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VARIATIONS ON PRODUCTS AND QUOTIENTS OF DARBOUX FUNCTIONS

I. Let us establish some of the terminology to be used. R denotes the real line and N denotes the set of natural numbers. If $a,b\in R$, then (a,b) denotes the open interval with the end-points a,b. For $A\subset R$, we shall say that I is an open interval of A iff $I=(a,b)\cap A$ for some $a,b\in R$. If B is a planar set, we shall denote its x - projection by dom B and its y - projection by rng B. If A,B are subsets of R then $A\ B=\{a\cdot b\ :\ a\in A,\ b\in B\}$, $a\ B=\{a\}\cdot B$ and $A^{-1}=\{1/a\ :\ a\in A\setminus\{0\}\}$. For $A\subset R$, $a\in R$, and $f:A\to R$, we define the set $[f<a]$ as $\{x\in A\ :\ f(x)<a\}$. Analogously, we define the sets $[f>a]$ and $[f=a]$. Let $A\subset R$ be a c-dense set in itself (where c denotes the cardinality of the continuum) and let B be a subset of R. We say that $f:A\to B$ is an (A,B)-Darboux function iff f has the intermediate value property, i.e. $(f(x),f(y))\cap B\subset f((x,y)\cap A)$ for each $x,y\in A$. Let $\mathcal{D}(A,B)$ denote the class of all (A,B)-Darboux functions. Let $\mathcal{D}^*(A,B)$ denote the class of all functions $f:A\to B$ which take on every $y\in B$ in every non-empty interval I of A. Let $\mathcal{D}^{**}(A,B)$ denote the class of all functions $f:A\to B$ which take on every $y\in B$ c times in every interval of A. It is clear that $\mathcal{D}^{**}(A,B)\subset\mathcal{D}^*(A,B)\subset\mathcal{D}(A,B)$ for every bilaterally c-dense subset A of R and every subset B of R. For A=B=R, we shall denote the classes $\mathcal{D}(A,B)$, $\mathcal{D}^*(A,B)$, and $\mathcal{D}^{**}(A,B)$ by $\mathcal{D}$, $\mathcal{D}^*$, $\mathcal{D}^{**}$ (see [3]).
A.M. Bruckner and J. Ceder proved the following theorem.

THEOREM 1.[3]. Let $f\in\mathcal{D}$ be constant on no subinterval of R and let M be a set of real numbers whose complement is dense. Then for each countable dense subset D of $R\setminus M$ there exists a function $d\in\mathcal{D}^*$ such that the range of f+d is D.

In the same way, we can prove the following result.

THEOREM 1*. If $A \subset R$ is bilaterally c-dense set in itself, D is a countable dense subset of R, and if $f \in \mathcal{D}(A,R)$ is constant on no interval of A, then there exists a function $d \in \mathcal{D}^*(A,R)$ such that for every interval I of A the range of $(f+d)|I$ is D.

THEOREM 2. Assume that D is a countable dense subset of R and $0 \in D$. Then for each $f \in \mathcal{D}$ there exists a function $d \in \mathcal{D}$ such that for each interval $I \subset R$ we have:

- if f is not constant on I, then $d(I)=R$ and $(f \cdot d)(I)=D$,
- if $f|I$ is constant, then $d|I$ is constant and $(f \cdot d)(I) \subset D$.

P r o o f . Let $B=\{(x,y) : y=r/f(x)$ if $f(x) \neq 0$ and $y=r$ if $f(x)=0$, $r \in D$, $x \in R\}$. As in the proof of Theorem 1 ([3]), we shall define a function $d \in \mathcal{D}$ such that $d \subset B$. (No distinction is made between a function and its graph).

Let us put $\mathcal{J}=\{I \subset R$: I is a maximal open interval such that $f|I$ is constant$\}$. Observe that sets from $\mathcal{J}$ are pairwise disjoint and hence the family $\mathcal{J}$ is countable.

Let $\mathcal{J}=\{J_n : n \in N\}$, $J_n=(a_n,b_n)$, $A_0=\{a_n,b_n : n \in N\}$, $J=\bigcup \mathcal{J}$ and $A=R \setminus J$. Notice that $A_0 \subset A$ and $f(a_n)=f(b_n)$ for each $n \in N$. Additionally, $f|A \in \mathcal{D}(A,R)$.

For $x \in R$, let $V(x)=\{x\} \times R$ and $H(x)=A \times \{x\}$. For each $x \in R$ it is clear that $V(x) \cap B$ is dense in $V(x)$ and it is easy to verify that $H(x) \cap B$ is dense in $H(x)$. Indeed, it is clear for $x=0$. Assume that $x \neq 0$ and I is an open interval of A. Then $f|I$ is non-constant and, since $f \in \mathcal{D}$, there exist $y, z \in I$ for which $f(y) \neq f(z)$ and $f(y) \cdot f(z)>0$. We may assume that $f(z)>f(y)>0$. Because D is dense in R, we have

$$\bigcup_{r \in D} r\,(f(y),f(z))^{-1} = \bigcup_{r \in D} (r/f(z),r/f(y)) = R,$$

so there exists an $r \in D$ such that $x \in (r/f(z),r/f(y))$, i.e. $r/x \in (f(y),f(z))$. Since $f|A \in \mathcal{D}(A,R)$, there exists a $t \in I$ for which $f(t)=r/x$, i.e. $x=r/f(t)$ and $(t,x) \in (I \times \{x\}) \cap B$.

For $x \in R$ let $N(x) = [\; A \times x\,(E \setminus \{0\}) \cdot (E \setminus \{0\})^{-1}\;] \cap B$, where

$$E=\bigcup_{n=1}^{\infty} \underbrace{D\cdot\ldots\cdot D}_{n\text{ - times}}$$

Observe that

(i) $N(0)=H(0)=A\times\{0\}$,

(ii) $\bigcup\{N(x) : x\in R\}=B\cap(A\times R)$,

(iii) $\text{card}(\text{rng } N(x))=\omega_o$ for $x\in R\setminus\{0\}$,

(iv) dom $N(x)$ is dense in A, for each $x\in R$,

(v) if $x \neq 0 \neq y$ and dom $N(x) \cap$ dom $N(y) \neq \emptyset$, then $N(x) = N(y)$.

Now we can define the function d. First we define for each $x\neq 0$ a function d_x such that $d_x=d_y$ if $N(x)=N(y)$, d_x is dense in any non-empty $H(y)\cap N(x)$, $a_n \in$ dom d_x iff $b_n \in$ dom d_x, and $d_x(a_n)=d_x(b_n)$. To do this, let O_n, $n=0,1,2,\ldots$ be an enumeration of all horizontal open intervals with rational ends which intersect $N(x)$. Put $w_o\in O_o\cap N(x)$,

$$v_o = \begin{cases} w_o & \text{if dom } w_o\notin A_o, \\ (b_k, \text{ rng } w_o) & \text{if dom } w_o=a_k, \\ (a_k, \text{ rng } w_o) & \text{if dom } w_o=b_k. \end{cases} \qquad k\in N$$

and $w_n\in O_n \cap N(x)\setminus \bigcup_{i<n} (V(w_i)\cup V(v_i))$,

$$v_n = \begin{cases} w_n & \text{if dom } w_n\notin A_o, \\ (b_k, \text{ rng } w_n) & \text{if dom } w_n=a_k, \\ (a_k, \text{ rng } w_n) & \text{if dom } w_n=b_k. \end{cases} \qquad k\in N$$

Then $d_x = \{ w_n, v_n : n\in N \}$ has the desired properties.

Next let $d_{1,1}$, $d_{1,2}$ be a partition of d_1 onto two sets, each dense in d_1 and such that $a_n \in d_{1,i}$ iff $b_n \in d_{1,i}$ for $n\in N$, $i=1,2$. Then $d_{1,1}$ is dense in $N(1)$, dom $d_{1,2}$ is dense in A and $d_o=\text{dom } d_{1,2}\times\{0\}$ is dense in $N(0)$.

Now enumerate the countable family of uncountable sets of the form $\{ (x,y) : x \in I\setminus A_o, y = r/f(x) \text{ and } f(x)\neq 0 \}$, where $I=(a,b)\cap A$ for some rationals a,b and $r\in D\setminus\{0\}$, as $\{C_i\}$. As in [3], we pick a sequence of points $\{e_i\}$ such that: $e_i\in C_i\setminus Z_i$, where $Z_i=\bigcup\{N(x) :$ there exists $j<i$ with $e_j\in N(x)\}\cup N(1)$. This

is possible because card (rng C_i)=c and card (rng Z_i)=ω_0. Since $d_0 \subset (\text{dom } N(1))\times\{0\}$, dom $d_0 \cap$ dom e_i =$\emptyset$. Let $e=\{e_i : i\in N\}$. Then card($e \cap d_x$)≤ 1 for each $x\in R$, $0 \notin$ rng e and dom e $\cap A_0=\emptyset$. Let $A_1 = A_0 \setminus \bigcup \{ \text{dom } d_z : z\in R \}$. Then $a_n \in A_1$ iff $b_n \in A_1$ for each $n\in N$.

Now we define a function d on R as follows.

$$d(x)=\begin{cases} e(x) & \text{if } x \in \text{dom } e, \\ d_z(x) & \text{if } x \in \text{dom } d_z \setminus \text{dom } e,\ z\in R, \\ 0 & \text{if } x \in A\setminus(\text{dom } e \cup \bigcup\{\text{dom } d_z : z\in R\}), \\ d(a_n)=d(b_n) & \text{if } x \in J_n,\ n\in N. \end{cases}$$

It is clear that $d\subset B$ and therefore $(f\cdot d)(I)\subset D$ for every open interval I. If I is an open interval for which $f|I$ is not constant and $y\in R$, then $I\cap A$ is non-empty and $(I\times\{y\}) \cap d_y \setminus e$ is infinite. It follows that $y\in d(I)$ and, consequently, $d(I)=R$. If $r\in D\setminus\{0\}$, then there exists $x\in I\setminus A_0$ such that $e(x)=r/f(x)$, $f(x)\ne 0$ and hence $r\in(f\cdot d)(I)$. Thus $D \subset (f\cdot d)(I)$. Since $(I\times\{0\}) \cap d_0 \ne \emptyset$, $0 \in (f\cdot d)(I)$.

If $f|I$ is constant, then $I\subset(a_n,b_n)$ for some $n\in N$. Then $d(x)=d(a_n)=d(b_n)$ for $x\in I$. Finally, if $x,y\in R$, $d(x)\ne d(y)$, then f is non-constant on (x,y) and the range of d on (x,y) is R. Thus $d\in\mathcal{D}$. This finishes the proof.

REMARKS. 1) If D satisfies all assumptions of Theorem 2 and $f\in\mathcal{D}$ is constant on no interval, then there exists a $d\in\mathcal{D}^*$ such that for any interval I of R the range of $(f\cdot d)|I$ is D.
2) In the same way as Theorem 2, we can generalize Theorem 1.

THEOREM 1**. Assume that $D\subset R$ is a countable dense set and $f\in\mathcal{D}$. Then there exists a $d\in\mathcal{D}$ such that for each interval $I\subset R$ we have:

- if f is not constant on I then d(I)=R and (f+d)(I)=D,
- if f is constant on I then d(I)={y} for some $y\in D$.

THEOREM 3. Let D be a countable dense subset of R with $0 \in D$ and let $f \in \mathcal{D}$ be constant on no interval. Then there exists a function $d\in\mathcal{D}^*(R,(0,\infty))$ such that for every interval I we have:

- if $I \subset [f>0]$ then $(f/d)(I) = D \cap (0,\infty)$,
- if $I \subset [f<0]$ then $(f/d)(I) = D \cap (-\infty,0)$,
- if f changes sign on I then $(f/d)(I) = D$.

P r o o f . Let us put $B = [f>0]$, $C = [f<0]$, $D^+ = D \cap (0,\infty)$, $D^- = D \cap (-\infty,0)$, $f_1 = \ln(f|B)$ and $f_2 = \ln(-f|C)$.
By Theorem 1*, there exists a function $d_1 \in \mathcal{D}^*(B,R)$ such that $(f_1 + d_1)(I) = \ln D^+$ for every interval I of B. Then $[\exp(f_1+d_1)](I)=D^+$ and hence the range of $(f|I)\cdot(\exp d_1|I)$ is D^+. Observe that $d_+ = \exp(-d_1) \in \mathcal{D}^*(B,(0,\infty))$. In the same way, we define $d_2 \in \mathcal{D}^*(C,R)$ such that $(f_2 + d_2)(I) = \ln(-D^-)$ for every interval I of C. Then $d_- = \exp(-d_2) \in \mathcal{D}^*(C,(0,\infty))$ and the range of $(f|I)/(d_-|I)$ is D^-.
Let us define $d: R \to (0,\infty)$ by

$$d(x) = \begin{cases} d_+(x) & \text{for } x\in B, \\ d_-(x) & \text{for } x\in C, \\ 1 & \text{if } f(x)=0. \end{cases}$$

It is easy to verify that such a defined function d satisfies the conditions of Theorem 3.

II. The following result is proved in [6].

THEOREM 4. Assume that $A,B,C\subset R$, $F: A\times B \to R$ and $f: R \to A$. Then there exists a $d\in\mathcal{D}^{**}(R,B)$ such that $F(f,d)\in\mathcal{D}^{**}(R,C)$ iff the following conditions hold:

(1) for every $x\in R$ there exists $y\in B$ such that $F(f(x),y)\in C$,

(2) card $(\{x\in I : F(f(x),y)=c \text{ for some } y\in B\})=\mathfrak{c}$ for every $c\in C$ and every interval I,

(3) card $(\{x\in I : F(f(x),y)\in C\})=\mathfrak{c}$ for every $y\in B$ and every interval I.

Observe that for $A=B=R$, $F(x,y)=x\cdot y$, $f: R \to R$ and $0\in C$ we obtain the following.

COROLLARY . There exists a $d\in\mathcal{D}^{**}$ such that $f\cdot d\in\mathcal{D}^{**}(R,C)$ iff card $(\{x\in I : f(x)\neq 0\}) = \mathfrak{c}$ for every interval I and card$(x\in I : f(x)\cdot y\in C\})=\mathfrak{c}$ for every $y\in R$ and every interval I.

III. Let $\mathcal{A}$ be a family of real functions. A subfamily $\mathcal{B}$ of

$\mathcal{A}$ is called the maximal multiplicative (additive) family for $\mathcal{A}$ provided $\mathcal{B}$ is the set of all functions in $\mathcal{A}$ such that $f\,g\in\mathcal{A}$ ($f+g\in\mathcal{A}$, respectively) whenever $f\in\mathcal{B}$ and $g\in\mathcal{A}$. (See [2], p. 14).

As an immediate consequence of Theorem 2 (respectively Theorem 1^{**}), we obtain that the maximal multiplicative (additive) family for $\mathcal{D}$ is the class of all constant functions ([7],[2]).

Using a method similar to that used by J. Jastrzębski in [5], we can porove the following results.

THEOREM 5. Let $g\in\mathcal{D}$ and $g\not\equiv 0$. Then $f\,g\in\mathcal{D}$ for every $f\in\mathcal{D}^*$ iff theree exists a sequence a of open intervals $\{I_k\}$ such that:

(1) $\bigcup_{k=1}^{\infty} I_k$ is dense in R,

(2) $g|I_k$ is constant and $g|I_k \not\equiv 0$.

P r o o f . Assume that for $g\in\mathcal{D}$ there exists a sequence $\{I_k\}$ which satisfies the conditions (1) and (2). Let $f\in\mathcal{D}^*$ and let I be an open interval. Then $\emptyset \neq J = I_k \cap I \subset I$ for some $k\in N$, $g|J$ is constant and $g|J \not\equiv 0$. Consequently, $f(J)=R$ and $g\cdot f(I)=g\cdot f(J)=R$. Thus $f\cdot g\in\mathcal{D}^*$.

Assume that $g\in\mathcal{D}$, I is an open interval and g is not constant on every subinterval of I. It follows from Theorem 2 that there exists a function $f \in \mathcal{D}^*$ such that $f\cdot g \notin \mathcal{D}$. Now assume that there exists an open interval I and a sequence of pairwise disjoint, open subintervals of I, $\{I_k\}$ such that $\bigcup_{k=1}^{\infty} I_k$ is dense in I and $g(x) = 0$ for each $x \in \bigcup_{k=1}^{\infty} I_k$. Since $g \in \mathcal{D}$ and $I \neq \bigcup_{k=1}^{\infty} I_k$, there exist $y, z \in I \setminus \bigcup_{k=1}^{\infty} I_k$ with $g(y) \neq g(z)$. Choose $f_k \in \mathcal{D}^*(I_k, R)$ for $k\in N$ and put

$$f(x)=\begin{cases} f_k(x) & \text{for } x\in I_k,\ k\in N,\\ 1 & \text{for } x\in\{y,z\},\\ 0 & \text{otherwise.}\end{cases}$$

Then $f \in \mathcal{D}^*$ and $f\cdot g(y) = g(y) \neq g(z) = f\cdot g(z)$, $f\cdot g(x) = 0$

for $x\notin\{y,z\}$, i.e. $f\cdot g\notin\mathcal{D}$.

Of course, the condition (2) can not be satisfied for any $g\in\mathcal{D}^*$. Hence the maximal multiplicative family for $\mathcal{D}^*$ is empty.

THEOREM 6. Let $g:R\to R$, $g\not\equiv 0$. Then $f\cdot g\in\mathcal{D}$ for every $f\in\mathcal{D}^{**}$ iff there exist a sequence of open intervals $\{I_k\}$ and a set $A\subset R$ such that:

(3) $\bigcup_{k=1}^{\infty} I_k$ is dense in R,

(4) card $(A)<c$,

(5) $g|(I_k\setminus A)$ is constant for every k and $g|(I_k\setminus A)\not\equiv 0$.

P r o o f . Assume that for $g:R\to R$ there exist a set A and a sequence $\{I_k\}$ which satisfy the conditions (3), (4) and (5). Let I be an open interval, $f\in\mathcal{D}^{**}$, and $y\in R$. Then $\emptyset\neq J=I_k\cap I\subset I$ for some $k\in N$ and $g(x)=a\neq 0$ for each $x\in J\setminus A$. Since $f\in\mathcal{D}^{**}$, card $(\{x\in J: f(x)=y/a\})=c$. Thus card $(\{x\in I: f(x)\cdot g(x)=y\})\geq$ card $(\{x\in J\setminus A: f(x)=y/a\})=c$ and $f\cdot g\in\mathcal{D}^{**}$.

Assume that $g:R\to R$, I is an open interval and $g|(J\setminus A)$ is not constant for every subinterval J of I and every subset A of J with card$(A)<c$. Let $C=R\setminus\{1\}$. It follows from the Corollary to Theorem 2, that there exists a $d\in\mathcal{D}^{**}$ such that $f\cdot d\in\mathcal{D}^{**}(R,C)$, i.e. $f\cdot d\notin\mathcal{D}$.

Now assume that there exist an open interval I, a sequence of pairwise disjoint, open subintervals of I, $\{I_k\}$ and a subset A of I such that card$(A)<c$, $\bigcup_{k=1}^{\infty} I_k$ is dense in I and $g(x)=0$ for each $x\in\bigcup_{k=1}^{\infty} I_k\setminus A$. Notice that there exist y,z $\in I\setminus\bigcup_{k=1}^{\infty} I_k$ with $g(y)\neq g(z)$. Choose $f_k\in\mathcal{D}^{**}(I_k,R)$ for $k\in N$ and put

$$f(x)=\begin{cases} f_k(x) & \text{for } x\in I_k\setminus A,\ k\in N,\\ 1 & \text{for } x\in\{y,z\},\\ 0 & \text{otherwise.}\end{cases}$$

Then $f\in\mathcal{D}^{**}$ and $f\cdot g\notin\mathcal{D}$.

Evidently, the conditions (4) and (5) can not be satisfied for any $g\in\mathcal{D}^{**}$. Therefore the maximal multiplicative family for $\mathcal{D}^{**}$ is empty.

IV. J. Ceder in [4] has characterized those functions which can be factored into a product of two Darboux functions. In the same paper, the author stated that a function f is a quotient of two Darboux functions iff $[f\neq 0]$ is bilaterally c-dense in itself ([4], Theorem 2). Unfortunately, this result is not true. For example, for the function $f:R\to R$, defined by $f(x)=1$ if $x\neq 0$ and $f(x)=-1$ if $x=0$, the set $[f\neq 0]$ is bilaterally c-dense in itself, and evidently, f is not a quotient of two Darboux functions.
We shall prove the following theorem.

THEOREM 7. A function $f:R\to R$ is a quotient of two Darboux functions iff f satisfies the following conditions:

(i) if $a<b$ and $f(a)\cdot f(b)<0$ then $f(c)=0$ for some $c\in(a,b)$,

(ii) the sets $[f>0]$ and $[f<0]$ are bilaterally c-dense in itself.

P r o o f . Assume that $h_1, h_2 \in \mathcal{D}$ and $f = h_1/h_2$. Then $h_2<0$ or $h_2>0$. Thus, if $f(a)\cdot f(b)<0$, then $h_1(a)\cdot h_1(b)<0$ and, since $h_1\in\mathcal{D}$, we have $h_1(c)=0$ for some $c\in(a,b)$. Then $f(c)=0$ and (i) holds.
We may assume that $h_2>0$. Then $[f>0]=[h_1>0]$ and $[f<0]=[h_1<0]$, and by $h_1\in\mathcal{D}$ we obtain that $[f>0]$ and $[f<0]$ are bilaterally c-dense in itself. The condition (ii) holds too.
Now notice that if A is bilaterally c-dense in itself then $\mathcal{D}^*(A,B) \neq \emptyset$ ([4]). Assume that f satisfies the conditions (i) and (ii). Let us decompose $[f>0]$ into disjoint sets T_1 and T_2 each c-dense in $[f>0]$. (See [1] or [4]) Similarly, let us decompose $[f<0]$ into disjoint sets T_3 and T_4 each c-dense in $[f<0]$.
Let us define h_1, h_2 as follows:

on $[f=0]$,	$h_1=0$	$h_2=1$,
on T_1,	$h_1\in\mathcal{D}^*(T_1,(0,\infty))$,	$h_2=h_1/f$,
on T_2,	$h_2\in\mathcal{D}^*(T_2,(0,\infty))$,	$h_1=f\cdot h_2$,

on T_3, $\quad h_1 \in \mathcal{D}^*(T_3, (-\infty, 0))$, $\quad h_2 = h_1/f$,

on T_4, $\quad h_2 \in \mathcal{D}^*(T_4, (0, \infty))$, $\quad h_1 = f \cdot h_2$.

Let us observe that $f = h_1/h_2$ and $h_2 > 0$. We shall prove that $h_1 \in \mathcal{D}$. Let $h_1(a) < h_1(b)$ and $y \in (h_1(a), h_1(b))$. There are five possible cases:

(a) if $h_1(a) \geq 0$, then $f(b) > 0$. Since the set $[f>0]$ is bilaterally c-dense in itself, we obtain that $[f>0] \cap (a,b) \neq \emptyset$ and consequently $h_1(x) = y$ for some $x \in T_1 \cap (a,b)$,

(b) if $h_1(b) \leq 0$, then $f(a) < 0$ and hence there exists $x \in T_3 \cap (a,b)$ such that $h_1(x) = y$,

(c) if $h_1(a) < y < 0 \leq h_1(b)$, then $f(a) < 0$ and $h_1(x) = y$ for some $x \in T_3 \cap (a,b)$,

(d) if $h_1(a) < 0 = y < h_1(b)$, then it follows from (i) that there exists $x \in (a,b)$ such that $h_1(x) = f(x) = 0$,

(e) if $h_1(a) \leq 0 < y < h_1(b)$, then $f(b) > 0$ and $h_1(x) = y$ for some $x \in T_1 \cap (a,b)$.

Thus $h_1 \in \mathcal{D}$. Now we shall show that $h_2 \in \mathcal{D}$. Assume that $h_2(a) < h_2(b)$ and $y \in (h_2(a), h_2(b))$. Then $h_2(a) > 0$ and $(a,b) \cap [f>0] \neq \emptyset$ or $(a,b) \cap [f<0] \neq \emptyset$. If $(a,b) \cap [f>0] \neq \emptyset$, then $h_2(x) = y$ for some $x \in T_2 \cap (a,b)$. If $(a,b) \cap [f<0] \neq \emptyset$, then $h_2(x) = y$ for some $x \in T_4 \cap (a,b)$. Thus $h_2 \in \mathcal{D}$ and this finishes the proof of Theorem 7.

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