must investigate the structure of the set of fixed points of $\mathcal{C}_{\mathcal{T}}(F_{A(E,f,n,a,b)})$

(see Lemma 2.4) for $\mathcal{T}=\mathcal{O}$. The following lemma shows that this is possible when $A(E,f,n,a,b)$ is of suitable type.

Lemma 3.2. Let $\mathcal{A}\subset 2^R$ be a σ-additive and multiplicative family, $\mathcal{O}\subset\mathcal{A}$, $S\subset R\times R$, $S\in\sigma(\mathcal{A}\times 2^R)$. Let $\emptyset\notin\mathcal{E}\subset 2^R$, $\mathcal{E}\neq\emptyset$ have the following property: if $\bigcup_{n=1}^{\infty} A_n\in\mathcal{E}$, then there is an n such that $A_n\in\mathcal{E}$. Then $F(\mathcal{E}_{\mathcal{O}}(F_S))\in\delta\mathcal{A}$.

PROOF. Let $A_n=\{(x,y): x+1/n>y>x-1/n\}$. Then $x\in F(\mathcal{E}_{\mathcal{O}}(F_S))$ if and only if for all $n\in N$ there is $A(n)\in\mathcal{E}$ such that $A(n)\subset S_x\cap(x-1/n,x+1/n)\setminus\{x\} = F_S(x)\cap(x-1/n,x+1/n)\setminus\{x\} = F_{S\cap A_n}(x)\setminus\{x\}$.

If $T\subset R\times R$, define $\mathcal{E}(F_T)=\{x\in R:$ there is $A\in\mathcal{E}$, $A\subset F_T(x)\setminus\{x\}\}$. That means $F(\mathcal{E}_{\mathcal{O}}(F_S))=\bigcap_{n=1}^{\infty}\mathcal{E}(F_{S\cap A_n})$. We shall show that $\mathcal{E}(F_{S\cap A_n})\in\mathcal{A}$. It is easy to verify that for any sequence $\{T_n\}_{n=1}^{\infty}$, $T_n\subset R\times R$, the equality $\mathcal{E}F_{\bigcup_{n=1}^{\infty}T_n})=\bigcup_{n=1}^{\infty}\mathcal{E}(F_{T_n})$ holds. Since $\mathcal{O}\subset\mathcal{A}$ and $\mathcal{A}$ is a σ-additive and multiplicative family, the set $S\cap A_n$ can be expressed as the union of a sequence of sets $R_n^i\times S_n^i$ where $R_n^i\in\mathcal{A}$, $S_n^i\subset 2^R$. That means $\mathcal{E}(F_{S\cap A_n})=\bigcup_{i=1}^{\infty}(F_{R_n^i\times S_n^i})$. Since $\mathcal{E}(F_{R_n^i\times S_n^i})=\{x\in R:$ there is $A\in\mathcal{E}$, $A\subset F_{R_n^i\times S_n^i}(x)\setminus\{x\}=S_n^i\setminus\{x\}\}$, $\mathcal{E}(F_{R_n^i\times S_n^i})=R_n^i$ if $x\in R_n^i$ and there is $A\in\mathcal{E}$ such that $A\subset S_n^i\setminus\{x\}$ and $\mathcal{E}(F_{R_n^i\times S_n^i})=\emptyset$ otherwise. Hence $\mathcal{E}(F_{S\cap A_n})\in\mathcal{A}$.

THEOREM 3.3. Let $\mathcal{A}$ be a σ-additive and multiplicative family, $\mathcal{O}\subset\mathcal{A}$, $f:R\longrightarrow R$. Let $\mathcal{E}=(R,\mathcal{O},E,\mathscr{E})$ be a DS in which $Gr(E)\in\sigma(\mathcal{A}\times 2^R)$ and $\mathscr{E}$ has the property of Lemma 3.2.

(a) If $f\in\mathcal{A}^-$ ($f\in\mathcal{A}^+$) and $\mathcal{E}$ is left-sided, then $D_{f,\mathcal{E}}\in c\delta\mathcal{A}^+$ ($D_{f,\mathcal{E}}\in c\delta\mathcal{A}^-$) and $\bar{f}'_{\mathcal{E}}\in c\delta\mathcal{A}^+$ ($\underline{f}'_{\mathcal{E}}\in c\delta\mathcal{A}^-$).

(b) If $f\in\mathcal{A}^-$ ($f\in\mathcal{A}^+$) and $\mathcal{E}$ is right-sided, then $D_{f,\mathcal{E}}\in c\delta\mathcal{A}^-$ ($D_{f,\mathcal{E}}\in c\delta\mathcal{A}^+$) and $\underline{f}'_{\mathcal{E}}\in c\delta\mathcal{A}^-$ ($\bar{f}'_{\mathcal{E}}\in c\delta\mathcal{A}^+$).

(c) If $f\in\mathcal{A}^+\cap\mathcal{A}^-$, then $D^-_{f,\mathcal{E}}([a,b])\in\delta\mathcal{A}$ for any $a,b\in R^*$, $a<b$. Hence $D_{f,\mathcal{E}}\in c\delta\mathcal{A}^+\cap c\delta\mathcal{A}^-$ and $\bar{f}'_{\mathcal{E}}\in c\delta\mathcal{A}^+$, $\underline{f}'_{\mathcal{E}}\in c\delta\mathcal{A}^-$.

PROOF. (a) Since $\mathcal{E}$ is left-sided, $Gr(E)=Gr(E_-)\in\sigma(\mathcal{A}\times 2^R)$. Let $f\in\mathcal{A}^-$ ($f\in\mathcal{A}^+$). By Lemma 2.5, $S_{a-1/n}\cap Gr(E_-)\in\sigma(\mathcal{A}\times 2^R)$ ($T_{a+1/n}\cap Gr(E_-)\in\sigma(\mathcal{A}\times 2^R)$). By Lemma 2.6(a) (Lemma 2.6(b)), $A(E,f,n,a,\infty)=f_o^{-1}((a-1/n,\infty))\cap Gr(E_-)=S_{a-1/n}\cap Gr(E_-)\in\sigma(\mathcal{A}\times 2^R)$ ($A(E,f,n,-\infty,a)=f_o^{-1}((-\infty,a+1/n))\cap Gr(E_-)=T_{a+1/n}\cap Gr(E_-)\in\sigma(\mathcal{A}\times 2^R)$) for any $a\in R$. By Lemmas 3.2 and 2.4, $D^-_{f,\mathcal{E}}([a,\infty])\in\delta\mathcal{A}$ ($D^-_{f,\mathcal{E}}([-\infty,a])\in\delta\mathcal{A}$). Hence $D_{f,\mathcal{E}}\in c\delta\mathcal{A}^+$ ($D_{f,\mathcal{E}}\in c\delta\mathcal{A}^-$). Since $\bar{f}'^{-1}_{\mathcal{E}}([a,\infty])=D^-_{f,\mathcal{E}}([a,\infty])\in\delta\mathcal{A}$ ($\underline{f}'^{-1}_{\mathcal{E}}([-\infty,a])=D^-_{f,\mathcal{E}}([-\infty,a])\in\delta\mathcal{A}$) for all $a\in R$, $\bar{f}'_{\mathcal{E}}\in c\delta\mathcal{A}^+$ ($\underline{f}'_{\mathcal{E}}\in c\delta\mathcal{A}^-$).

(b) Since $\mathcal{E}$ is right-sided, $Gr(E)=Gr(E_+)\in\sigma(\mathcal{A}\times 2^R)$. Let $f\in\mathcal{A}^-$ ($f\in\mathcal{A}^+$). By Lemma 2.5, $S_{a+1/n}\cap Gr(E_+)\in\sigma(\mathcal{A}\times 2^R)$

$(T_{a-1/n} \cap Gr(E_+) \in \sigma(\mathcal{a} \times 2^R))$. By Lemma 2.6(d) (Lemma 2.6(c)), $A(E,f,n,-\infty,a) = f_o^{-1}((-\infty,a+1/n)) \cap Gr(E_+) = S_{a+1/n} \cap Gr(E_+) \in \sigma(\mathcal{a} \times 2^R)$ $(A(E,f,n,a,\infty) = f_o^{-1}((a-1/n,\infty)) \cap Gr(E_+) = T_{a-1/n} \cap Gr(E_+) \in \sigma(\mathcal{a} \times 2^R))$ for all $a \in R$. By Lemmas 3.2 and 2.4, $D^-_{f,\mathcal{E}}([-\infty,a]) \in \delta\mathcal{a}$ $(D^-_{f,\mathcal{E}}([a,\infty]) \in \delta\mathcal{a})$ hence $D_{f,\mathcal{E}}$, $\underline{f}'_{\mathcal{E}} \in c\delta\mathcal{a}^-$ $(D_{f,\mathcal{E}}$, $\bar{f}'_{\mathcal{E}} \in c\delta\mathcal{a}^+)$.

(c) Let $N^- = \{(x,y): x>y\}$, $N^+ = \{(x,y): x<y\}$. Since $N^-, N^+ \in \sigma(\mathcal{a} \times 2^R)$, $Gr(E_-) = Gr(E) \cap N^- \in \sigma(\mathcal{a} \times 2^R)$, $Gr(E_+) = Gr(E) \cap N^+ \in \sigma(\mathcal{a} \times 2^R)$. $f \in \mathcal{a}^- \cap \mathcal{a}^+$. Hence $S_{a-1/n}, T_{a-1/n}, S_{b+1/n}, T_{b+1/n} \in \sigma(\mathcal{a} \times 2^R)$ for all $a,b \in R$, $n \in N$ by Lemma 2.5.

There are four cases. (1) $a,b \in R$, $a<b$. (2) $a = -\infty$, $b \in R$. (3) $a \in R$, $b = \infty$. (4) $a = -\infty$, $b = \infty$.

Case (1). By Lemma 2.6 (e),(f) $A(E,f,n,a,b) = (f_o^{-1}((a-1/n, b+1/n)) \cap Gr(E_-)) \cup (f_o^{-1}((a-1/n,b+1/n)) \cap Gr(E_+)) = (S_{a-1/n} \cap T_{b+1/n} \cap Gr(E_-)) \cup (S_{b+1/n} \cap T_{a-1/n} \cap Gr(E_+)) \in \sigma(\mathcal{a} \times 2^R)$. By Lemmas 3.2 and 2.4, $D^-_{f,\mathcal{E}}([a,b]) \in \delta\mathcal{a}$.

Case (2). By Lemma 2.6 (b), (d), $A(E,f,n,-\infty,b) = (f_o^{-1}((-\infty,b+1/n)) \cap Gr(E_+)) \cup (f_o^{-1}((-\infty,b+1/n)) \cap Gr(E_-)) = (S_{b+1/n} \cap Gr(E_+)) \cup (T_{b+1/n} \cap Gr(E_-)) \in \sigma(\mathcal{a} \times 2^R)$. By Lemmas 3.2 and 2.4, $D^-_{f,\mathcal{E}}([-\infty,b]) \in \delta\mathcal{a}$.

Case (3). By Lemma 2.6 (c) ,(a), $A(E,f,n,a,\infty) = (f_o^{-1}((a-1/n,\infty)) \cap Gr(E_+)) \cup (f_o^{-1}((a-1/n,\infty)) \cap Gr(E_-)) = (T_{a-1/n} \cap$

$+ \quad {}_{a-1/n}\ \mathrm{Gr}(E_-)) \in \sigma(\mathcal{A}\times 2^R)$. By Lemmas 3.2 and 2.4, $D^-_{f,\mathcal{E}}([a,\infty]) \in \delta\mathcal{A}$.

Case (4). $A(E,f,n,-\infty,\infty) = \mathrm{Gr}(E) \in \sigma(\mathcal{A}\times 2^R)$ and by Lemmas 3.2 and 2.4, $D^-_{f,\mathcal{E}}([-\infty,\infty]) \in \delta\mathcal{A}$.

Now we obtain the following consequence of Theorem 3.3 for the semi-Borel classification of $D_{f,\mathcal{E}}$.

<u>COROLLARY 3.4</u>. Let $\mathcal{E}$ be an ordinary or qualitative DS in which $\mathrm{Gr}(E) \in \sigma(\mathcal{A}_\alpha\times 2^R)$.

(a) If f is a lower (upper) semi-Borel function of class α and $\mathcal{E}$ is left-sided, then the upper (lower) extreme $\mathcal{E}$-derivative of f is an upper (lower) semi-Borel function of class $\alpha+1$ and $D_{f,\mathcal{E}}$ is an upper (lower) semi-Borel multifunction of class $\alpha+1$.

(b) If f is a lower (upper) semi-Borel function of class α and $\mathcal{E}$ is right-sided, then the lower (upper) extreme $\mathcal{E}$-derivative of f is a lower (upper) semi-Borel function of class $\alpha+1$ and $D_{f,\mathcal{E}}$ is a lower (upper) semi-Borel multifunction of class $\alpha+1$.

(c) If f is a Borel function of class α, then the upper (lower) extreme $\mathcal{E}$-derivative of f is an upper (lower) semi-Borel function of class $\alpha+1$ and $D_{f,\mathcal{E}}$ is a Baire multifunction of class $\alpha+1$.

REMARK 3.5. The results of Theorem 3.3. and Corollary 3.4 can be remembered very easily. If the sing + (−) corresponds to a right (left)-sided DS, then the assertions of Theorem 3.3 and Corollary 3.4 can be read out of the following tables by the rules concerning the multiplication of negative and positive numbers.

f	$\wp$	$D_{f,\wp}$
$\mathcal{A}^+$	+	$c\delta\mathcal{A}^+$
$\mathcal{A}^-$	−	$c\delta\mathcal{A}^+$
$\mathcal{A}^+$	−	$c\delta\mathcal{A}^-$
$\mathcal{A}^-$	+	$c\delta\mathcal{A}^-$

f	$\wp$	$D_{f,\wp}$
$\mathcal{A}_\alpha^+$	+	$\mathcal{A}_{\alpha+1}^+$
$\mathcal{A}_\alpha^-$	−	$\mathcal{A}_{\alpha+1}^+$
$\mathcal{A}_\alpha^+$	−	$\mathcal{A}_{\alpha+1}^-$
$\mathcal{A}_\alpha^-$	+	$\mathcal{A}_{\alpha+1}^-$

For example, if $f \in \mathcal{A}^+$ and $\wp$ is left-sided, then $D_{f,\wp} \in c\delta\mathcal{A}^-$ ("+ times − = −" see line 3).

For the classification of the extreme $\wp$-derivatives we have another rule. If the sign + (−) corresponds to the upper (lower) extreme $\wp$-derivative, then we can read out of the following tables: If $f \in \mathcal{A}^-$ and $\wp$ is right-sided, then the lower extreme $\wp$-derivative is lower semi-$c\delta\mathcal{A}$-measurable (" − times + = − " see line 4).

f	$\wp$	$\bar{f}'_\wp$	$\underline{f}'_\wp$
$\mathcal{A}^+$	+	$c\delta\mathcal{A}^+$	
$\mathcal{A}^-$	−	$c\delta\mathcal{A}^+$	
$\mathcal{A}^+$	−		$c\delta\mathcal{A}^-$
$\mathcal{A}^-$	+		$c\delta\mathcal{A}^-$

f	$\wp$	$\bar{f}'_\wp$	$\underline{f}'_\wp$
$\mathcal{A}_\alpha^+$	+	$\mathcal{A}_{\alpha+1}^+$	
$\mathcal{A}_\alpha^-$	−	$\mathcal{A}_{\alpha+1}^+$	
$\mathcal{A}_\alpha^+$	−		$\mathcal{A}_{\alpha+1}^-$
$\mathcal{A}_\alpha^-$	+		$\mathcal{A}_{\alpha+1}^-$

Our next consequence of Theorem 3.3 deals with measurability of $D_{f,\mathcal{F}}$, $\bar{f}'_{\mathcal{F}}$, $\underline{f}'_{\mathcal{F}}$.

COROLLARY 3.6. Let $\mathcal{F}$ be an ordinary or qualitative DS in which $Gr(E) \in \sigma(\mathcal{B}r \times 2^R)$, $\sigma(\mathcal{L} \times 2^R)$, $\sigma(\mathcal{B} \times 2^R)$ respectively. If f is Borel, Lebesgue, Baire measurable respectively, then $\bar{f}'_{\mathcal{F}}$, $\underline{f}'_{\mathcal{F}}$, $D_{f,\mathcal{F}}$ are Borel, Lebesgue, Baire measurable respectively.

PROOF. By Theorem 3.3(c), $D^-_{f,\mathcal{F}}([a,b]) \in \mathcal{B}r$, $\mathcal{L}$, $\mathcal{B}$, respectively, for all $a,b \in R^*$, $a<b$. Since $D^-_{f,\mathcal{F}}(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} D^-_{f,\mathcal{F}}(A_n)$ for all $A_n \subset R^*$, $D^-_{f,\mathcal{F}}(G) \in \mathcal{B}r$, $\mathcal{L}$, $\mathcal{B}$, respectively, for all $G \in \mathcal{O}^*$.

The following corollary improves Professor Mišík's results [6].

COROLLARY 3.7.

(a) If f is a lower semi-Borel function of class α , then D^-f , $\bar{f}^-_q$ (D_+f , $\underline{f}^+_q$) are upper (lower) semi-Borel functions of class $\alpha+1$.

(b) If f is an upper semi-Borel function of class α , then D_-f , $\underline{f}^-_q$ (D^+f , $\bar{f}^+_q$) are lower (upper) semi-Borel functions of class $\alpha+1$.

(c) If f is a Baire function of class α , then D^+f, D^-f, $\bar{f}^+_q$, $\bar{f}^-_q$ (D_+f, D_-f, $\underline{f}^+_q$, $\underline{f}^-_q$) are upper (lower) semi-Borel functions of class $\alpha+1$.

In the remainder of this section we will recall some results concerning the ordinary DS which can be found in [5].

As we will see the continuity of E assumed in [1] can be replaced by semi continuity.

DEFINITION 3.8. A multifunction $E:R \longrightarrow 2^R$ is lower (upper) semi continuous if $F^-(G) \in \mathcal{O}$ $(F^+(G) \in \mathcal{O})$ for all $G \in \mathcal{O}$.

The proof of the following theorems can be found in [5].

THEOREM 3.9 (see Theorem 4.11 in [5]). Let $\mathcal{E}$ be an ordinary derivation system and let f be a Baire function of class one. If Gr(E) is an F_σ set and E(x) has x as a point of $\mathcal{O}$-accumulation, then $D_{f,\mathcal{E}} \in \mathcal{A}_2^- \cap \mathcal{A}_2^+$, $\bar{f}'_{\mathcal{E}} \in \mathcal{A}_2^+$, $\underline{f}'_{\mathcal{E}} \in \mathcal{A}_2^-$.

Consequently, if E is a closed valued upper semi continuous multifunction (that means Gr(E) is closed), then Theorem 3.9 holds.

THEOREM 3.10 (see Theorem 4.14 in [5]). Let $\mathcal{E}$ be an ordinary DS and let f be a continuous function. If E is a lower semi continuous multifunction and E(x) has x as a point of $\mathcal{O}$-accumulation, then $D_{f,\mathcal{E}} \in \mathcal{A}_1^- \cap \mathcal{A}_1^+$, $\bar{f}'_{\mathcal{E}} \in \mathcal{A}_1^+$, $\underline{f}'_{\mathcal{E}} \in \mathcal{A}_1^-$.

COROLLARY 3.11 (see also Corollary 10 of [1]). Let $\mathcal{E}$ be a congruent and ordinary DS. If f is a continuous function, then $\bar{f}'_{\mathcal{E}} \in \mathcal{A}_1^+$, $\underline{f}'_{\mathcal{E}} \in \mathcal{A}_1^-$ and $D_{f,\mathcal{E}} \in \mathcal{A}_1^- \cap \mathcal{A}_1^+$.

4. The classification of $D_{f,\mathcal{E}}$ and the extreme $\mathcal{E}$-derivatives for the approximate derivation systems

This section is devoted to the semi-Borel classification and Lebesgue measurability of the multifunction of all approximate $\mathcal{E}$-derived numbers and $\bar{f}^+_{ap}$, $\underline{f}^+_{ap}$, $\bar{f}^-_{ap}$, $\underline{f}^-_{ap}$. First of all we must solve the structure of the set of fixed points of certain multifunctions for the density topology $\mathcal{D}$. This is solved by the following three lemmas.

LEMMA 4.1. Let $\mathcal{A} \subset 2^R$ be a σ-additive and multiplicative family, $\emptyset, R \in \mathcal{A}$, $S = \bigcup_{n=1}^{J} A_i \times B_i$, $A_i \in \mathcal{A}$, $B_i \in \mathcal{L}$. If $f: R \to R^*$ is defined as $f(x) = \lambda(F_S(x))$ (λ is Lebesgue measure), then $f^{-1}((a,\infty]) \in \mathcal{A}$ for all $a \in R$.

PROOF. Let $I = 2^{\{B_1,\dots,B_J\}}$, $I_a = \{ i \in I : \lambda(\bigcup_{B_k \in i} B_k) \le a \}$.

We shall show that $\{x : f(x) \le a\} = \bigcup_{i \in I_a} F_S^+(\bigcup_{B_k \in i} B_k)$. If $x \in f^{-1}([-\infty, a])$, then $\lambda(F_S(x)) = \lambda((\bigcup_{i=1}^{J} A_i \times B_i)_x) = \lambda(\bigcup_k \{B_k : x \in A_k\}) \le a$.

Hence $i_o := \{B_k : x \in A_k\} \in I_a$ and $x \in F_S^+(\bigcup_{B_k \in i_o} B_k)$.

If $x \in \bigcup_{i \in I_a} F_S^+(\bigcup_{B_k \in i} B_k)$, then there is an $i_o \in I_a$ such that $F_S(x) \subset \bigcup_{B_k \in i_o} B_k$. Hence $\lambda(F_S(x)) = f(x) \le a$.

The following two equalities finish the proof:

$$F_S^+(\bigcup_{B_k \in i} B_k) = R \setminus F_S^-(R \setminus \bigcup_{B_k \in i} B_k),$$

$$F_S^-(R \setminus \bigcup_{B_k \in i} B_k) = \bigcup_m \{A_m : B_m \cap (R \setminus \bigcup_{B_k \in i} B_k) \neq \emptyset\}.$$

LEMMA 4.2 Let $S = \bigcup_{i=1}^{\infty} A_i \times B_i$, $A_i \in \mathcal{A}$, $B_i \in \mathcal{L}$ ($\mathcal{A}$ is as in Lemma 4.1). If $f(x) = \lambda(F_S(x))$, then $f^{-1}((a,\infty]) \in \mathcal{A}$ for all $a \in R$.

PROOF. Let $f_j(x) = \lambda(F_{S_j}(x))$, $x \in R$, $j = 1,2,\ldots$ where $S_j = \bigcup_{i=1}^{j} A_i \times B_i$. Evidently, $F_{S_j}(x) \subset F_{S_{j+1}}(x)$ and $F_S(x) = \bigcup_{j=1}^{\infty} F_{S_j}(x)$ for all $x \in R$. Hence $f(x) = \lambda(F_S(x)) = \lim_{j\to\infty} \lambda(F_{S_j}(x)) = \lim_{j\to\infty} f_j(x)$ for all $x \in R$. Since $\{f_j\}_{j=1}^{\infty}$ is a nondecreasing sequence, $f^{-1}((a,\infty]) = \bigcup_{j=1}^{\infty} \{x: f_j(x) > a\} \in \mathcal{A}$.

LEMMA 4.3. Let $S = \bigcup_{i=1}^{\infty} A_i \times B_i$, $A_i \in \mathcal{A}$, $B_i \in \mathcal{L}$, $\mathcal{O} \subset \mathcal{A}$ ($\mathcal{A}$ is as in Lemma 4.1). If $\mathcal{E} = 2^R \setminus \{\emptyset\}$, then $P(\mathcal{E}_{\mathfrak{D}}(F_S)) \in \sigma\delta\mathcal{A}$.

PROOF. Let $C_n = \{(x,y): x-1/n < y < x+1/n\}$. Evidently $C_n \cap S \in \sigma(\mathcal{A} \times \mathcal{L})$. Since $x \in P(\mathcal{E}_{\mathfrak{D}}(F_S))$ if and only if

$$\limsup_{n\to\infty} n/2(\lambda(F_S(x) \cap (x-1/n, x+1/n))) =$$
$$\limsup_{n\to\infty} n/2(\lambda(F_{S\cap C_n}(x))) > 0,$$

$P(\mathcal{E}_{\mathfrak{D}}(F_S)) = g^{-1}((0,1])$ where $g: R \to [0,1]$ is defined as $g(x) = \limsup_{n\to\infty} n/2(\lambda(F_{S\cap C_n}(x)))$. By Lemma 4.2, for any $n \in N$ and $a \in R$ we have $\{x: n/2(\lambda(F_{S\cap C_n}(x))) > a\} \in \mathcal{A}$. The equality $g^{-1}((0,1]) = \bigcup_{r=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \{x: m/2(\lambda(F_{S\cap C_n}(x))) > 1/r\}$ finishes the proof.

A motivation for this section came from paper [7] where Professor Mišík showed that the upper (lower) unilateral (i.e. $E(x)=[x,\infty)$, $E(x)=(-\infty,x]$ respectively) approximate derivative of a function of Baire class α is a lower (upper) semi-Borel function of class $\alpha+2$. The following main theorem of this section and its consequences show that E can have more general form.

THEOREM 4.4. Let $\mathcal{A}$ be a σ-additive and multiplicative family, $\mathcal{O} \subset \mathcal{A} \subset \mathcal{L}$, $f: R \longrightarrow R$ and let $\mathcal{E}$ be an approximate DS in which $Gr(E) \in \sigma(\mathcal{A} \times \mathcal{L})$.

(a) If $f \in \mathcal{A}^-$ ($f \in \mathcal{A}^+$) and $\mathcal{E}$ is left-sided, then $D_{f,\mathcal{E}} \in c\delta\sigma\delta\mathcal{A}^+$ ($D_{f,\mathcal{E}} \in c\delta\sigma\delta\mathcal{A}^-$) and $\bar{f}'_{\mathcal{E}} \in c\delta\sigma\delta\mathcal{A}^+$ ($\underline{f}'_{\mathcal{E}} \in c\delta\sigma\delta\mathcal{A}^-$).

(b) If $f \in \mathcal{A}^-$ ($f \in \mathcal{A}^+$) and $\mathcal{E}$ is right-sided, then $D_{f,\mathcal{E}} \in c\delta\sigma\delta\mathcal{A}^-$ ($D_{f,\mathcal{E}} \in c\delta\sigma\delta\mathcal{A}^+$) and $\underline{f}'_{\mathcal{E}} \in c\delta\sigma\delta\mathcal{A}^-$ ($\bar{f}'_{\mathcal{E}} \in c\delta\sigma\delta\mathcal{A}^+$).

(c) If $f \in \mathcal{A}^- \cap \mathcal{A}^+$, then $D^-_{f,\mathcal{E}}([a,b]) \in \delta\sigma\delta\mathcal{A}$ for all $a,b \in R^*$, $a<b$, and hence $D_{f,\mathcal{E}} \in c\delta\sigma\delta\mathcal{A}^- \cap c\delta\sigma\delta\mathcal{A}^+$ and $\bar{f}'_{\mathcal{E}} \in c\delta\sigma\delta\mathcal{A}^+$, $\underline{f}'_{\mathcal{E}} \in c\delta\sigma\delta\mathcal{A}^-$.

f	$\mathcal{E}$	$D_{f,\mathcal{E}}$	$\bar{f}'_{\mathcal{E}}$	$\underline{f}'_{\mathcal{E}}$
$\mathcal{A}^+$	+	$c\delta\sigma\delta\mathcal{A}^+$	$c\delta\sigma\delta\mathcal{A}^+$	
$\mathcal{A}^-$	-	$c\delta\sigma\delta\mathcal{A}^+$	$c\delta\sigma\delta\mathcal{A}^+$	
$\mathcal{A}^+$	-	$c\delta\sigma\delta\mathcal{A}^-$		$c\delta\sigma\delta\mathcal{A}^-$
$\mathcal{A}^-$	+	$c\delta\sigma\delta\mathcal{A}^-$		$c\delta\sigma\delta\mathcal{A}^-$

Considering Lemma 4.3 the proof is similar to that of Theorem 3.3.

COROLLARY 4.5. Let $\mathcal{E}$ be an approximate DS in which $Gr(E) \in \sigma(\mathcal{A}_d \times \mathcal{L})$. The semi-Borel classification of $D_{f,\mathcal{E}}$, $\bar{f}'_{\mathcal{E}}$, $\underline{f}'_{\mathcal{E}}$ can be read out of the following table

f	$\mathcal{E}$	$D_{f,\mathcal{E}}$	$\bar{f}'_{\mathcal{E}}$	$\underline{f}'_{\mathcal{E}}$
$\mathcal{A}^{+}_{\alpha}$	+	$\mathcal{A}^{+}_{\alpha+3}$	$\mathcal{A}^{+}_{\alpha+3}$	
$\mathcal{A}^{-}_{\alpha}$	-	$\mathcal{A}^{+}_{\alpha+3}$	$\mathcal{A}^{+}_{\alpha+3}$	
$\mathcal{A}^{+}_{\alpha}$	-	$\mathcal{A}^{-}_{\alpha+3}$		$\mathcal{A}^{-}_{\alpha+3}$
$\mathcal{A}^{-}_{\alpha}$	+	$\mathcal{A}^{-}_{\alpha+3}$		$\mathcal{A}^{-}_{\alpha+3}$

COROLLARY 4.6.

(a) If $f \in \mathcal{A}^{-}_{\alpha}$, then $\bar{f}^{-}_{ap} \in \mathcal{A}^{+}_{\alpha+3}$, $\underline{f}^{+}_{ap} \in \mathcal{A}^{-}_{\alpha+3}$.

(b) If $f \in \mathcal{A}^{+}_{\alpha}$, then $\underline{f}^{-}_{ap} \in \mathcal{A}^{-}_{\alpha+3}$, $\bar{f}^{+}_{ap} \in \mathcal{A}^{+}_{\alpha+3}$.

(c) (see Theorem 3 of [7]) If $f \in \mathcal{A}^{+}_{\alpha} \cap \mathcal{A}^{-}_{\alpha}$, then $\bar{f}^{+}_{ap}, \bar{f}^{-}_{ap} \in \mathcal{A}^{+}_{\alpha+3}$, $\underline{f}^{+}_{ap}, \underline{f}^{-}_{ap} \in \mathcal{A}^{-}_{\alpha+3}$.

THEOREM 4.7. (for $\bar{f}^{-}_{ap}$ see Theorem 3 of [7]). Let $\mathcal{E}$ be an approximate DS in which Gr(E) is Lebesgue measurable. If f is a Lebesgue measurable function, then $D_{f,\mathcal{E}}$, $\bar{f}'_{\mathcal{E}}$, $\underline{f}'_{\mathcal{E}}$ are Lebesgue measurable.

PROOF. Let $A_n = \{(x,y)\colon x-1/n<y<x+1/n\}$. By Lemmas 2.6 and 2.5, $A(E,f,n,a,b) \cap A_n$ is Lebesgue measurable for all $a,b \in R^*$ $a<b$. Consider the functions

$$f_{m,n}(x) = n/2(\lambda^*(F_{A_n \cap A(E,f,m,a,b)}(x)))$$

$$\text{and } g_m = \limsup_{n\to\infty} f_{m,n}$$

where λ^* is Lebesgue outer measure ($m,n=1,2,3,\ldots$). Evidently g_m is Lebesgue measurable. Since $P(\mathscr{E}_{\mathfrak{D}}(F_{A(E,f,m,a,b)})) = g_m^{-1}((0,1])$, $P(\mathscr{E}_{\mathfrak{D}}(F_{A(E,f,m,a,b)}))$ is Lebesgue measurable and by Lemma 2.4, $D^-_{f,\mathscr{E}}([a,b]) \in \mathscr{L}$ for all $a,b \in R^*$, $a<b$. Hence $D^-_{f,\mathscr{E}}(G) \in \mathscr{L}$ for all $G \in \mathscr{O}^*$.

5. The properties of $\mathscr{E}$-primitives

The results of the previous sections show that the properties of the extreme $\mathscr{E}$-derivatives depend on the structure of Gr(E). On the other hand, as we shall see in this section, the properties of the $\mathscr{E}$-primitives depend on the values of E. The following facts can be found in [5].

DEFINITION 5.1. A function $f:(R,\mathcal{T}) \longrightarrow R$ is $\mathcal{T}$-quasicontinuous at a point $x \in R$ if for any $V \in \mathscr{O}$, $U \in \mathcal{T}$, $f(x) \in V$, $x \in U$ there is a set $H \in \mathcal{T}$, $\emptyset \neq H \subset U$ such that $H \subset f^{-1}(V)$. A function f is $\mathcal{T}$-quasicontinuous if it is $\mathcal{T}$-quasicontinuous at every $x \in R$. A function f has the $\mathcal{T}$-Baire property if $f^{-1}(G)$ has the $\mathcal{T}$-Baire property for any $G \in \mathscr{O}$.

THEOREM 5.2 (see Theorem 2.7 in [5]). Let $\mathscr{E} = (R,\mathcal{T},E,\mathscr{C})$ be a DS, $\mathscr{C} = \{A \subset R:$ A is of the $\mathcal{T}$-second category and A has the $\mathcal{T}$-Baire property$\}$.

(a) If there is a $\mathcal{T}$-dense set S such that for all $x \in S$ f has at least one finite $\mathscr{E}$-derived number at x, then f has the $\mathcal{T}$-Baire property.

(b) If for any $x \in R$ f has at least one finite $\mathscr{E}$-derived number at x, then f is $\mathcal{T}$-quasicontinuous.

THEOREM 5.3 (see Theorem 3.13 in [5]). Let $(R,\mathcal{T})$ be a Baire space having no isolated points, $\mathcal{O} \subset \mathcal{T}$. Let $\mathcal{E}$ be a Baire essential and unilateral DS. If $\bar{f}'_{\mathcal{E}}(x) < \infty$ ($\underline{f}'_{\mathcal{E}}(x) > -\infty$) except for a set of the $\mathcal{T}$-first category, then f has the $\mathcal{T}$-Baire property.

COROLLARY 5.4. Let $\mathcal{E} = (R,\mathcal{D},E,2^R\setminus\{\emptyset\})$ be a unilateral and approximate DS in which $E(x) \in \mathcal{L}$ for all $x \in R$. If $\bar{f}'_{\mathcal{E}}(x) < \infty$ ($\underline{f}'_{\mathcal{E}}(x) > -\infty$) except for a set of Lebesgue measure zero, then f is Lebesgue measurable.

PROOF. Let $A = \{x: \bar{f}'_{\mathcal{E}}(x) < \infty\}$ ($A = \{x: \underline{f}'_{\mathcal{E}}(x) > -\infty\}$). Let $\mathcal{E}_1 = (R,\mathcal{D},E_1,\mathcal{C})$, $\mathcal{C} = \{A : \lambda^*(A) > 0\}$, $E_1(x) = E(x)$ if $x \in A$, $E_1(x) = [x,\infty)$ if $x \notin A$ and $\mathcal{E}$ is right-sided, $E_1(x) = (-\infty,x]$ if $x \notin A$ and $\mathcal{E}$ is left-sided. Since $\mathcal{E}_1$ is a Baire unilateral and essential DS and $D_{f,\mathcal{E}}(x) = D_{f,\mathcal{E}_1}(x)$ for all $x \in A$, f is Lebesgue measurable by Theorem 5.3.

THEOREM 5.5 (see Theorem 3.14 in [5]). Let $(R,\mathcal{T})$ be a Baire space having no isolated points, $\mathcal{O} \subset \mathcal{T}$. Let $\mathcal{E}$ be a unilateral and ordinary DS in which E(x) has the $\mathcal{T}$-Baire property and E(x) is of the $\mathcal{T}$-second category at x for all $x \in R$. If $\bar{f}'_{\mathcal{E}}(x) < \infty$ ($\underline{f}'_{\mathcal{E}}(x) > -\infty$) except for a set of the $\mathcal{T}$-second category, then f has the $\mathcal{T}$-Baire property.

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