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A MULTIDIMENSIONAL VARIATIONAL INTEGRAL AND ITS EXTENSIONS[1]

Abstract. We define a variational integral in the m–dimensional Euclidean space so that the Gauss–Green theorem holds for each vector field which is everywhere differentiable (not necessary continuously). The variational integral is then extended by a transfinite sequence of improper integrals, and the Gauss–Green theorem is proved for vector fields which are differentiable only outside fairly large exceptional sets. The variational integral and its extensions are invariant with respect to a continuously differentiable change of coordinates, and hence suitable for integration on differentiable manifolds.

0. Introduction. As the divergence of a *noncontinuously* differentiable vector field need not be Lebesgue integrable, it is clear that the full–strength Gauss–Green theorem must be formulated by means of a more general integral than that of Lebesgue. This was recognized a long time ago by Denjoy (1912) and Perron (1914), who independently and by different means, defined a suitable extension of the Lebesgue integral in dimension one. While many higher–dimensional analogs of the Denjoy–Perron integral were subsequently

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produced by various authors, none of these integrated the divergence of an arbitrary differentiable vector field. Consequently, the classical Gauss–Green theorem in higher dimensions remained essentially unimproved. The situation changed only recently when following the work of Henstock (see [H]) and Kurzweil (see [K]), more sophisticated generalizations of the multidimensional Lebesgue integral were obtained in [Ma_1], [Ma_2], [JKS], [P_1], [JK], [P_2], and [P_3]. However, even these integrals have much to be desired: they are either coordinates dependent, and hence unusable on manifolds ([Ma_1], [Ma_2], [JKS], [P_1], and [P_2]), or unable to integrate vector fields with larger sets of singularities ([JK] and [P_3]).

Our goal is to define a coordinate free extension of the Lebesgue integral in the m–dimensional Euclidean space so that the Gauss–Green theorem holds for every bounded vector field, continuous outside a compact set of (m–1)–dimensional Hausdorff measure zero, and differentiable outside a compact set which is a countable union of compact sets whose (m–1)–dimensional Hausdorff measures are finite (Theorem 5.12). This is accomplished in several steps.

Elaborating on ideas of Henstock (see [H]), we begin with a simple variational integral, and the Gauss–Green theorem for continuous vector fields, differentiable outside compact sets of finite (m–1)–dimensional Hausdorff measure (Theorem 3.12). This is already an improvement of [P_3, Theorem 5.6] where the exceptional compact set is only of finite (m–1)–dimensional *upper Minkowski's content* (see [Fe, Section 3.2.37, p.273]). The reasons why we chose the variational rather than generalized Riemann integral are partly a personal preference, and partly our desire to investigate carefully the behavior of variational integrals defined by means of *additive* majorants. Naturally, *superadditive* majorants could have been employed too, but there appears to be no appreciable advantage in using them.

To enlarge the exceptional sets for the differentiability of vector fields, and relax the continuity requirement, we have extended the variational integral by the method of Marik (see [M], [HM], [KM], and [MM]). In early sixties, Marik and his collaborators devised a general two–stage process of forming improper integrals, and used it to extend the multidimensional Lebesgue integral. In [HM], both stages of the extension are treated simultaneously in an abstract setting of additive maps from Boolean rings into abelian groups. This general approach is very elegant, but it often obliterates the underlying intuition. In particular, it conceals the fact that the first stage of the extension is transfinite, while the second has only one step. Thus we considered it worthwhile to reformulate Marik's method for our specific purpose, and use the transfinite induction and recursion whenever convenient. We believe that proceeding in this way makes our treatment more intuitive.

Our presentation is local (e.g., in a Euclidean space equipped with a fixed coordinate system), however when a new integral is defined, we always prove its invariance with respect to a continuously differentiable change of coordinates (Theorems 3.14, 4.28, and 5.16). Thus using standard techniques, the integrals can be lifted to differentiable manifolds, and appropriate Stokes theorems can be obtained (see [P_3, Section 7]).

Our exposition is organized as follows. After some necessary preliminaries in Section 1, we prove in Section 2 the basic lemma (Lemma 2.2) about lower continuous additive functions of sets. The importance of this lemma for variational integrals is the same as that of Cousin's lemma (see [P_3, Proposition 2.5]) for generalized Riemann integrals; in fact, it is not hard to see that both results are actually equivalent. Section 3 is devoted to the development of the variational integral. The transfinite extension of the variational integral is presented in Section 4, and the final, nontransfinite, extension in Section 5.

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1. **Preliminaries.** By $\mathbb{R}$ we denote the set of all real numbers. Unless stated otherwise, all functions considered in this work are real–valued. The algebraic operations, partial order, and convergence among functions on the same set are defined pointwise.

On several occasions we shall employ arguments by transfinite induction and recursion in which the ordinal numbers are used extensively. As usual an ordinal is identified with the set of all smaller ordinals, and cardinals are the initial ordinals. The first uncountable cardinal is denoted by ω_1 .

Throughout, $m \geq 1$ is a fixed integer, and $\mathbb{R}^m$ denotes the m–dimensional Euclidean space. For $x = (\xi_1,\dots,\xi_m)$ and $y = (\eta_1,\dots,\eta_m)$ in $\mathbb{R}^m$, we let $x \cdot y = \xi_1\eta_1 + \cdots + \xi_m\eta_m$, and set $\|x\| = \sqrt{x \cdot x}$ and $|x| = \max(|\xi_1|,\dots,|\xi_m|)$. Unless specified otherwise, in $\mathbb{R}^m$ we use exclusively the metric induced by the norm $|x|$. The distance between a point $x \in \mathbb{R}^m$ and a set $E \subset \mathbb{R}^m$ is denoted by $\mathrm{dist}(x,E)$. If $E \subset \mathbb{R}^m$, then E^-, E°, $E^\cdot$, and $d(E)$ denote, respectively, the closure, interior, boundary, and diameter of E .

An interval $\Pi_{i=1}^m[k_i2^{-n},(k_i+1)2^{-n})$, where $n \geq 0$ and $k_1,\dots,k_m$ are integers, is called a *dyadic cube.* Often we shall use the simple observation that any family of dyadic cubes contains a disjoint subfamily which has the same union as the original family.

By $\mathcal{H}$ we denote the (m–1)–dimensional outer Hausdorff measure in $\mathbb{R}^m$ as defined in [Fe, Section 2.10.2]. If $k \geq 1$ is an integer, then λ_k denotes the k–dimensional outer Lebesgue measure in $\mathbb{R}^k$. We write λ instead of λ_1 , and $|E|$ instead of $\lambda_m(E)$

for each $E \subset \mathbb{R}^m$. The words "outer measure", "measure", and "measurable", as well as the expressions "almost all", "almost everywhere", always refer to λ_m.

Note that the measure $\mathscr{H}$ is defined so that $\mathscr{H}(E) = \lambda_{m-1}(E)$ for each set $E \subset \mathbb{R}^{m-1}$. In particular, $\mathscr{H}$ is a constant multiple (by a constant different from one) of the measure $\mathscr{H}^{m-1}$ defined in [Fa, Section 1.2] – cf. [Fa, Theorem 1,12, p.13].

A compact set $T \subset \mathbb{R}^m$ with $\mathscr{H}(T) < +\infty$ is called *thin*. In view of [Fe, Section 3.2.40, p. 276], our thin sets are *larger* than the thin sets defined in $[P_3]$. A bounded set $A \subset \mathbb{R}^m$ is called *admissible* if its boundary is thin. Admissible subsets form a ring, denoted by $\mathfrak{A}$, which is central to all our further work. The collection of all thin subsets of $\mathbb{R}^m$ is a subfamily of $\mathfrak{A}$, which is closed with respect to finite unions. If $A \subset \mathbb{R}^m$, we set $\mathfrak{A}(A) = \{B \in \mathfrak{A} : B \subset A\}$ and $\mathfrak{A}_o(A) = \{B \in \mathfrak{A} : B^- \subset A\}$.

It follows from [KM, Theorem 26] and [M, Theorem 18] that if $A \in \mathfrak{A}$, on $A^\cdot$ there is a unique finite Borel measure ρ and a ρ–almost everywhere unique vector field ν such that

$$\int_A \operatorname{div} v \, d\lambda_m = \int_{A^\cdot} v \cdot \nu \, d\rho$$

for each vector field v continuously differentiable in a neighborhood of A^-. According to [Fe, Chapter 4], to each $A \in \mathfrak{A}$ we can also associate an $\mathscr{H}$–measurable vector field n_A on $\mathbb{R}^m$ (usually referred to as the *Federer exterior normal*) such that

$$\int_A \operatorname{div} v \, d\lambda_m = \int_{A^\cdot} v \cdot n_A \, d\mathscr{H}$$

for each vector field v continuously differentiable in a neighborhood of A^-. If $\partial^* A = \{x \in A^\cdot : \|n_A\| = 1\}$, then $\mathscr{H}(\partial^* A) = \rho(A^\cdot)$ and $n_A = \nu$ $\mathscr{H}$–almost everywhere on $\partial^* A$. For a proof of this well–known but nontrivial fact, we refer the reader to [De G_1] and [De G_2]. The number $\|A\| = \mathscr{H}(\partial^* A)$ is called the *perimeter* of A. Note that if A is a cube, then $\|A\|$ is the surface area of A. Following [P_3], we set

$$r(A) = \begin{cases} \dfrac{|A|}{d(A)\|A\|} & \text{if } d(A)\|A\| > 0\ , \\ 0 & \text{otherwise.} \end{cases}$$

We close this section by proving a simple lemma, which will be needed later.

1.1. LEMMA. There is a constant α, depending only on m, with the following property: for every set $E \subset \mathbb{R}^m$ with $\mathscr{H}(E) < +\infty$, and for each $\eta > 0$, there is a countable disjoint family $\mathfrak{C}$ of dyadic cubes such that $E \subset (\cup\mathfrak{C})^\circ$, $|\cup\mathfrak{C}| < \eta$, and

$$\sum_{C \in \mathfrak{C}} \|C\| < \alpha \mathscr{H}(E) + \eta\ .$$

PROOF. It follows from [Fa, Theorem 5.1, p. 65] that there is a constant $a > 0$, depending only on m, such that for each $\delta > 0$, we can find a countable family $\mathfrak{K}$ of dyadic cubes such that $E \subset (\cup\mathfrak{K})^\circ$,

$$\sum_{K \in \mathfrak{K}} [d(K)]^{m-1} < a\mathscr{H}(E) + \frac{\eta}{2m}\ ,$$

and $d(K) < \delta$ for each $K \in \mathfrak{K}$. Let $\alpha = 2ma$, and choose $\delta \in (0,1)$ so that $a\mathscr{H}(E)\delta < \eta/2$. Then

$$\sum_{K\in\mathfrak{K}} \|K\| = 2m \sum_{K\in\mathfrak{K}} [d(K)]^{m-1} < \alpha\mathscr{H}(E) + \eta$$

and

$$|\cup\mathfrak{K}| \le \sum_{K\in\mathfrak{K}} |K| \le \delta \sum_{K\in\mathfrak{K}} [d(K)]^{m-1} < \delta\left(a\mathscr{H}(E) + \frac{\eta}{2m}\right) < \eta .$$

Now it suffices to select a disjoint family $\mathfrak{C} \subset \mathfrak{K}$ with $\cup\mathfrak{C} = \cup\mathfrak{K}$. $\square$

2. Lower continuous additive functions. A *division* of an admissible set A is a finite disjoint family $\mathfrak{D} \subset \mathfrak{A}$ with $\cup\mathfrak{D} = A$.

2.1. DEFINITION. Let $A \in \mathfrak{A}$, and let F be a function on $\mathfrak{A}(A)$. We say that F is :

(i) *additive* if $F(A) = \Sigma_{D\in\mathfrak{D}} F(D)$ for each division $\mathfrak{D}$ of A ;

(ii) *lower continuous* if given $\epsilon > 0$, there is a $\delta > 0$ such that $F(B) > -\epsilon$ for each $B \in \mathfrak{A}(A)$ with $|B| < \delta$ and $\|B\| < 1/\epsilon$.

(iii) *continuous* if both F and $-F$ are lower continuous.

2.2. LEMMA. Let $A \in \mathfrak{A}$, let $T \subset A^-$ be thin, and let F be an additive lower continuous function on $\mathfrak{A}(A)$. If $F(A) < 0$, then there is a strictly decreasing sequence $\{C_n\}$ of dyadic cubes such that $C_n^- \subset A^\circ - T$ and $F(C_n) < 0$ for $n = 1,2,\dots$.

PROOF. We may assume that $A^\bullet \subset T$. Select $\eta > 0$, so that $F(B) > F(A)$ for each $B \in \mathfrak{A}(A)$ with $|B| < 3^m\eta$ and $\|B\| < \|A\| + 3^m[\alpha\mathscr{H}(T) + \eta]$; here α is the constant defined in Lemma 1.1. Now by Lemma 1.1, there is a sequence $\{T_n\}$ of dyadic cubes such that $T \subset \cup_n T_n$, $\Sigma_n |T_n| < \eta$, and $\Sigma_n \|T_n\| < \alpha\mathscr{H}(T) + \eta$. For $n = 1,2,\dots$, let $\mathfrak{T}_n$ be the collection of those dyadic cubes C for which $d(C) = d(T_n)$ and $C^- \cap T_n^- \neq \emptyset$. If $T_n^\star = (\cup \mathfrak{T}_n)^\circ$, then $\{T_n^\star\}$ is an open cover of T. As T is compact, $T \subset \cup_{n=1}^N T_n^\star$ for some integer $N \geq 1$, and we let $\mathfrak{T} = \cup_{n=1}^N \mathfrak{T}_n$ and $B = \cup \mathfrak{T}$. Since $\mathfrak{T}_n$ contains 3^m cubes congruent to T_n, we see that

$$|A \cap B| \leq 3^m \sum_n |T_n| < 3^m \eta ,$$

and by [M, Theorem 35],

$$\|A \cap B\| \leq \|A\| + \|B\| \leq \|A\| + 3^m \sum_n \|T_n\| < \|A\| + 3^m [\alpha \mathscr{H}(T) + \eta] .$$

It follows that $F(A \cap B) > F(A)$, and consequently

$$F(A - B) = F(A) - F(A \cap B) < 0 .$$

Now $A - B$ is the union of finitely many disjoint dyadic cubes each of which is congruent to a cube in $\mathfrak{T}$ of the smallest diameter. By the additivity of F, for one of these cubes, say C_1, we have $F(C_1) < 0$. As $C_1 \cap B = \emptyset$ and $A^{\cdot} \subset T \subset B^\circ$, it follows that $C_1^- \subset A^\circ - T$. Since C_1 is the disjoint union of dyadic cubes whose diameters are equal to $d(C_1)/2$, by the additivity of F, for one of these cubes, say C_2, we have $F(C_2) < 0$. An obvious induction completes the proof. $\square$

2.3. Lemma. Let $A \in \mathfrak{A}$, let f be a measurable function on A with $\int_A |f| \, d\lambda_m < +\infty$, and let v be a continuous vector field on A^-. For each $B \in \mathfrak{A}(A)$ set $F(B) = \int_B f \, d\lambda_m$ and $G(B) = \int_{B^{\cdot}} v \cdot n_B \, d\mathscr{H}$. Then F and G are additive continuous functions on $\mathfrak{A}(A)$.

PROOF. The additivity of F and G is easy to show, and the continuity of F follows from the absolute continuity of the indefinite Lebesgue integral. To prove the continuity of G , we proceed as in [KM, Section 12]. Choose an $\epsilon > 0$, and using the Stone–Weierstrass theorem, find a vector field w with polynomial coordinates and such that $\|v(x) - w(x)\| < \epsilon^2/2$ for each $x \in A^-$. Let

$$M = \sup \{ |\operatorname{div} w(x)| : x \in A^- \} ,$$

and find a $\delta > 0$ with $\delta M < \epsilon/2$. Now if $B \in \mathfrak{A}(A)$, $\|B\| < 1/\epsilon$, and $|B| < \delta$, then

$$\left| G(B) \right| = \left| \int_{B^\cdot} (v - w) \cdot n_B \, d\mathscr{H} \right| + \left| \int_{B^\cdot} w \cdot n_B \, d\mathscr{H} \right|$$

$$\le \frac{\epsilon^2}{2} \int_{B^\cdot} \|n_B\| \, d\mathscr{H} + \int_B |\operatorname{div} w| \, d\lambda_m \le \frac{\epsilon^2 \|B\|}{2} + M|B| < \epsilon . \quad \square$$

3. **The variational integral.** Let $A \in \mathfrak{A}$, and let f and F be functions defined on A and $\mathfrak{A}(A)$, respectively. Given $\epsilon > 0$ and $E \subset A$, an *ϵ–majorant* of the pair (f,F) in E is a nonnegative additive function M on $\mathfrak{A}(A)$ which satisfies the following conditions:

(i) $M(A) < \epsilon$;

(ii) for every $x \in E$ there is a $\delta > 0$ such that

$$\left| f(x) |B| - F(B) \right| \le M(B)$$

for each $B \in \mathfrak{A}_o(E)$ with $x \in B^-$, $d(B) < \delta$, and $r(B) > \epsilon$.

3.1. DEFINITION. Let $A \in \mathfrak{A}$ and $f: A \to \mathbb{R}$. We say that f is *variationally integrable*, or simply *v–integrable*, in A if there is an additive continuous function F on $\mathfrak{A}(A)$ which satisfies the following condition: for each $\epsilon > 0$ there is a thin set $T \subset A^-$ such that the pair (f,F) has an ϵ–majorant in $A - T$.

The family of all v–integrable functions on a set $A \in \mathfrak{A}$ is denoted by $\mathscr{V}(A)$. If $f \in \mathscr{V}(A)$, then each function F on $\mathfrak{A}(A)$ which satisfies the conditions of Definition 3.1 is called an *indefinite v–integral* of f in A. Our first aim is to show that every $f \in \mathscr{V}(A)$ has precisely *one* indefinite v–integral.

3.2. LEMMA. Let $A \in \mathfrak{A}$, and let F_i be an indefinite v–integral in A of $f_i \in \mathscr{V}(A)$, $i = 1,2$. If $f_1 \le f_2$, then $F_1 \le F_2$.

PROOF. Since for each $B \in \mathfrak{A}(A)$, the restriction $F_i \restriction \mathfrak{A}(B)$ is an indefinite v–integral in B of $f_i \restriction B$, $i = 1,2$, it suffices to show that $F_1(A) \le F_2(A)$. Working towards contradiction, suppose that $F_2(A) < F_1(A)$, and choose an $\epsilon > 0$ so that $\epsilon < 1/2m$ and $F_2(A) + 2\epsilon < F_1(A)$. For $i = 1,2$, there is a thin set $T_i \subset A^-$, and an ϵ–majorant M_i of (f_i,F_i) in $A - T_i$. The function

$$F = F_2 - F_1 + M_1 + M_2$$

is additive and lower continuous on $\mathfrak{A}(A)$. As $T = T_1 \cup T_2$ is a thin set and

$$F(A) < F_2(A) - F_1(A) + 2\epsilon < 0 ,$$

it follows from Lemma 2.2 that there is a strictly decreasing sequence $\{C_n\}$ of dyadic cubes such that $C_n^- \subset A - T$ and $F(C_n) < 0$ for $n = 1,2,\dots$. If $\bigcap_{n=1}^{\infty} C_n^- = \{x\}$, then $x \in A - T$ and there is $\delta_i > 0$ such that

$$\left| f_i(x)\,|B| - F_i(B) \right| \le M_i(B)$$

for each $B \in \mathfrak{A}_o(A - T)$ with $x \in B^-$, $d(B) < \delta_i$, $i = 1,2$, and $r(B) > \epsilon$. Find an integer $N \ge 1$ so that $d(C_N) < \min(\delta_1,\delta_2)$. As $r(C_N) = 1/2m > \epsilon$, we have

$$F_1(C_N) - M_1(C_N) \le f_1(x)\,|C_N| \le f_2(x)\,|C_N| \le F_2(C_N) + M_2(C_N) ,$$

and hence $F(C_N) \ge 0$; a contradiction. $\square$

3.3. Corollary. If $A \in \mathfrak{A}$ and $f \in \mathscr{V}(A)$, then all indefinite v–integrals of f in A are equal. $\square$

Let $A \in \mathfrak{A}$ and $f \in \mathscr{V}(A)$. In view of Corollary 3.3, we can talk about *the* indefinite v–integral of f in A , denoted by $I_v(A;f,\cdot)$. The number $I_v(A;f,A)$ is called the *v–integral* of f over A . Since $I_v(B;f,\cdot) = I_v(A;f,\cdot)\restriction\mathfrak{A}(B)$ for each $B \in \mathfrak{A}(A)$, no confusion will arise if instead of $I_v(A;f,\cdot)$ and $I_v(A;f,A)$, we write simply $I_v(f,\cdot)$ and $I_v(f,A)$, respectively.

3.4. Proposition. If $A \in \mathfrak{A}$, then $\mathscr{V}(A)$ is a linear space, and the map $f \mapsto I_v(f,A)$ is a nonnegative linear functional on $\mathscr{V}(A)$.

Proof. If $f \in \mathscr{V}(A)$ and $f \ge 0$, then it follows from Lemma 3.2 that $I_v(f,A) \ge 0$. The rest of the proposition follows easily from Definition 3.1. $\square$

3.5. PROPOSITION. Let $A \in \mathfrak{A}$, $f\colon A \to \mathbb{R}$, and let $\mathfrak{D}$ be a division of A. Then $f \in \mathscr{V}(A)$ if and only if $f{\upharpoonright}D \in \mathscr{V}(D)$ for each $D \in \mathfrak{D}$.

PROOF. If $f \in \mathscr{V}(A)$, $F = I_v(f,\cdot)$, and $D \in \mathfrak{D}$, then it is clear that $F{\upharpoonright}\mathfrak{A}(D)$ is the indefinite v–integral of $f{\upharpoonright}D$ in D, and so $f{\upharpoonright}D \in \mathscr{V}(D)$.

Conversely, suppose that $f{\upharpoonright}D \in \mathscr{V}(D)$ for each $D \in \mathfrak{D}$, and let $F_D = I_v(f{\upharpoonright}D,\cdot)$. For every $B \in \mathfrak{A}(A)$, we set

$$F(B) = \sum_{D \in \mathfrak{D}} F_D(B \cap D) ,$$

and we show that F is the indefinite v–integral of f in A. It is easy to see that F is an additive continuous function on $\mathfrak{A}(A)$. Given $\epsilon > 0$, let $\epsilon_D = \epsilon|D|/(1 + |A|)$ for each $D \in \mathfrak{D}$, and find a thin set $T_D \subset D^-$ for which the pair $(f{\upharpoonright}D,F_D)$ has an ϵ_D–majorant M_D in $D - T_D$. For every $B \in \mathfrak{A}(A)$, set

$$M(B) = \sum_{D \in \mathfrak{D}} M_D(B \cap D)$$

and let $T = \bigcup_{D \in \mathfrak{D}}(D^{\cdot} \cup T_D)$. Fix an $x \in A - T$. Then $x \in D^{\circ}$ for some $D \in \mathfrak{D}$, and we can find $\eta > 0$ so that $E^- \subset D$ whenever $x \in E^-$ and $d(E) < \eta$. Moreover, there is a $\delta > 0$ such that

$$\Big| f(x)\,|B| - F_D(B) \Big| \le M_D(B)$$

for each $B \in \mathfrak{A}_o(D - T_D)$ with $x \in B^-$, $d(B) < \delta$, and $r(B) > \epsilon_D$. Thus if $B \in \mathfrak{A}_o(A - T)$, $x \in B^-$, $r(B) > \epsilon \ge \epsilon_D$, and $d(B) < \min(\eta,\delta)$, then $B \in \mathfrak{A}_o(D - T_D)$

and we have

$$\left|f(x)|B| - F(B)\right| = \left|f(x)|B| - F_D(B)\right| \le M_D(B) = M(B) .$$

Since T is a thin set, and

$$M(A) = \sum_{D\in\mathfrak{D}} M_D(D) < \sum_{D\in\mathfrak{D}} \epsilon_D = \frac{\epsilon|A|}{1+|A|} < \epsilon ,$$

the proposition is proved. $\square$

If $E \subset \mathbb{R}^m$ is a measurable set, we denote by $\mathscr{L}(E)$ the family of all measurable functions f on E for which the *finite* $\int_E f\, d\lambda_m$ exists.

3.6. PROPOSITION. If $A \in \mathfrak{A}$, then $\mathscr{L}(A) \subset \mathscr{V}(A)$ and $I_v(f,A) = \int_A f\, d\lambda_m$ for each $f \in \mathscr{L}(A)$.

PROOF. Let $f \in \mathscr{L}(A)$, and set $F(B) = \int_B f\, d\lambda_m$ for each $B \in \mathfrak{A}(A)$. According to Lemma 2.3, F is an additive continuous function on $\mathfrak{A}(A)$, and we show that F is the indefinite v–integral of f in A. Given $\epsilon > 0$, there are *extended* real–valued functions g and h on A which are, respectively, upper and lower semicontinuous, and such that $g \le f \le h$ and $\int_A (h-g)\, d\lambda_m < \epsilon/2$ (see [Ru, Theorem 2.25, p.56]). For every $B \in \mathfrak{A}(A)$, set

$$M(B) = \frac{\epsilon|B|}{2(1+|A|)} + \int_B (h-g)\, d\lambda_m ,$$

and fix an $x \in A$. There is a $\delta > 0$ such that

$$g(y) < f(x) + \frac{\epsilon}{2(1+|A|)} \quad \text{and} \quad h(y) > f(x) - \frac{\epsilon}{2(1+|A|)}$$

for each $y \in A$ with $|x-y| < \delta$. Thus if $B \in \mathfrak{A}_o(A)$, $x \in B^-$, and $d(B) < \delta$, then

$$\int_B g \, d\lambda_m - \frac{\epsilon |B|}{2(1+|A|)} \leq f(x)|B| \leq \int_B h \, d\lambda_m + \frac{\epsilon |B|}{2(1+|A|)} .$$

Since also

$$\int_B g \, d\lambda_m \leq F(B) \leq \int_B h \, d\lambda_m ,$$

we have

$$\left| f(x)|B| - F(B) \right| \leq M(B) .$$

As $M(A) < \epsilon$, we conclude that M is an ϵ–majorant of the pair (f,F) in A. $\square$

A set $C = \Pi_{i=1}^m [a_i, a_i+h]$, where $a_1,\ldots,a_m$, and h are positive real numbers, is called a *closed cube*. Let $A \in \mathfrak{A}$, $x \in A^o$, and let F be a function on $\mathfrak{A}(A)$. We say that F is *derivable* at x if a finite $\lim[F(C_n)/|C_n|]$ exists for each sequence $\{C_n\}$ of closed subcubes of A such that $x \in C_n$ for $n = 1,2,\ldots$, and $\lim d(C_n) = 0$. If all these limits exist, they have the same value, denoted by $F'(x)$.

3.7. LEMMA. Let $A \in \mathfrak{A}$, $f \in \mathscr{V}(A)$, and let $F = I_V(f,\cdot)$. Then for almost all $x \in A^o$ the function F is derivable at x and $F'(x) = f(x)$.

PROOF. Let E be the set of all $x \in A^{\circ}$ for which either F is not derivable at x, or $F'(x) \neq f(x)$. Then given $x \in E$, we can find a $\beta(x) > 0$ so that for each $\delta > 0$ there is a closed cube $C \subset A$ with $x \in C$, $d(C) < \delta$, and

$$\left| \frac{F(C)}{|C|} - f(x) \right| \geq \beta(x)$$

Fix integers $n \geq 1$ and $k > 2m$, and let $E_n = \{x \in E : \beta(x) \geq 1/n\}$. There is a thin set $T_k \subset A^-$ such that in $A - T_k$ the pair (f,F) has a $(1/nk)$–majorant M. Thus for each $x \in A - T_k$ there is a $\delta(x) > 0$ so that

$$\left| f(x)|C| - F(C) \right| \leq M(C)$$

for each closed cube $C \subset A - T_k$ with $x \in C$ and $d(C) < \delta(x)$; for $r(C) = 1/2m > 1/nk$. Let $\mathfrak{C}$ be the family of all closed cubes $C \subset A - T_k$ such that $d(C) < \delta(x_c)$ for some $x_c \in C$, and

$$\left| F(C) - f(x_c)|C| \right| \geq \frac{|C|}{n}.$$

It is easy to see that $\mathfrak{C}$ covers $E_n - T_k$ in the sense of Vitali. By [Sa, Chapter IV, Theorem 3.1, p.112], there are disjoint cubes $C_1, C_2, \ldots$ in $\mathfrak{C}$ such that $|(E_n - T_k) - \cup_{i=1}^{\infty} C_i| = 0$. We have

$$\sum_{i=1}^{p} |C_i| \leq n \sum_{i=1}^{p} \left| F(C_i) - f(x_{C_i})|C_i| \right| \leq n \sum_{i=1}^{p} M(C_i)$$

$$= nM\left|\bigcup_{i=1}^{p} C_i\right| \le nM(A) < 1/k$$

for each $p = 1,2,\dots$, and hence

$$|E_n - T_k| \le \left|\bigcup_{i=1}^{\infty} C_i\right| = \sum_{i=1}^{\infty} |C_i| \le 1/k .$$

Since $|T_k| = 0$, we obtain that $|E_n| \le 1/k$ for all $k > 2m$, and consequently $|E_n| = 0$. As $E = \cup_{n=1}^{\infty} E_n$, also $|E| = 0$. □

3.8. COROLLARY. If $A \in \mathfrak{A}$, then each $f \in \mathscr{V}(A)$ is measurable.

PROOF. Since $|A^{\bullet}| = 0$, the corollary follows from Lemma 3.7 by standard arguments (see, e.g., [Sa, Chapter IV, Theorem (4.2), p.112]). □

Next we establish a fairly general result concerning a broad class of integrals.

If $(X,\mathfrak{M},\mu)$ is a measure space, we denote by $\mathscr{L}_1(\mu)$ the family of all $\mathfrak{M}$–measurable functions f on X with $\int_X |f|\, d\mu < +\infty$.

3.9. PROPOSITION. Let $(X,\mathfrak{M},\mu)$ be a σ–finite measure space, and let $\mathscr{F}$ be a linear space of $\mathfrak{M}$–measurable functions on X which contains $\mathscr{L}_1(\mu)$. Further, let L be a nonnegative linear functional on $\mathscr{F}$ such that $L(f) = \int_X f\, d\mu$ for each $f \in \mathscr{L}_1(\mu)$. Then a function f on X belongs to $\mathscr{L}_1(\mu)$ whenever both f and $|f|$ belong to $\mathscr{F}$. Moreover, if $f_n \in \mathscr{F}$, $n = 1,2,\dots$, and $\lim f_n = f$, then $f \in \mathscr{F}$ and $L(f) = \lim L(f_n)$ whenever either of the following conditions holds:

(i) $f_n \le f_{n+1}$, $n = 1,2,\ldots$, and $\lim L(f_n) < +\infty$;

(ii) $g \le f_n \le h$ for some g , h in $\mathscr{F}$ and $n = 1,2,\ldots$.

PROOF. Since μ is σ–finite , $\mathscr{L}_1(\mu)$ contains a strictly positive function w (see [Ru, Lemma 6.9, p.121]). If f and $|f|$ belong to $\mathscr{F}$, then

$$g_n = \min(|f|,nw)$$

belongs to $\mathscr{L}_1(\mu)$, and hence to $\mathscr{F}$, for $n = 1,2,\ldots$. As $w(x) > 0$ for each $x \in X$, we have

$$\int_X |f|\, d\mu = \lim \int_X g_n\, d\mu = \lim L(g_n) \le L(|f|) < +\infty .$$

Since f is $\mathfrak{M}$–measurable, it belongs to $\mathscr{L}_1(\mu)$.

Now the rest of the proposition follows from the monotone and dominated convergence theorems applied to the sequences $\{f_n - f_1\}$ and $\{f_n - g\}$, respectively. $\square$

In view of Proposition 3.6 and Corollary 3.8, Proposition 3.9 applies to the variational integral $I_v(\cdot,A)$ on $\mathscr{V}(A)$, where $A \in \mathfrak{A}$. We also have the following corollary.

3.10. COROLLARY. Let f be a function $A \in \mathfrak{A}$. Then $f = 0$ almost everywhere if and only if $f \in \mathscr{V}(A)$ and $I_v(f,B) = 0$ for each $B \in \mathfrak{A}(A)$.

PROOF. If $f \in \mathscr{V}(A)$ and $I_v(f,\cdot) = 0$, then it follows directly from Definition 3.1

that $|f| \in \mathscr{V}(A)$ and $I_v(|f|,\cdot) = 0$. Now it suffices to apply Proposition 3.9. $\square$

NOTE. In view of previous corollary, if a function f is defined almost everywhere on a set A in $\mathbb{R}^m$, then the function can be extended to A and the v–integral of the extended function depends only on f and not on the way f is extended.

The next lemma is proved in [P_3 , Lemma 5.5]. We quote it here for completeness.

3.11. LEMMA. Let v be a continuous vector field in an open set $U \subset \mathbb{R}^m$ which is differentiable at $x \in U$. Then given $\epsilon > 0$, there is a $\delta > 0$ such that

$$\left| \operatorname{div} v(x) |B| - \int_{B^{\cdot}} v \cdot n_B \, d\mathscr{H} \right| < \epsilon |B|$$

for each $B \in \mathfrak{A}_o(U)$ with $x \in B^-$, $d(B) < \delta$, and $r(B) > \epsilon$. $\square$

If $v = (f_1,\dots,f_m)$ is a vector field defined in $E \subset \mathbb{R}^m$, we let

$$\operatorname{div} v(x) = \sum_{i=1}^{m} \frac{\partial f_i(x)}{\partial \xi_i}$$

for each $x \in E^o$ at which v is differentiable.

NOTE. We use the usual definition of a differentiable map (see, e.g. , [Ru, Definition 7.22, p.150]). In particular, differentiable does *not* mean continuously differentiable.

3.12. THEOREM. Let $A \in \mathfrak{A}$, and let $T \subset A^-$ be a thin set. Let v be a continuous vector field in A^- which is differentiable in $A^\circ - T$. Then $\operatorname{div} v(x)$ is defined for almost all $x \in A$, is v–integrable in A and

$$I_v(\operatorname{div} v\,,\, A) = \int_{A^\cdot} v\cdot n_A \, d\mathscr{H}\,.$$

PROOF. For each $B \in \mathfrak{A}(A)$, let $F(B) = \int_{B^\cdot} v\cdot n_B \, d\mathscr{H}$. By Lemma 2.3, F is an additive continuous function on $\mathfrak{A}(A)$, and we show that it is an indefinite v–integral of $f = \operatorname{div} v$ in A. To this end, choose an $\epsilon > 0$, and let $M(B) = \epsilon|B|/(1+|A|)$ for each $B \in \mathfrak{A}(A)$. Clearly, M is nonnegative and additive, and $M(A) < \epsilon$. If $x \in A^\circ - T$, then by Lemma 3.11, there is a $\delta > 0$ such that

$$\left| f(x)\,|B| - F(B) \right| < \frac{\epsilon}{1+|A|}|B| = M(B)$$

for each $B \in \mathfrak{A}_o(A^\circ - T)$ with $x \in B^-$, $d(B) < \delta$, and

$$r(B) > \epsilon > \frac{\epsilon}{1+|A|}\,.$$

Thus M is an ϵ–majorant of the (f,F) in $A^\circ - T$, and since $A^\cdot \cup T$ is thin subset of A^-, the theorem is proved. □

Let $E \subset \mathbb{R}^m$ and $\Phi\colon E \to \mathbb{R}^m$. We say that Φ is a *regular* map of E if it can be extended to a C^1–diffeomorphism (also denoted by Φ) of an open neighborhood of E^-. For a regular map Φ, we denote by $\det \Phi$ the determinant of its Jacobi matrix. Note that if $\Phi\colon E \to \mathbb{R}^m$ is regular, then Φ is defined uniquely on E^-, and $\det \Phi$ is defined uniquely on E° whenever E° is nonempty. Since $|E^\cdot| = 0$, we see that $\det \Phi$ is defined uniquely almost everywhere in E^-. If Φ is regular, then Φ and $\det \Phi$ are both extended continuously to a neighborhood of E^-.

3.13. LEMMA. Let Φ be a regular map of $E \subset \mathbb{R}^m$. If E is thin or $E \in \mathfrak{A}$, then so is $\Phi[E]$, respectively.

PROOF. This follows immediately from [Ro, Theorem 29, p.53] and the equality $(\Phi[E])^\bullet = \Phi[E^\bullet]$. $\square$

3.14. THEOREM. Let $A \in \mathfrak{A}$, and let $\Phi: A \to \mathbb{R}^m$ be a regular map. If $f \in \mathscr{V}(\Phi[A])$, then $f \circ \Phi \cdot |\det \Phi|$ belongs to $\mathscr{V}(A)$ and

$$I_V(f \circ \Phi \cdot |\det \Phi|, A) = I_V(f, \Phi[A]) .$$

PROOF. There are positive real numbers a,b,b', and c such that the following inequalities hold:

(i) $c \geq 1$ and $b'/ac < 1/2$;

(ii) $|\Phi(x) - \Phi(y)| \leq a|x - y|$ for each $x, y \in A^-$;

(iii) $b'|B| \leq |\Phi[B]| \leq b|B|$ for each measurable set $B \subset A^-$;

(iv) $\|\Phi[B]\| \leq c\|B\|$ for each $B \in \mathfrak{A}(A)$.

Inequality (ii) is a direct consequence of the regularity of Φ. Inequalities (iii) and (iv) follow from [Ru, Theorem 7.26, p.153] and [M, Theorem 50], respectively. Finally, by enlarging c, we obtain (i).

For each $B \in \mathfrak{A}(A)$, let $F(B) = I_V(f, \Phi[B])$. Clearly, F is an additive function on $\mathfrak{A}(A)$, and we show that it is also continuous. To this end, choose an $\epsilon > 0$, and using the continuity of $I_V(f, \cdot)$, find a $\delta > 0$ such that $|I_V(f,C)| < \epsilon/c$ for each $C \in \mathfrak{A}(\Phi[A])$

with $|C| < b\delta$ and $\|C\| < c/\epsilon$. Now if $B \in \mathfrak{A}(A)$, $|B| < \delta$, and $\|B\| < 1/\epsilon$, then by (iii) and (iv), respectively, $|\Phi[B]| \le b|B| < b\delta$ and $\|\Phi[B]\| \le c\|B\| < c/\epsilon$. Consequently,

$$|F(B)| = |I_V(f,\Phi[B])| < \epsilon/c \le \epsilon ,$$

and the continuity of F is established.

We prove the theorem by showing that F is the indefinite v–integral of $f\circ\Phi \cdot |\det \Phi|$ in A. Select an $\epsilon > 0$, and let $\epsilon' = \epsilon b'/ac$. There is a thin set $T \subset \Phi[A^-]$ such that the pair $(f, I_V(f,\cdot))$ has an ϵ'–majorant, say M, in $\Phi[A] - T$. Set $S = A^{\cdot} \cup \Phi^{-1}[T]$, and for each $B \in \mathfrak{A}(A)$, let

$$N(B) = M(\Phi[B]) + \frac{\epsilon|B|}{2(|A|+1)} .$$

Then S is a thin subset of A^-, and N is a nonnegative additive function on $\mathfrak{A}(A)$ with $N(A) < \epsilon$ (see (i)). Choose an $x \in A - S$, let $y = \Phi(x)$, and find a $\delta > 0$ so that

$$\Big| f(y)|C| - I_V(f,C) \Big| \le M(C)$$

for each $C \in \mathfrak{A}_o(\Phi[A] - T)$ for which $y \in C^-$, $d(C) < a\delta$, and $r(C) > \epsilon'$. By making δ smaller, if necessary, we may also assume that

$$\left| \frac{|\Phi[B]|}{|B|} - |\det \Phi(x)| \right| < \frac{\epsilon}{2[|f(y)|+1](|A|+1)}$$

for each $B \in \mathfrak{A}_o(A^o)$ with $x \in B^-$ and $d(B) < \delta$; for $|\det \Phi|$ is a continuous function on A°, and $|\Phi[B]| = \int_B |\det \Phi| \, d\lambda_m$ (see [Ru, Theorem 7.26, p.153]).

Fix a $B \in \mathfrak{A}_o(A-S)$ with $x \in B^-$, $d(B) < \delta$, and $r(B) > \epsilon$, and let $C = \Phi[B]$. Then $y \in C^-$, and by (i)–(iv), we have $d(C) < a\delta$ and

$$r(C) \geq \frac{b'}{ac} r(B) > \epsilon' .$$

Thus

$$\Big| f \circ \Phi(x) \cdot |\det \Phi(x)| \cdot |B| - F(B) \Big| \leq |f(y)| \Big| |\det \Phi(x)| \cdot |B| - |\Phi(B)| \Big|$$

$$+ \Big| f(y)|C| - I_v(f,C) \Big| < \frac{\epsilon |B|}{2(|A|+1)} + M(C) = N(B) ,$$

and we see that N is an ϵ–majorant of the pair $(f \circ \Phi \cdot |\det \Phi|, F)$ in $A-S$. $\square$

Let $A \in \mathfrak{A}$, $f \in \mathcal{V}(A)$, and $\epsilon > 0$. We say that an $x \in A$ is an *ϵ–point* of f if there is a disjoint sequence $\{B_n\}$ in $\mathfrak{A}(A)$ such that $r(B_n \cup \{x\}) > \epsilon$ for $n = 1,2,\ldots$, $\lim d(B_n \cup \{x\}) = 0$, and

$$\sum_{n=1}^{\infty} |I_v(f,B_n)| = +\infty .$$

The set of all ϵ–points of f is denoted by $V_\epsilon(f,A)$, or simply by V_ϵ.

The next proposition gives a useful necessary condition for v–integrability in terms of ϵ–points.

3.15. PROPOSITION. Let $A \in \mathfrak{A}$ and $f \in \mathscr{V}(A)$. Then the set V_ϵ^- is thin for each $\epsilon > 0$.

PROOF. Proceeding towards contradiction, suppose that V_ϵ^- is not thin for some $\epsilon > 0$. Let $F = I_V(f,\cdot)$, and find a thin set $T \subset A^-$ and an ϵ–majorant M of the pair (f,F) in $A - T$. Since V_ϵ^- is not thin, there is an $x \in V_\epsilon - (A^\bullet \cup T)$. Let $\{B_n\}$ be a sequence in $\mathfrak{A}(A)$ associated with the ϵ–point x. The open set $A - (A^\bullet \cup T)$ contains disjoint countable sets C_n with $x \in C_n^-$, $n = 1,2,\dots$, and we let $D_n = (B_n \cup C_n) - T$. As $|T \cup C_n| = 0$, it follows from Proposition 3.5 and Corollary 3.10 that $F(D_n) = F(B_n)$. Moreover, by making the C_n's sufficiently small, we may assume that $d(D_n) = d(B_n \cup \{x\})$, and hence also $r(D_n) = r(B_n \cup \{x\})$. Consequently, there is an integer $N \geq 1$ such that

$$\left| f(x)\,|D_n| - F(D_n) \right| \leq M(D_n)$$

for each $n \geq N$. From this we obtain that

$$\sum_{n=N}^{p} |F(B_n)| = \sum_{n=N}^{p} |F(D_n)| \leq f(x) \sum_{n=N}^{p} |D_n| + \sum_{n=N}^{p} M(D_n)$$

$$= f(x) \left| \bigcup_{n=N}^{p} D_n \right| + M\left(\bigcup_{n=N}^{p} D_n \right) \leq f(x)\,|A| + M(A) < f(x)\,|A| + \epsilon$$

for all $p \geq N$, which is a contradiction, for $\Sigma_{n=1}^{\infty} |F(B_n)| = +\infty$.

3.16. REMARK. Let $A \in \mathfrak{A}$ and $f \colon A \to \mathbb{R}$. We say that f is *weakly integrable*, or simply *w–integrable*, in A if there is an additive continuous function F on $\mathfrak{A}(A)$ and a thin set $T \subset A^-$ such that the pair (f,F) has an ϵ–majorant in $A-T$ for each $\epsilon > 0$. By repeating verbatim the arguments of this section, it is easy to show that the w–integral has properties identical to those of the v–integral. It follows directly from the definitions that the v–integral is an extension of the w–integral. Whether the v– or w–integral (or their extensions which will be described in Sections 4 and 5) actually coincide appears unknown.

3.17. REMARK. Let f be a w–integrable function in an admissible set A. Then the proof of Proposition 3.15 reveals that $(\cup_{\epsilon>0} V_\epsilon)^-$ is a thin set. Whether the same can be proved for each v–integrable function in A is unclear, and it is likely related to the more general problem stated in Remark 3.16.

4. **The integral.** We say that a sequence $\{A_n\}$ in $\mathfrak{A}$ *converges* to a set $A \in \mathfrak{A}$, in writing $\{A_n\} \to A$, if $A_n \subset A$ for $n = 1,2,\dots$, $\sup_n \|A_n\| < +\infty$, and $\lim |A - A_n| = 0$. A family $\mathfrak{E} \subset \mathfrak{A}$ is called *closed* if $E \in \mathfrak{E}$ for each $E \in \mathfrak{A}$ for which there is a sequence $\{E_n\}$ in $\mathfrak{E}$ with $\{E_n\} \to E$. The *closure* of a family $\mathfrak{E} \subset \mathfrak{A}$, denoted by $\mathrm{cl}\,\mathfrak{E}$, is the intersection of all closed subfamilies of $\mathfrak{A}$ containing $\mathfrak{E}$. It is easy to verify that for each $\mathfrak{E} \subset \mathfrak{A}$, the closure of $\mathfrak{E}$ is a closed subfamily of of $\mathfrak{A}$.

4.1. REMARK. If $\mathfrak{E} \subset \mathfrak{A}$ is closed and $E \in \mathfrak{E}$, then also $E^- \in \mathfrak{E}$; indeed, as $\lambda_m(E^\cdot) = 0$, the constant sequence $\{E\}$ converges to each set B with $E \subset B \subset E^-$. In particular, the family $\mathfrak{A}(A)$ with $A \in \mathfrak{A}$ is closed if and only if A is a closed set.

If $\mathfrak{E} \subset \mathfrak{A}$, we denote by $\mathrm{cl}_1(\mathfrak{E})$ the collection of all $E \in \mathfrak{A}$ for which there is a sequence $\{E_n\}$ in $\mathfrak{E}$ with $\{E_n\} \to E$. As the constant sequence $\{A\}$ in $\mathfrak{A}$ converges to A, we see that $\mathfrak{E} \subset \mathrm{cl}_1(\mathfrak{E})$ for each $\mathfrak{E} \subset \mathfrak{A}$. However, the following example shows that $\mathrm{cl}_1(\mathfrak{E})$ need not be closed.

4.2. EXAMPLE. For $n = 1,2,\dots$, let $E_n = (2^{-n}, 2^{-n+1})$, and let $E = (0,1]$. If $\mathfrak{E} = \mathfrak{A}_o(\cup_{n=1}^{\infty} E_n)$, then $\mathrm{cl}_1(\mathfrak{E}) = \mathfrak{A}_o(E)$ is not closed; for $E^- = [0,1]$ belongs to $\mathrm{cl}[\mathfrak{A}_o(E)]$.

We show that the closure of a family $\mathfrak{E} \subset \mathfrak{A}$ can be described by a transfinite construction.

Let $\mathfrak{E} \subset \mathfrak{A}$, and let $\mathrm{cl}_0(\mathfrak{E}) = \mathfrak{E}$. Assuming that $\mathrm{cl}_\alpha(\mathfrak{E})$ has been defined for each ordinal $\alpha < \beta \le \omega_1$, we define $\mathrm{cl}_\beta(\mathfrak{E})$ as follows:

(i) if β is a limit ordinal, let $\mathrm{cl}_\beta(\mathfrak{E}) = \cup_{\alpha<\beta} \mathrm{cl}_\alpha(\mathfrak{E})$;

(ii) if $\beta = \alpha + 1$, let $\mathrm{cl}_\beta(\mathfrak{E}) = \mathrm{cl}_1[\mathrm{cl}_\alpha(\mathfrak{E})]$.

4.3. PROPOSITION. For each $\mathfrak{E} \subset \mathfrak{A}$, we have $\mathrm{cl}\,\mathfrak{E} = \mathrm{cl}_{\omega_1}(\mathfrak{E})$.

PROOF. Since $\mathfrak{E} \subset \mathrm{cl}_{\omega_1}(\mathfrak{E}) \subset \mathrm{cl}\,\mathfrak{E}$, it suffices to show that $\mathrm{cl}_{\omega_1}(\mathfrak{E})$ is closed. Let $\{E_n\}$ be a sequence in $\mathrm{cl}_{\omega_1}(\mathfrak{E})$ which converges to an $E \in \mathfrak{A}$. For $n = 1,2,\dots$, there is an $\alpha_n < \omega_1$ with $E_n \in \mathrm{cl}_{\alpha_n}(\mathfrak{E})$. If $\alpha = \sup_n \alpha_n$, then $\{E_n\}$ is a sequence in $\mathrm{cl}_\alpha(\mathfrak{E})$, and so $E \in \mathrm{cl}_{\alpha+1}(\mathfrak{E})$. As $\alpha + 1 < \omega_1$, the proposition follows. $\square$

4.4. LEMMA. Let A and B belong to $\mathfrak{A}$, and let $\{A_n\}$ and $\{B_n\}$ be sequences in $\mathfrak{A}$ which converge to A and B, respectively. Then $\{A_n \cup B_n\} \to A \cup B$ and $\{A_n \cap B_n\} \to A \cap B$.

PROOF. Since $A \cup B - (A_n \cup B_n)$ and $A \cap B - (A_n \cap B_n)$ are both contained in $(A - A_n) \cup (B - B_n)$, the lemma follows from [M, Theorem 35]. $\square$

4.5. PROPOSITION. If $\mathfrak{I}$ is an ideal in $\mathfrak{A}$, then so is $\mathrm{cl}\,\mathfrak{I}$.

PROOF. By Lemma 4.4, $cl_1(\mathfrak{I})$ is closed with respect to unions. Let $A \in \mathfrak{A}$ and $B \in cl_1(\mathfrak{I})$. Then there is a sequence $\{B_n\}$ in $\mathfrak{I}$ with $\{B_n\} \to B$. By Lemma 4.4, $\{A \cap B_n\} \to A \cap B$; for the constant sequence $\{A\}$ converges to A. As $\{A \cap B_n\}$ is a sequence in $\mathfrak{I}$, we have $A \cap B \in cl_1(\mathfrak{I})$. Thus $cl_1(\mathfrak{I})$ is an ideal in $\mathfrak{A}$. Since the union of any increasing collection of ideals in $\mathfrak{A}$ is also an ideal in $\mathfrak{A}$, the proposition follows from Proposition 4.3 by transfinite induction. $\square$

Let $A \in \mathfrak{A}$ and $f\colon A \to \mathbb{R}$. We denote by $\mathfrak{V}(A,f)$ the family of all $B \in \mathfrak{A}(A)$ on which f is v–integrable. By Proposition 3.5, the family $\mathfrak{V}(A,f)$ is an ideal in $\mathfrak{A}$.

4.6. DEFINITION. Let $A \in \mathfrak{A}$ and $f\colon A \to \mathbb{R}$. We say that f is *integrable* in A if $A \in cl[\mathfrak{V}(A,f)]$ and there is a continuous additive function F on $\mathfrak{A}(A)$ such that $F(B) = I_v(f,B)$ for each $B \in \mathfrak{V}(A,f)$.

The family of all integrable functions on a set $A \in \mathfrak{A}$ is denoted by $\mathcal{I}(A)$. The following fact is an immediate consequence of Propositions 3.5 and 4.5.

4.7. PROPOSITION. If $A \in \mathfrak{A}$ and $f \in \mathcal{I}(A)$, then $f \restriction B \in \mathcal{I}(B)$ for each $B \in \mathfrak{A}(A)$. $\square$

4.8. REMARK. If $A \in \mathfrak{A}$, then each additive continuous function F on $\mathfrak{A}(A)$ has a unique additive continuous extension to $\mathfrak{A}(A^-)$. Indeed, as $\lambda_m(A^\cdot) = 0$, the extension $\hat{F}$ of F is obtained by setting $\hat{F}(B) = 0$ for each $B \in \mathfrak{A}(A^- - A)$. From this and Remark 4.1, it follows that any extension to A^- of an integrable function f in A is integrable in A^-.

If $A \in \mathfrak{A}$ and $f \in \mathcal{I}(A)$, then each function F on $\mathfrak{A}(A)$ which satisfies the conditions of Definition 4.6 is called an *indefinite integral* of f in A. As with the variational integral, our task is to show that every $f \in \mathcal{J}(A)$ has precisely *one* indefinite integral. The proof requires three lemmas.

4.9. LEMMA. Let $A \in \mathfrak{A}$, and let F be an additive function on $\mathfrak{A}(A)$. Then F is continuous if and only if $\lim F(B_n) = F(B)$ for each sequence $\{B_n\}$ in $\mathfrak{A}(A)$ which converges to $B \in \mathfrak{A}(A)$.

PROOF. Let F be continuous, $B \in \mathfrak{A}(A)$, and let $\{B_n\}$ be a sequence in $\mathfrak{A}(A)$ which converges to B. Choose an $\epsilon > 0$ with $1/\epsilon > \|B\| + \sup_n \|B_n\|$, and find a $\delta > 0$ so that $|F(C)| < \epsilon$ for each $C \in \mathfrak{A}(A)$ for which $|C| < \delta$ and $\|C\| < 1/\epsilon$. There is an integer $N \geq 1$ such that $|B - B_n| < \delta$ for each $n \geq N$. By [M, Theorem 35],

$$\|B - B_n\| \leq \|B\| + \|B_n\| < 1/\epsilon$$

for $n = 1,2,\ldots$, and so

$$|F(B) - F(B_n)| = |F(B - B_n)| < \epsilon$$

for each $n \geq N$.

Conversely, if F is not continuous, then there is an $\epsilon > 0$ and a sequence $\{B_n\}$ in $\mathfrak{A}(A)$ such that $|B_n| < 1/n$, $\|B_n\| < 1/\epsilon$, and

$$\epsilon \leq |F(B_n)| = |F(A) - F(A - B_n)|$$

for $n = 1,2,\ldots$. Yet, it follows from [M, Theorem 35] that $\{A - B_n\} \to A$. $\square$

4.10. LEMMA. Let $\mathfrak{I}$ and $\mathfrak{J}$ be two ideals in $\mathfrak{A}$. Then $\mathrm{cl}(\mathfrak{I} \cap \mathfrak{J}) = (\mathrm{cl}\, \mathfrak{I}) \cap (\mathrm{cl}\, \mathfrak{J})$.

PROOF. As $\mathfrak{I} \cap \mathfrak{J} \subset \mathrm{cl}\, \mathfrak{I}$, we have $\mathrm{cl}(\mathfrak{I} \cap \mathfrak{J}) \subset \mathrm{cl}\, \mathfrak{I}$, and by symmetry, also $\mathrm{cl}(\mathfrak{I} \cap \mathfrak{J}) \subset (\mathrm{cl}\, \mathfrak{I}) \cap (\mathrm{cl}\, \mathfrak{J})$. Let $\beta \leq \omega_1$, and assume that for each $\alpha < \beta$, $\mathrm{cl}_\alpha(\mathfrak{I})$ and $\mathrm{cl}_\alpha(\mathfrak{J})$ are ideals in $\mathfrak{A}$ with $\mathrm{cl}_\alpha(\mathfrak{I}) \cap \mathrm{cl}_\alpha(\mathfrak{J}) \subset \mathrm{cl}(\mathfrak{I} \cap \mathfrak{J})$; this is indeed true for $\beta = 1$. If β is a limit ordinal, then $\mathrm{cl}_\beta(\mathfrak{I}) = \cup_{\alpha<\beta} \mathrm{cl}_\alpha(\mathfrak{I})$ and $\mathrm{cl}_\beta(\mathfrak{J}) = \cup_{\alpha<\beta} \mathrm{cl}_\alpha(\mathfrak{J})$. Since $\alpha < \alpha' < \beta$ implies $\mathrm{cl}_\alpha(\mathfrak{I}) \subset \mathrm{cl}_{\alpha'}(\mathfrak{I})$ and $\mathrm{cl}_\alpha(\mathfrak{J}) \subset \mathrm{cl}_{\alpha'}(\mathfrak{J})$, it is easy to verify that $\mathrm{cl}_\beta(\mathfrak{I})$ and $\mathrm{cl}_\beta(\mathfrak{J})$ are ideals in $\mathfrak{A}$ with $\mathrm{cl}_\beta(\mathfrak{I}) \cap \mathrm{cl}_\beta(\mathfrak{J}) \subset \mathrm{cl}(\mathfrak{I} \cap \mathfrak{J})$. Now let $\beta = \alpha + 1$, and let $A \in \mathrm{cl}_\beta(\mathfrak{I}) \cap \mathrm{cl}_\beta(\mathfrak{J})$. Then there are sequences $\{A_n\}$ in $\mathrm{cl}_\alpha(\mathfrak{I})$ and $\{B_n\}$ in $\mathrm{cl}_\alpha(\mathfrak{J})$ with $\{A_n\} \to A$ and $\{B_n\} \to A$. By Lemma 4.4, $\{A_n \cap B_n\} \to A$, and as $\mathrm{cl}_\alpha(\mathfrak{I})$ and $\mathrm{cl}_\alpha(\mathfrak{J})$ are ideals, $\{A_n \cap B_n\}$ is a sequence in $\mathrm{cl}_\alpha(\mathfrak{I}) \cap \mathrm{cl}_\alpha(\mathfrak{J}) \subset \mathrm{cl}(\mathfrak{I} \cap \mathfrak{J})$. Thus $A \in \mathrm{cl}(\mathfrak{I} \cap \mathfrak{J})$ which proves again that $\mathrm{cl}_\beta(\mathfrak{I}) \cap \mathrm{cl}_\beta(\mathfrak{J}) \subset \mathrm{cl}(\mathfrak{I} \cap \mathfrak{J})$. The lemma follows from Proposition 4.3. $\square$

4.11. LEMMA. Let $A \in \mathfrak{A}$, and let F_i be an indefinite integral in A of $f_i \in \mathscr{I}(A)$, $i = 1,2$. If $f_1 \leq f_2$, then $F_1 \leq F_2$.

PROOF. By Lemma 3.2,

$$F_1(B) = I_V(f_1,B) \leq I_V(f_2,B) = F_2(B)$$

for each B in $\mathfrak{I} = \mathfrak{V}(A,f_1) \cap \mathfrak{V}(A,f_2)$. By Remark 4.8, we may assume that A is a closed set. It follows from Remark 4.1, Proposition 4.5, and Lemma 4.10 that $\mathrm{cl}\, \mathfrak{I} = \mathfrak{A}(A)$. By Lemma 4.9, $F_1 \leq F_2$ on $\mathrm{cl}_1(\mathfrak{I})$, and by transfinite induction $F_1 \leq F_2$ on $\mathrm{cl}_{\omega_1}(\mathfrak{I}) = \mathrm{cl}\, \mathfrak{I}$ (see Proposition 4.3). $\square$

4.12. COROLLARY. If $A \in \mathfrak{A}$ and $f \in \mathcal{I}(A)$, then all indefinite integrals of f in A are equal. $\square$

Let $A \in \mathfrak{A}$ and $f \in \mathcal{I}(A)$. In view of Corollary 4.12, f has a *unique* indefinite integral in A, denoted by $I(A;f,\cdot)$. The number $I(A;f,A)$ is called the *integral* of f over A. Since $I(B;f,\cdot) = I(A;f,\cdot)\restriction\mathfrak{A}(B)$ for each $B \in \mathfrak{A}(A)$, no confusion will arise if instead of $I(A;f,\cdot)$ and $I(A;f,A)$, we write simply $I(f,\cdot)$ and $I(f,A)$, respectively. It is a direct consequence of Definition 4.6 that $\mathcal{V}(A) \subset \mathcal{I}(A)$ and $I(g,A) = I_V(g,A)$ for each $g \in \mathcal{V}(A)$. We shall see later (Examples 4.29 and 4.30) that the inclusion $\mathcal{V}(A) \subset \mathcal{I}(A)$ is *proper*. Now we show that the integral has properties similar to those we established for the variational integral in Section 3.

4.13. PROPOSITION. If $A \in \mathfrak{A}$, then $\mathcal{I}(A)$ is a linear space, and the map $f \mapsto I(f,A)$ is a nonnegative linear functional on $\mathcal{I}(A)$.

PROOF. The nonnegativity of the map $f \mapsto I(f,A)$ follows from Lemma 4.11. The remaining properties are established transfinitely by arguments similar to that employed in the proof of Lemma 4.11. $\square$

If $\mathfrak{E}$ and $\mathfrak{H}$ are families of sets, we set

$$\mathfrak{E} \vee \mathfrak{H} = \{E \cup H : E \in \mathfrak{E} \;\&\; H \in \mathfrak{H}\} .$$

A transfinite induction argument, similar to the proof of Lemma 4.10, yields the following lemma.

4.14. LEMMA. If $\mathfrak{E}$ and $\mathfrak{H}$ are subfamilies of $\mathfrak{A}$, then we have

$(\mathrm{cl}\,\mathfrak{E}) \vee (\mathrm{cl}\,\mathfrak{H}) \subset \mathrm{cl}(\mathfrak{E} \vee \mathfrak{H})$. □

4.15. PROPOSITION. Let $A \in \mathfrak{A}$, $f\colon A \to \mathbb{R}$, and let $\mathfrak{D}$ be a division of A . Then $f \in \mathcal{J}(A)$ if and only if $f\upharpoonright D \in \mathcal{J}(D)$ for each $D \in \mathfrak{D}$.

PROOF. We may assume that $\mathfrak{D} = \{E,H\}$. As the converse follows from Proposition 4.7, suppose that f , properly restricted, belongs to $\mathcal{J}(E)$ and $\mathcal{J}(H)$, and let $\mathfrak{E} = \mathfrak{V}(E,f)$ and $\mathfrak{H} = \mathfrak{V}(H,f)$. By Proposition 3.5, $\mathfrak{E} \vee \mathfrak{H} \subset \mathfrak{V}(A,f)$, and it follows from Lemma 4.14 that $A \in \mathrm{cl}[\mathfrak{V}(A,f)]$. Setting

$$F(B) = I(f, B \cap E) + I(f, B \cap H)$$

for each $B \in \mathfrak{A}(A)$, it is easy to check that F is the indefinite integral of f in A . □

4.16. LEMMA. Let $\mathfrak{E} \subset \mathfrak{A}$, and let $E \in \mathrm{cl}\,\mathfrak{E}$. Then there are $E_n \in \mathfrak{E}$ such that $E_n \subset E$, $n = 1,2,\ldots$ and $|E - \cup_n E_n| = 0$.

PROOF. Let $\mathfrak{G}$ be the family of all sets $A \in \mathfrak{A}$ for which there are $E_n \in \mathfrak{G}$ such that $E_n \subset A$, $n=1,2,\ldots$, and $|A - \cup E_n| = 0$. It is easy to see that $\mathfrak{E} \subset \mathfrak{G}$ and that $\mathfrak{G}$ is closed. Hence $\mathrm{cl}\,\mathfrak{E} \subset \mathfrak{G}$. □

4.17. COROLLARY. If $A \in \mathfrak{A}$ and $f \in \mathcal{J}(A)$, then f is measurable.

PROOF. By Corollary 3.8, $f\upharpoonright E$ is measurable for each $E \in \mathfrak{V}(A,f)$. Since $A \in \mathrm{cl}[\mathfrak{V}(A,f)]$, the corollary follows from Lemma 4.16. □

4.18. REMARK. By Corollary 4.17, we see that Proposition 3.9 applies to the integral

$I(\cdot,A)$ on $\mathcal{J}(A)$, where $A \in \mathfrak{A}$. We also see that an almost everywhere statement completely analogous to Corollary 3.10 holds for the integral $I(f,A)$.

4.19. **Lemma.** Let $E \subset \mathbb{R}^m$, and let $\mathfrak{G}$ be the family of all open sets $G \subset \mathbb{R}^m$ for which $\mathfrak{A}_o(G) \subset \mathrm{cl}[\mathfrak{A}_o(E)]$. Then $G_o = \cup\, \mathfrak{G}$ belongs to $\mathfrak{G}$.

Proof. For each $x \in G_o$ find a $G_x \in \mathfrak{G}$ containing x, and a closed neighborhood U_x of x for which $U_x \in \mathfrak{A}_o(G_x)$. If $A \in \mathfrak{A}_o(G_o)$, then there are $x_1,\dots,x_n$ in A^- such that $A^- \subset \cup_{i=1}^n U_{x_i}$; for A^- is compact, and $\{U_x^o : x \in A^-\}$ is an open cover of A^-. By our choice of the U_x's, we see that $A \cap U_{x_i}$ belongs to $\mathfrak{A}_o(G_{x_i})$, and hence to $\mathrm{cl}[\mathfrak{A}_o(E)]$ for $i = 1,\dots,n$. Since $\mathrm{cl}[\mathfrak{A}_o(E)]$ is an ideal in $\mathfrak{A}$, (see Proposition 4.5), we conclude that $A = \cup_{i=1}^n (A \cap U_{x_i})$ belongs to $\mathrm{cl}[\mathfrak{A}_o(E)]$. The lemma follows. $\square$

4.20. **Lemma.** If $A \in \mathfrak{A}$ and $T \subset A^-$ is thin, then $A \in \mathrm{cl}_1[\mathfrak{A}_o(A - T)]$.

Proof. By Lemma 1.1, for each integer $k \geq 1$, there is a sequence $\{T_{k,n}\}_n$ of dyadic cubes such that $A^{\cdot} \cup T \subset \cup_n T_{k,n}$, $\Sigma_n |T_{k,n}| < 1/k$, and

$$\sum_n \|T_{k,n}\| < \alpha \mathscr{H}(A^{\cdot} \cup T) + 1$$

where α is the constant defined in Lemma 1.1. Fix an integer $k \geq 1$, and for $n = 1,2,\dots$, let $\mathfrak{T}_{k,n}$ be the collection of those dyadic cubes C for which $d(C) = d(T_{k,n})$ and $C^- \cap T_{k,n}^- \neq \emptyset$. If $T_{k,n}^\star = (\cup \mathfrak{T}_{k,n})^o$, then $\{T_{k,n}^\star\}_n$ is an open cover of $A^{\cdot} \cup T$. As T is compact, $A^{\cdot} \cup T \subset \cup_{n=1}^{n_k} T_{k,n}^\star$ for some integer $n_k \geq 1$. Letting $\mathfrak{T}_k = \cup_{n=1}^{n_k} \mathfrak{T}_{k,n}$ and $B_k = \cup \mathfrak{T}_k$, we see that $A^{\cdot} \cup T \subset B_k^o$, and

$$|B_k| \le 3^m \sum_n |T_{k,n}| < 3^m/k \,,$$

$$\|B_k\| \le 3^m \sum_n \|T_{k,n}\| < 3^m[\alpha\mathscr{H}(A^{\cdot} \cup T) + 1] \,;$$

for $\mathfrak{T}_{k,n}$ contains 3^m cubes congruent to $T_{k,n}$. It follows that $\{A - B_k\}$ is a sequence in $\mathfrak{A}_o(A - T)$ and $\{A - B_k\} \to A$ (see [M , Theorem 35]).

4.21. Corollary. Let $H \subset \mathbb{R}^m$ be an open set, and let T be a thin set. Then $\mathfrak{A}(H) \subset \mathrm{cl}_1[\mathfrak{A}_o(H - T)]$.

Proof. If $B \in \mathfrak{A}(H)$, then $B \in \mathrm{cl}_1[\mathfrak{A}_o(B - T)]$ by Lemma 4.20. Since $B - T \subset H - T$, we have $\mathrm{cl}_1[\mathfrak{A}_o(B - T)] \subset \mathrm{cl}_1[\mathfrak{A}_o(H - T)]$, and hence $B \in \mathrm{cl}_1[\mathfrak{A}_o(H - T)]$. $\square$

A compact set $T \subset \mathbb{R}^m$ is called *σ–thin* if it is a countable union of thin sets.

4.22. Proposition. Let $A \in \mathfrak{A}$, and let $T \subset A^-$ be σ–thin. Then $A \in \mathrm{cl}[\mathfrak{A}_o(A - T)]$.

Proof. Let G_o be the union of all open sets $G \subset \mathbb{R}^m$ for which $\mathfrak{A}_o(G) \subset \mathrm{cl}[\mathfrak{A}_o(A - T)]$. Then G_o is an open subset of $\mathbb{R}^m$, and by Lemma 4.19, $\mathfrak{A}_o(G_o) \subset \mathrm{cl}[\mathfrak{A}_o(A - T)]$.

Suppose that A^o is not contained in G_o. Then $E = A^o - G_o$ is a nonempty locally compact subspace of $\mathbb{R}^m$ (see [D, Chapter XI, Theorem 6.5(2), p. 239]). Since $A^o - T \subset G_o$, we have $E \subset T$. If $T = \cup_n T_n$ where $T_1, T_2, \ldots$ are thin sets, then by the Baire category theorem (see [D, Chapter XI, Theorem 10.3, p.250), there is an integer

$N \geq 1$ such that the interior of $E \cap T_N$ relative to E is nonempty. This means that there is an open set $G \subset \mathbb{R}^m$ such that $G \cap E \neq \emptyset$ and $G \cap E \subset T_N$. In particular, $H = G \cap A^o$ is a nonempty open set which is not a subset of G_o. Moreover,

$$H - T_N \subset G \cap A^o - G \cap E = G \cap (A^o \cap G_o) \subset G_o ,$$

and so by Corollary 4.21,

$$\mathfrak{A}_o(H) \subset \mathrm{cl}_1[\mathfrak{A}_o(H - T_N)] \subset \mathrm{cl}_1[\mathfrak{A}_o(G_o)] \subset \mathrm{cl}_1\{\mathrm{cl}[\mathfrak{A}_o(A - T)]\} = \mathrm{cl}[\mathfrak{A}_o(A - T)] .$$

As H is not contained in G_o, this contradicts the definition of G_o, and we conclude that $A^o \subset G_o$. It follows that

$$\mathrm{cl}_1[\mathfrak{A}_o(A^o)] \subset \mathrm{cl}_1[\mathfrak{A}_o(G_o)] \subset \mathrm{cl}_1\{\mathrm{cl}[\mathfrak{A}_o(A - T)]\} = \mathrm{cl}[\mathfrak{A}_o(A - T)] .$$

Since $A^{\boldsymbol{\cdot}}$ is thin, Lemma 4.20 implies that A belongs to $\mathrm{cl}_1[\mathfrak{A}_o(A - A^{\boldsymbol{\cdot}})] = \mathrm{cl}_1[\mathfrak{A}_o(A^o)]$, and the proposition follows. $\square$

The next example shows that for no ordinal $\alpha < \omega_1$ can $\mathrm{cl}_\alpha[\mathfrak{A}_o(A - T)]$ replace $\mathrm{cl}[\mathfrak{A}_o(A - T)]$ in Proposition 4.22.

4.23. Example. As we agreed in Section 1, we identify an ordinal with the set of all smaller ordinals, and we give ω_1 the order topology. Let $\beta < \omega_1$, and suppose that for each $\alpha < \beta$ and each nonempty interval (a,b) we have defined an order preserving homeomorphism $\varphi_\alpha\colon \alpha + 1 \to (a,b)$; if $\beta = 1$, this can be done by letting $\varphi_0(0) = x$ for any $x \in (a,b)$. If $\beta = \alpha + 1$, we can define $\varphi_\beta\colon \beta + 1 \to (a,b)$ by letting $\varphi_\beta(\gamma) = \varphi_\alpha(\gamma)$ for each $\gamma \leq \alpha$, and $\varphi_\beta(\beta) = x$ for any $x \in (\varphi_\alpha(\alpha),b)$. If β is a limit ordinal, find ordinals $\alpha_1 < \alpha_2 < \ldots < \beta$ with $\sup_n \alpha_n = \beta$, and points $x_0 < x_1 < \ldots < y$ in (a,b) with $\sup_n x_n = y$. By the induction hypothesis, for $n = 1,2,\ldots$, there is an order preserving homeomorphism $\psi_n\colon \alpha_n + 1 \to (x_{n-1},x_n)$. Setting $\varphi_\beta(\gamma) = \psi_1(\gamma)$ if $\gamma \le \alpha_1$, $\varphi_\beta(\gamma) = \psi_n(\gamma)$ if $\alpha_{n-1} < \gamma \le \alpha_n$, $n = 2,3,\ldots$, and $\varphi_\beta(\beta) = y$, it is easy to check that $\varphi_\beta\colon \beta + 1 \to (a,b)$ is an order preserving homeomorphism.

Now given an ordinal $\alpha < \omega_1$, we define ω^α according to the usual rules for ordinal arithmetic (see [Si, Chapter XIV, Sections 8 and 9, pp. 287 – 290]). As $\omega^\alpha < \omega_1$, by the previous paragraph, there is an order preserving homeomorphism φ from $\omega^\alpha + 1$ into the interval $A = (0,1)$. The set $T = \varphi(\omega^\alpha + 1)$ is a σ–thin subset of A, and it is not difficult to verify by transfinite induction that $A \notin \mathrm{cl}_\alpha[\mathfrak{A}_o(A - T)]$ (cf. Example 4.2).

4.24. THEOREM. Let $A \in \mathfrak{A}$, and let $T \subset A^-$ be a σ–thin set. Let v be a continuous vector field in A^- which is differentiable in $A^\circ - T$. Then $\operatorname{div} v(x)$ is defined for almost all $x \in A$, is integrable in A and

$$I(\operatorname{div} v, A) = \int_{A^\cdot} v \cdot n_A \, d\mathscr{H} .$$

PROOF. For each $B \in \mathfrak{A}(A)$, let $F(B) = \int_{B^\cdot} v \cdot n_B \, d\mathscr{H}$. By Lemma 2.3, F is an additive continuous function on $\mathfrak{A}(A)$. Let $f = \operatorname{div} v$ and $B \in \mathfrak{V}(A,f)$. By Theorem 3.12, $F(C) = I_v(f,C)$ for each $C \in \mathfrak{A}_o(B - T)$. Using Lemma 4.9 and Propositions 4.3 and 4.22, a simple transfinite induction yields that $F(B) = I_v(f,B)$. From Theorem 3.12 it also follows that $\mathfrak{A}_o(A - T) \subset \mathfrak{V}(A,f)$. Thus $A \in \mathrm{cl}[\mathfrak{V}(f,A)]$ by Proposition 4.22, and the Theorem is established. □

Let $A \in \mathfrak{A}$ and $f : A \to \mathbb{R}$. We denote by $\mathfrak{I}(A,f)$ the family of all $B \in \mathfrak{A}(A)$ on which f is integrable. By Proposition 4.15, the family $\mathfrak{I}(A,f)$ is an ideal in $\mathfrak{A}(A)$.

4.25. **Proposition.** Let $A \in \mathfrak{A}$ and $f\colon A \to \mathbb{R}$. Then $f \in \mathcal{I}(A)$ if and only if the following conditions are satisfied:

(i) there is a sequence $\{A_n\}$ in $\mathfrak{I}(A,f)$ converging to A;

(ii) a finite $\lim I(f,B_n)$ exists for each sequence $\{B_n\}$ in $\mathfrak{I}(A,f)$ which converges to A.

Proof. Suppose that conditions (i) and (ii) are fulfilled, and for $B \in \mathfrak{I}(A,f)$, set $F(B) = I(f,B)$.

First we observe that if $\{X_n\}$ and $\{Y_n\}$ are sequences in $\mathfrak{I}(A,f)$ converging to A, then $\lim F(X_n) = \lim F(Y_n)$. Indeed, if $\lim F(X_n) \neq \lim F(Y_n)$, we let $Z_{2n-1} = X_n$ and $Z_{2n} = Y_n$ for $n = 1,2,\ldots$. Then $\{Z_n\} \to A$, and contrary to (ii), $\lim F(Z_n)$ does not exist.

Let $C \in \mathfrak{A}(A)$, $D = A - C$, and let $\{X_n\}$ be a sequence in $\mathfrak{I}(A,f)$ converging to A. If $\limsup F(X_n \cap C) = +\infty$, then for each integer $k \geq 1$ there is an integer $n_k \geq 1$ such that

$$F(X_{n_k} \cap C) \geq -F(X_n \cap D) + k .$$

Thus letting $B_k = (X_{n_k} \cap C) \cup (X_n \cap D)$, we see that $\lim F(B_k) = +\infty$. This contradicts (ii), for $\{B_k\} \to A$ by Lemma 4.4. From this and symmetric arguments, we conclude that the sequences $\{F(X_n \cap C)\}$ and $\{F(X_n \cap D)\}$ are bounded. Now suppose that

$$\liminf F(X_n \cap C) < \limsup F(X_n \cap C)$$

or

$$\liminf F(X_n \cap D) < \limsup F(X_n \cap D) \,,$$

and choose subsequences $\{C_n^i\}$ and $\{D_n^i\}$, $i = 1,2$, of $\{X_n \cap C\}$ and $\{X_n \cap D\}$, respectively, so that

$$\liminf F(X_n \cap C) = \lim F(C_n^1) \,, \qquad \limsup F(X_n \cap C) = \lim F(C_n^2) \,,$$

$$\liminf F(X_n \cap D) = \lim F(D_n^1) \,, \qquad \limsup F(X_n \cap D) = \lim F(D_n^2) \,.$$

Then $\{C_n^i \cup D_n^i\} \to A$, $i = 1,2$, and

$$\lim F(C_n^1 \cup D_n^1) = \lim F(C_n^1) + \lim F(D_n^1) < \lim F(C_n^2) + \lim F(D_n^2) = \lim F(C_n^2 \cup D_n^2) \,,$$

contrary to our previous observation. Thus a finite $\lim F(X_n \cap C)$ exist. Moreover, by arguing as before, we observe that $\lim F(X_n \cap C)$ does not depend on the choice of the sequence $\{X_n\}$ converging to A. If $C \in \mathfrak{I}(A,f)$, then it follows from Lemma 4.9 that $F(C) = \lim F(X_n \cap C)$. We conclude that F can be extended from $\mathfrak{I}(A,f)$ to $\mathfrak{A}(A)$ by setting $F(B) = \lim F(A_n \cap B)$ for each $B \in \mathfrak{A}(A)$. This extension is clearly additive, and we complete the proof by showing that it is also continuous.

Let $\{B_k\}$ be a sequence in $\mathfrak{A}(A)$ which converges to a set $B \in \mathfrak{A}(A)$, and let $C = A - B$. By the definition of $F(B_k)$, there is an integer $n_k \geq 1$ such that $|F(A_{n_k} \cap B_k) - F(B_k)| < 1/k$, $k = 1,2,\ldots$. The sets $X_k = (A_{n_k} \cap B_k) \cup (A_{n_k} \cap C)$ belong to $\mathfrak{I}(A,f)$, and $\{X_k\} \to A$ by Lemma 4.4. Thus

$$F(B) = \lim F(X_k \cap B) = \lim F(A_{n_k} \cap B_k) = \lim F(B_k) ,$$

and the continuity of F is established.

The converse follows directly from Lemma 4.9. $\square$

The previous proposition shows that the integral I is closed with respect to the formation of *improper integrals*. We show next that the integral I is actually *the smallest* extension of the variational integral I_V which has this property.

Let $A \in \mathfrak{A}$ and $f\colon A \to \mathbb{R}$. We let $\mathfrak{I}_0 = \mathfrak{V}(A,f)$, and for each $B \in \mathfrak{I}_0$, set $F(B) = I_V(f,B)$. Assuming that $\mathfrak{I}_\alpha$ and F_α have been defined for each ordinal $\alpha < \beta \le \omega_1$ so that $\mathfrak{I}_\alpha \subset \mathfrak{I}_{\alpha'}$ and $F_\alpha = F_{\alpha'} \restriction \mathfrak{I}_\alpha$ whenever $\alpha < \alpha' < \beta$, we define $\mathfrak{I}_\beta$ and F_β as follows:

(i) If β is a limit ordinal, let $\mathfrak{I}_\beta = \bigcup_{\alpha<\beta} \mathfrak{I}_\alpha$ and let F_β be the unique function on $\mathfrak{I}_\beta$ such that $F_\alpha = F_\beta \restriction \mathfrak{I}_\alpha$ for each $\alpha < \beta$.

(ii) If $\beta = \alpha + 1$, let $\mathfrak{I}_\beta$ consist of all $B \in \mathrm{cl}_1(\mathfrak{I}_\alpha)$ such that a finite $\lim F_\alpha(B_n)$ exists for each sequence $\{B_n\}$ in $\mathfrak{I}_\alpha$ converging to B. It is easy to see that all such limits have the same value, and we declare it equal to $F_\beta(B)$.

A simple transfinite argument shows that any extension of the variational integral $I_V(f,\cdot)$ on $\mathfrak{V}(A,f)$ which is closed with respect to the formation of improper integrals (in the sense of Proposition 4.25) also extends F_{ω_1}.

4.26. Proposition. Let $A \in \mathfrak{A}$ and $f\colon A \to \mathbb{R}$. Then $\mathfrak{I}_{\omega_1} = \mathfrak{I}(A^-,f)$ and $I(f,B) = F_{\omega_1}(B)$ for each $B \in \mathfrak{I}(A^-,f)$.

PROOF. In view of Remark 4.8, we may assume that A is closed. By Proposition 4.25, a straightforward transfinite induction shows that $\mathfrak{I}_{\omega_1} \subset \mathfrak{I}(A,f)$ and $F_{\omega_1}(B) = I(f,B)$ for each $B \in \mathfrak{I}_{\omega_1}$.

Now let $B \in \mathfrak{I}(A,f)$. Then $B \in \mathrm{cl}[\mathfrak{V}(B,f)]$, and we show inductively that $\mathrm{cl}_\alpha[\mathfrak{V}(B,f)] \subset \mathfrak{I}_{\omega_1}$ for each $\alpha \le \omega_1$. This is true if $\alpha = 0$, and we assume that it is true for all $\alpha < \beta \le \omega_1$. If β is a limit ordinal, then trivially $\mathrm{cl}_\beta[\mathfrak{V}(B,f)] \subset \mathfrak{I}_{\omega_1}$. Let $\beta = \alpha + 1$, and let $C \in \mathrm{cl}_\beta[\mathfrak{V}(B,f)]$. By the inductive hypothesis, there is a sequence $\{C_n\}$ in $\mathfrak{I}_{\omega_1}$ which converges to C. There is a $\gamma < \omega_1$ such that $\{C_n\}$ is a sequence in $\mathfrak{I}_\gamma$, and hence $C \in \mathrm{cl}_1(\mathfrak{I}_\gamma)$. If $\{D_n\}$ is a sequence in $\mathfrak{I}_\gamma$ which converges to C, then

$$\lim F_\gamma(D_n) = \lim I(f,D_n) = I(f,C) \neq \pm\infty .$$

It follows that $C \in \mathfrak{I}_{\gamma+1}$, and we see again $\mathrm{cl}_\beta[\mathfrak{V}(B,f)] \subset \mathfrak{I}_{\omega_1}$. $\square$

4.27. LEMMA. Let $A \in \mathfrak{A}$, and let $\Phi: A \to \mathbb{R}^m$ be a regular map. Then the following statements are true.

(i) If $\{B_n\}$ is a sequence in $\mathfrak{A}(A^-)$ converging to a set $B \in \mathfrak{A}(A^-)$, then $\{\Phi[B_n]\} \to \Phi[B]$.

(ii) If $\mathfrak{C} \subset \mathfrak{A}(A^-)$, then $\mathrm{cl}(\{\Phi[E] : E \in \mathfrak{C}\}) = \{\Phi[E] : E \in \mathrm{cl}\,\mathfrak{C}\}$.

PROOF. (i) Clearly, $\Phi[B_n] \subset \Phi[B]$, and there are positive real numbers b and c such that

$$|\Phi[C]| \leq b|C| \quad \text{and} \quad \|\Phi[C]\| \leq c\|C\|$$

for each $C \in \mathfrak{A}(A^{-})$ (see [Ru, Theorem 7.26, p.153] and [M, Theorem 50]). Thus

$$0 \leq \lim |\Phi[B] - \Phi[B_n]| = \lim |\Phi[B - B_n]| \leq b \lim |B - B_n| = 0$$

and

$$\sup_n \|\Phi[B_n]\| \leq c \sup_n \|B_n\| < +\infty .$$

(ii) Applying (i) to Φ^{-1}, we see that the family $\mathfrak{C} = \{\Phi[E] : E \in \mathrm{cl}\, \mathfrak{E}\}$ is closed. As $\mathfrak{C}$ contains $\{\Phi[E] : E \in \mathfrak{E}\}$, it contains also $\mathfrak{D} = \mathrm{cl}\{\Phi[E] : E \in \mathfrak{E}\}$. To establish the reverse inclusion, let $\mathfrak{C}_\alpha = \{\Phi[E] : E \in \mathrm{cl}_\alpha(\mathfrak{E})\}$ for $\alpha \leq \omega_1$. Clearly $\mathfrak{C}_0 \subset \mathfrak{D}$, and we assume inductively that $\mathfrak{C}_\alpha \subset \mathfrak{D}$ for all $\alpha < \beta \leq \omega_1$. If β is a limit ordinal, we see immediately that $\mathfrak{C}_\beta \subset \mathfrak{D}$. Let $\beta = \alpha + 1$, and let $B \in \mathfrak{C}_\beta$. Then $B = \Phi(E)$ for some $E \in \mathrm{cl}_\beta(\mathfrak{E})$, and we can find a sequence $\{E_n\}$ in $\mathrm{cl}_\alpha(\mathfrak{E})$ which converges to E. By our assumption $\{\Phi(E_n)\}$ is a sequence in $\mathfrak{D}$, which converges to B by (i). As $\mathfrak{D}$ is closed, $B \in \mathfrak{D}$ and we have again that $\mathfrak{C}_\beta \subset \mathfrak{D}$. Now the inclusion $\mathfrak{C}_\beta \subset \mathfrak{D}$ follows from Proposition 4.3. $\square$

4.28. **Theorem.** Let $A \in \mathfrak{A}$, and let $\Phi: A \to \mathbb{R}^m$ be a regular map. If $f \in \mathcal{J}(\Phi[A])$, then $f \circ \Phi \cdot |\det \Phi|$ belongs to $\mathcal{J}(A)$ and

$$I(f \circ \Phi \cdot |\det \Phi|, A) = I(f, \Phi[A]) .$$

Proof. By Theorem 3.14, we have

$$\mathfrak{V}(A, f\circ\Phi\cdot|\det\Phi|) = \{\Phi^{-1}[B] : B \in \mathfrak{V}(\Phi[A], f)\} .$$

As $\Phi[A] \in \mathrm{cl}[\mathfrak{V}(\Phi[A],f)]$, it follows from Lemma 4.27, (ii) applied to Φ^{-1} that A belongs to $\mathrm{cl}[\mathfrak{V}(A,f\circ\Phi\cdot|\det\Phi|]$. For each $B \in \mathfrak{V}(A,f\circ\Phi\cdot|\det\Phi|)$, let $F(B) = I(f,\Phi[B])$. Then by Lemma 4.27, (i), F is an additive continuous function on $\mathfrak{A}(A)$, and by Theorem 3.14,

$$F(B) = I_v(f\circ\Phi\cdot|\det\Phi|, B)$$

for each $B \in \mathfrak{V}(A,f\circ\Phi\cdot|\det\Phi|)$. The theorem follows. $\square$

4.29. EXAMPLE. We shall construct a function $f: \mathbb{R} \to \mathbb{R}$, which is integrable but not v–integrable in $A = [0,1]$.

For each nonempty open interval $U \subset \mathbb{R}$, we fix a continuous function $\varphi_U: \mathbb{R} \to [0,\infty)$ such that $\varphi_U(s) = 0$ for each $s \in \mathbb{R} - U$, and $\int_U \varphi_U \, d\lambda = 1$.

Given a nondegenerate compact interval $C = [a,b]$, we set

$$C_+(n) = (a + 2^{-2n+1}|C| \ , \ a + 2^{-2n+2}|C|) ,$$

$$C_-(n) = (a + 2^{-2n}|C| \ , \ a + 2^{-2n+1}|C|) ,$$

$n = 1,2,\dots$, and let

$$f_C = |C| \sum_{n=1}^{\infty} n^{-1} [\varphi_{C_+(n)} - \varphi_{C_-(n)}]$$

The function f_C is continuous in $\mathbb{R} - \{a\}$, and $f_C(s) = 0$ for each $s \in \mathbb{R} - (a,b)$. If

$$F_C(s) = -\int_s^b f_C \, d\lambda$$

for $s \in (a,b)$, and $F_C(s) = 0$ otherwise, then F_C is continuous in $\mathbb{R}$, $F_C'(s) = f_C(s)$ for each $s \in \mathbb{R} - \{a\}$, and

$$|F_C(s)| \le -F_C(a+2^{-1}|C|) = |C|$$

for each $s \in \mathbb{R}$. Interpreting F_C as a vector field in $\mathbb{R}$, we obtain from Theorem 3.12 that f_C is v–integrable in C, and $I_v(f_C,C) = 0$.

Claim. The point a is an ϵ–point of f_C for each $\epsilon \in (0,1/2)$.

Proof. Recall that ϵ–points were defined in the paragraph preceding Proposition 3.15. Fix an odd integer $p \ge 1$, and for $n = 1,2,\dots$, let

$$B_n = (a + 2^{-np}|C| \ , \ a + 2^{-(n-1)p}|C|) \ .$$

Since

$$d(B_n \cup \{a\}) = 2^{-(n-1)p}|C| \ , \qquad |B_n \cup \{a\}| = [2^{-(n-1)p} - 2^{-np}]|C| \ ,$$

and $\|B_n \cup \{a\}\| = 2$, we see that $\lim d(B_n \cup \{a\}) = 0$ and $r(B_n \cup \{a\}) = (1 - 2^{-p})/2$. Moreover, it is easy to verify that

$$\sum_{n=1}^{\infty} |I_v(f_C,B_n)| = 2|C| \sum_{n=1}^{\infty} \frac{2}{(2n-1)p+1} = +\infty .$$

As $\{B_n\}$ is a disjoint sequence in $\mathfrak{A}(C)$, the claim is established. $\square$

Now let $A = [0,1]$, and let $C_k = [2^{-k}, 2^{-k+1}]$, $k = 1,2,\ldots$. Set

$$f = \sum_{k=1}^{\infty} f_{C_k} \qquad \text{and} \qquad F = \sum_{k=1}^{\infty} F_{C_k} .$$

Since $|F_{C_k}| \le |C_k| = 2^{-k}$, we see that F is continuous in $\mathbb{R}$; note that only the continuity of F at 0 requires a proof. If $S = \{0\} \cup \{2^{-k} : k = 1,2,\ldots\}$, then $F'(s) = f(s)$ for each $s \in \mathbb{R} - S$. As S is a σ–thin subset of A (see the the paragraph preceding Proposition 4.22), it follows from Theorem 4.24 that f is integrable in A , and $I(f,A) = 0$; for F vanishes outside $(0,1)$. On the other hand, $[V_{1/3}(f,A)]^{-} = S$ is not thin, and consequently, f is not v–integrable in A by Proposition 3.15.

4.30. Example. If $K \subset \mathbb{R}^m$ is a cube, we say that $g: K \to \mathbb{R}^m$ is *R–integrable* in K if it is integrable in K according to [P_3, Definition 3.1]. By [P_3, Proposition 8.3], the function f from Example 4.29 is R–integrable in $A = [0,1]$. Let $g = f \otimes 1$, i.e. , $g(s,t) = f(s)$ for each $(s,t) \in \mathbb{R}^2$. We show that in $K = A \times A$, the function g is integrable, but it is neither v– nor R–integrable. For the R–integral, this provides a negative answer to the Problem 6.4 in [P_4].

We shall use freely the notation of Example 4.29. If $v(s,t) = [F(s),0]$ for each $(s,t) \in \mathbb{R}^2$, then v is a continuous vector field in $\mathbb{R}^2$, which is differentiable in

$\mathbb{R}^2 - (S \times A)$. As $S \times A$ is a σ–thin subset of K and $g = \operatorname{div} v$, by Proposition 4.24, we see that g is integrable in K and $I(g,K) = 0$.

If g is, respectively, v– or R–integrable, then it follows from Theorem 3.12 or [P_3, Theorem 5.6] that the indefinite v– or R–integral G of g in K coincides with the function $B \mapsto \int_{B^{\cdot}} v \cdot n_B \, d\mathscr{H}$ for each interval $B \subset K$ for which $B^{-} \subset (0,1] \times A$.

Assume first that g is v–integrable. For $\epsilon = 1/9$, find a thin set $T \subset K$, and an ϵ–majorant M of the pair (g,G) in $K - T$. Then for each $x \in K - T$, there is a $\delta(x) > 0$ such that

$$\left| g(x)\,|B| - G(B) \right| \leq M(B)$$

for each $B \in \mathfrak{A}_o(K - T)$ with $x \in B^{-}$, $d(B) < \delta(x)$, and $r(B) > \epsilon$.

As $\mathscr{H}\,[\bigcup_{k=1}^{\infty}(\{2^{-k}\} \times A)] = +\infty$ and $\mathscr{H}(T) < +\infty$, there is an integer $k \geq 1$ such that $\mathscr{H}\,[(\{2^{-k}\} \times A) - T] > 0$. Since $\mathscr{H}$ is a Radon measure in $\{2^{-k}\} \times A$ (see [GP, Corollary 6.8]) the set $(\{2^{-k}\} \times A) - T$ contains a perfect subset P. By applying the Baire category theorem (see [D, Chapter XI, Theorem 6.5(2), p. 239] to P, we obtain an open set $U \subset \mathbb{R}^2$ with $P \cap U \neq \emptyset$ and a $\Delta > 0$ such that the set

$$Q = \{x \in P \cap U : \delta(x) \geq \Delta\}$$

is dense in $P \cap U$. As P and T are disjoint compact sets, there is an interval $[c,c+h]$ such that $0 < h < \Delta$, $\{2^{-k}\} \times [c,c+h] \subset P \cap U$, and

$$[2^{-k}, 2^{-k} + h] \times [c, c + h] \subset K^{o} - T.$$

Without loss of generality, we may assume that $h = 2^{-k-2N+2}$ for some integer $N \geq 2$. To simplify the the notation, we set $q = 2N-2$, and for $n = q,q+1,\dots$, let $r_n = 2^{2n-q-1}$ and

$$D_n = [2^{-k} + 2^{-k-2n+1}, 2^{-k} + 2^{-k-2n+2}) \times [c, c+h).$$

Each D_n is the disjoint union of r_n squares

$$D_{n,i} = [2^{-k} + 2^{-k-2n+1}, 2^{-k} + 2^{-k-2n+2}) \times [c + (i-1)2^{-k-2n+1}, c + i2^{-k-2n+1}),$$

for $i = 1,\dots,r_n$. In $(\{2^{-k}\} \times [c,c+h]) \cap \mathbb{Q}$, we can find distinct points $x_{n,i}$ so that $r(D_{n,i} \cup \{x_{n,i}\}) = 1/8$. Since the sets $E_{n,i} = D_{n,i} \cup \{x_{n,i}\}$ are disjoint, $d(E_{n,i}) < \delta(x_{n,i})$, $r(E_{n,i}) > \epsilon$, and $g(x_{n,i}) = 0$, we have

$$\epsilon > M(K) \geq M\left(\bigcup_{n=q}^{p} \bigcup_{i=1}^{r_n} E_{n,i}\right) = \sum_{n=q}^{p} \sum_{i=1}^{r_n} M(E_{n,i}) \geq \sum_{n=q}^{p} \sum_{i=1}^{r_n} \left| g(x_{n,i}) |E_{n,i}| - G(E_{n,i}) \right|$$

$$= \sum_{n=q}^{p} \sum_{i=1}^{r_n} \left| \int_{(E_{n,i})^{\cdot}} v \cdot n_{E_{n,i}} \, d\mathscr{H} \right| = \sum_{n=q}^{p} \left| \int_{(D_n)^{\cdot}} v \cdot n_{D_n} \, d\mathscr{H} \right|$$

$$= \sum_{n=q}^{p} \left| \int_{D_n} g \, d\lambda_2 \right| = \sum_{n=q}^{p} h \int_{(C_k)_+(n)} f \, d\lambda = h|C_k| \sum_{n=q}^{p} 1/n.$$

for $p = q,q+1,\dots$. This a contradiction, for $\sum_{n=q}^{\infty}(1/n) = +\infty$.

Now assume that g is R–integrable. Then there is a thin set $T \subset K$ and a

$\delta\colon K \to (0,+\infty)$ such that with $\epsilon = 1/9$,

$$\sum_{i=1}^{p} \left| g(x_i)\,|A_i| - G(A_i) \right| < \epsilon$$

for each δ–fine ϵ–partition $\{(A_1,x_1),\dots,(A_p,x_p)\}$ in $A - T$ (see [P_3, Section 2] for the definition of a δ–fine ϵ–partition in $A - T$). Proceeding exactly as before, we define the points $x_{n,i}$ and the sets $E_{n,i}$, and observe that for each $p = q,q+1,\dots$, the collection

$$\{(E_{n,i},x_{n,i}) : q \le n \le p \ , \ 1 \le i \le r_n\}$$

is a δ–fine ϵ–partition in $K - T$. Since

$$\sum_{n=q}^{p} \sum_{i=1}^{r_n} \left| g(x_{n,i})\,|E_{n,i}| - G(E_{n,i}) \right| = h\,|C_k| \sum_{n=q}^{p} 1/n \ ,$$

a contradiction follows.

5. **The star–integral.** We say that a sequence $\{A_n\}$ in $\mathfrak{A}$ *star–converges* to a set $A \in \mathfrak{A}$, in writing $\{A_n\} \overset{\star}{\to} A$, if $A_n \subset A$ for $n = 1,2,\dots$, and $\lim \|A - A_n\| = 0$. A family $\mathfrak{E} \subset \mathfrak{A}$ is called *star–closed* if $E \in \mathfrak{E}$ for each $E \in \mathfrak{A}$ for which there is a sequence $\{E_n\}$ in $\mathfrak{E}$ with $\{E_n\} \overset{\star}{\to} E$. The *star–closure* of a family $\mathfrak{E} \subset \mathfrak{A}$, denoted by $\mathrm{cl}^\star\, \mathfrak{E}$, is the intersection of all star–closed subfamilies of $\mathfrak{A}$ containing $\mathfrak{E}$. It is easy to verify that for each $\mathfrak{E} \subset \mathfrak{A}$, the star–closure of $\mathfrak{E}$ is a star–closed subfamily of $\mathfrak{A}$, which contains E^- for every $E \in \mathfrak{E}$ (cf. Remark 4.1). Instead of star–convergence, star–closed, and star–closure, we shall usually write $\star$–convergence, $\star$–closed, and $\star$–closure, respectively.

The following lemma indicates the relative simplicity of the $\star$–closure operation (cf. Examples 4.2 and 4.23).

5.1. LEMMA. Let $\mathfrak{E} \subset \mathfrak{A}$, and let $E \in \mathfrak{A}$. Then $E \in \mathrm{cl}^\star \mathfrak{E}$ if and only if there is a sequence $\{E_n\}$ in $\mathfrak{E}$ with $\{E_n\} \overset{\star}{\to} E$.

PROOF. Let $\mathfrak{E}^\star$ be the family of all sets $E \in \mathfrak{A}$ for which there is a sequence $\{E_n\}$ in $\mathfrak{E}$ with $\{E_n\} \overset{\star}{\to} E$. Clearly, $\mathfrak{E} \subset \mathfrak{E}^\star \subset \mathrm{cl}^\star \mathfrak{E}$, and the lemma will be proved by showing that $\mathfrak{E}^\star$ is a $\star$–closed family. To this end, let $\{A_n\}$ be a sequence in $\mathfrak{E}^\star$ which $\star$–converges to a set $A \in \mathfrak{A}$. For each A_n there is a sequence $\{E_{n,k}\}_k$ in $\mathfrak{E}$ with $\{E_{n,k}\} \overset{\star}{\to} A_n$. For $n = 1,2,\dots$, find an integer k_n so that $\|A_n - E_{n,k_n}\| < 1/n$, and set $E_n = E_{n,k_n}$. Then $E_n \in \mathfrak{E}$, $E_n \subset A_n \subset A$, and by [M, Theorem 35],

$$\|A - E_n\| = \|(A - A_n) \cup (A_n - E_n)\| \le \|A - A_n\| + \|A_n - E_n\| < \|A - A_n\| + 1/n .$$

Thus $\{E_n\} \overset{\star}{\to} A$, and $A \in \mathfrak{E}^\star$. $\square$

5.2. DEFINITION. Let $A \in \mathfrak{A}$, and let F be a function on $\mathfrak{A}(A)$. We say that F is *star–continuous* (or simply *$\star$–continuous*) if given $\epsilon > 0$, there is a $\delta > 0$ such that $|F(B)| < \epsilon$ for each $B \in \mathfrak{A}(A)$ with $\|B\| < \delta$.

5.3. LEMMA. Let $A \in \mathfrak{A}$, and let F be an additive function on $\mathfrak{A}(A)$. Then F is $\star$–continuous if and only if $\lim F(B_n) = F(B)$ for each sequence $\{B_n\}$ in $\mathfrak{A}(A)$ which $\star$–converges to $B \in \mathfrak{A}(A)$.

PROOF. Let F be $\star$–continuous, and let $\{B_n\}$ be a sequence in $\mathfrak{A}(A)$ which

$\star$—converges to $B \in \mathfrak{A}(A)$. Choose an $\epsilon > 0$, and find $\delta > 0$ so that $|F(C)| < \epsilon$ for each $C \in \mathfrak{A}(A)$ with $\|C\| < \delta$. There is an integer $N \geq 1$ such that $\|B - B_n\| < \delta$ for each $n \geq N$. Hence

$$|F(B) - F(B_n)| = |F(B - B_n)| < \epsilon$$

for each $n \geq N$, and we see that $\lim F(B_n) = F(B)$.

Conversely, if F is not $\star$—continuous, then there is an $\epsilon > 0$ and a sequence $\{B_n\}$ in $\mathfrak{A}(A)$ such that $\|B_n\| < 1/n$ and

$$\epsilon \leq |F(B_n)| = |F(A) - F(A - B_n)|$$

for $n = 1,2,\ldots$. Yet $\{A - B_n\} \overset{\star}{\to} A$, for

$$\lim \|A - (A - B_n)\| = \lim \|B_n\| = 0. \quad \square$$

5.4. Lemma. The following statements are true.

(i) Each sequence $\{A_n\}$ in $\mathfrak{A}$ which $\star$—converges to a set $A \in \mathfrak{A}$ also converges to A.

(ii) Each closed subfamily of $\mathfrak{A}$ is $\star$—closed; in particular, $\mathrm{cl}^\star\, \mathfrak{C} \subset \mathrm{cl}\, \mathfrak{C}$ for every $\mathfrak{C} \subset \mathfrak{A}$.

(iii) Each continuous function on $\mathfrak{A}(A)$ with $A \in \mathfrak{A}$ is $\star$—continuous.

Proof. Properties (i) and (iii) follow from [MM, Section 10], and property (ii) is a direct consequence of (i). $\square$

5.5. LEMMA. Let $A \in \mathfrak{A}$, and let v be a bounded $\mathscr{H}$–measurable vector field on A^-. If $F(B) = \int_{B^\cdot} v \cdot n_B \, d\mathscr{H}$ for each $B \in \mathfrak{A}(A)$, then F is an additive $\star$–continuous function on $\mathfrak{A}(A)$.

PROOF. The additivity of F is clear, and if $c = \sup\{\|v(x)\| : x \in A^-\}$, then

$$|F(B)| \le \int_{B^\cdot} |v \cdot n_B| \, d\mathscr{H} \le c \int_{B^\cdot} \|n_B\| \, d\mathscr{H} = c\|B\|$$

for each $B \in \mathfrak{A}(A)$. The $\star$–continuity of F follows. $\square$

5.6. LEMMA. Let A and B belong to $\mathfrak{A}$, and let $\{A_n\}$ and $\{B_n\}$ be sequences in $\mathfrak{A}$ which $\star$–converge to A and B, respectively. Then $\{A_n \cup B_n\} \overset{\star}{\to} A \cup B$ and $\{A_n \cap B_n\} \overset{\star}{\to} A \cap B$.

PROOF. Letting $C = A \cup B$, we have

$$A \cup B - (A_n \cup B_n) = [A \cup B - (A_n \cup B)] \cup [A_n \cup B - (A_n \cup B_n)] =$$

$$[(A - A_n) \cap (C - B)] \cup [(C - A_n) \cap (B - B_n)] =$$

$$[(A - A_n) \cap (C - B)] \cup [(C - A) \cap (B - B_n)] \cup [(A - A_n) \cap (B - B_n)]$$

and

$$[A \cap B - (A_n \cap B_n) = [A \cap B - (A_n \cap B)] \cup [A_n \cap B - (A_n \cap B_n)] =$$

$$[(A - A_n) \cap B] \cup [A_n \cap (B - B_n)] =$$

$$[(A - A_n) \cap B] \cup \{[A \cap (B - B_n)] - [(A - A_n) \cap (B - B_n)]\} .$$

According to [MM , Section 13], if $\{D_n\}$ is a sequence in $\mathfrak{A}$ with $\lim \|D_n\| = 0$, then $\lim \|D \cap D_n\| = 0$ for each $D \in \mathfrak{A}$. Using this, the lemma follows from [M , Theorem 35] . $\square$

5.7. COROLLARY. If $\mathfrak{I}$ is an ideal in $\mathfrak{A}$, then so is $\mathrm{cl}^\star \mathfrak{I}$. If $\mathfrak{J}$ is another ideal in $\mathfrak{A}$, then

$$\mathrm{cl}^\star(\mathfrak{I} \cap \mathfrak{J}) = (\mathrm{cl}^\star \mathfrak{I}) \cap (\mathrm{cl}^\star \mathfrak{J}) \quad \text{and} \quad (\mathrm{cl}^\star \mathfrak{I}) \vee (\mathrm{cl}^\star \mathfrak{J}) \subset \mathrm{cl}^\star(\mathfrak{I} \vee \mathfrak{J}) .$$

PROOF. In view of Lemmas 5.1 and 5.6, the proof is analogous to the proofs of Proposition 4.5 and Lemmas 4.10 and 4.14, except that no transfinite induction is needed. Moreover, the inclusion $(\mathrm{cl}^\star \mathfrak{I}) \vee (\mathrm{cl}^\star \mathfrak{J}) \subset \mathrm{cl}^\star(\mathfrak{I} \vee \mathfrak{J})$ is actually valid for *any* subfamilies $\mathfrak{I}$ and $\mathfrak{J}$ of $\mathfrak{A}$. $\square$

5.8. DEFINITION. Let $A \in \mathfrak{A}$ and $f\colon A \to \mathbb{R}$. We say that f is *star–integrable* (or simply *$\star$–integrable*) in A if $A \in \mathrm{cl}^\star[\mathfrak{I}(A,f)]$, and there is a $\star$–continuous additive function F on $\mathfrak{A}(A)$ such that $F(B) = I(f,B)$ for each $B \in \mathfrak{I}(A,f)$.

The family of all $\star$–integrable functions on a set $A \in \mathfrak{A}$ is denoted by $\mathcal{J}^\star(A)$.

5.9. PROPOSITION. Let $A \in \mathfrak{A}$ and $f \in \mathcal{J}^\star(A)$. Then $f{\restriction}B \in \mathcal{J}^\star(B)$ for each $B \in \mathfrak{A}$, and $\hat{f} \in \mathcal{J}^\star(A^-)$ for any extension $\hat{f}$ of f to A^- .

PROOF. This follows from Proposition 4.7, Corollary 5.7, and a remark analogous to Remark 4.8. $\square$

Proceeding as in Section 4, it is easy to show that each $\star$–integrable function f on a set $A \in \mathfrak{A}$ determines *uniquely* the $\star$–continuous additive function F from Definition 5.8. We call it *the indefinite $\star$–integral* of f in A, denoted by $I^\star(A;f,\cdot)$. The number $I^\star(A;f,A)$ is called the *$\star$–integral* of f over A. Since $I^\star(B;f,\cdot) = I^\star(A;f,\cdot)\restriction\mathfrak{A}(B)$ for each $B \in \mathfrak{A}(A)$, no confusion will arise if instead of $I^\star(A;f,\cdot)$ and $I^\star(A;f,A)$, we write simply $I^\star(f,\cdot)$ and $I^\star(f,A)$, respectively. Clearly, $\mathcal{I}(A) \subset \mathcal{I}^\star(A)$ and $I^\star(g,A) = I(g,A)$ for each $g \in \mathcal{I}(A)$. In the dimension one, i.e., for $m = 1$, the $\star$–convergence is trivial: by [M, Theorem 33] a sequence $\{E_n\}$ in $\mathfrak{A}$ $\star$–converges to a set $E \in \mathfrak{A}$ if and only if $|E - E_n| = 0$ for all sufficiently large n. Thus $\mathcal{I}(A) = \mathcal{I}^\star(A)$ if $m = 1$. However, Example 5.17 shows that the inclusion $\mathcal{I}(A) \subset \mathcal{I}^\star(A)$ is *proper* whenever $m \geq 2$.

The next proposition summarizes the basic properties of the $\star$–integral. Its proof is analogous to the proofs of Propositions 4.13, 4.15, and Corollary 4.17, except that no transfinite induction is required.

5.10. PROPOSITION. Let $A \in \mathfrak{A}$. Then $\mathcal{I}^\star(A)$ is a linear space of measurable functions on A, and the map $f \mapsto I^\star(f,A)$ is a nonnegative linear functional on $\mathcal{I}^\star(A)$. Moreover, if $\mathfrak{D}$ is a division of A, then a function f on A belongs to $\mathcal{I}^\star(A)$ if and only if $f\restriction D \in \mathcal{I}^\star(D)$ for each $D \in \mathfrak{D}$. $\square$

A compact set $S \subset \mathbb{R}^m$ with $\mathcal{H}(S) = 0$ is called *slight*. Again, it follows from [Fe, Section 3.2.40, p.276] that the slight sets we consider here are larger than those defined in $[P_3]$.

5.11. LEMMA. Let $A \in \mathfrak{A}$, let $S \subset A^-$ be slight, and let $\mathfrak{A}_S(A) = \{B \in \mathfrak{A}(A) : B^- \cap S = \emptyset\}$. Then $A \in \mathrm{cl}^\star[\mathfrak{A}_S(A)]$.

PROOF. By Lemma 1.1, for each integer $k \geq 1$, there is a sequence $\{S_{k,n}\}_n$ of dyadic cubes such that $S \subset \cup_n S_{k,n}$ and $\Sigma_n \|S_{k,n}\| < 1/k$. Fix an integer $k \geq 1$, and for $n = 1,2,\dots$, let $\mathfrak{S}_{k,n}$ be the collection of those dyadic cubes C for which $d(C) = d(S_{k,n})$ and $C^- \cap S_{k,n}^- \neq \emptyset$. If $S_{k,n}^\star = (\cup \mathfrak{S}_{k,n})^\circ$, then $\{S_{k,n}^\star\}_n$ is an open cover of S. As S is compact, $S \subset \cup_{n=1}^{n_k} S_{k,n}^\star$ for some integer $n_k \geq 1$. Letting $\mathfrak{S}_k = \cup_{n=1}^{n_k} \mathfrak{S}_{k,n}$ and $B_k = \cup \mathfrak{S}_k$, we have $S \subset B_k^\circ$, and

$$\|B_k\| \leq 3^m \sum_n \|S_{k,n}\| < 3^m/k\,;$$

for $\mathfrak{S}_{k,n}$ contains 3^m cubes congruent to $S_{k,n}$. It follows that $\{A - B_k\}$ is a sequence in $\mathfrak{A}_S(A)$ and $\{A - B_k\} \overset{\star}{\to} A$; for

$$\lim \|A - (A - B_k)\| = \lim \|A \cap B_k\| = 0$$

by [MM, Section 13]. $\square$

5.12. THEOREM. Let $A \in \mathfrak{A}$, and let T and S be, respectively, a σ–thin and a slight subset of A^-. Let v be a bounded vector field which is continuous in $A^- - S$ and differentiable in $A^\circ - T$. Then v is $\mathscr{H}$–measurable, $\operatorname{div} v(x)$ is defined for almost all $x \in A$, is $\star$–integrable in A, and

$$I^\star(\operatorname{div} v, A) = \int_{A^\cdot} v \cdot n_A \, d\mathscr{H}\,.$$

PROOF. As v is continuous $\mathcal{H}$–almost everywhere, it is $\mathcal{H}$–measurable. For each $B \in \mathfrak{A}(A)$, let $F(B) = \int_{B^\cdot} v \cdot n_B \, d\mathcal{H}$. By Lemma 5.5, F is an additive $\star$–continuous function on $\mathfrak{A}(A)$. Let $f = \operatorname{div} v$ and $B \in \mathfrak{I}(A,f)$. By Theorem 4.24, $F(C) = I(f,C)$ for each $C \in \mathfrak{A}_S(B)$ (see Lemma 5.11). It follows from Lemmas 5.1, 5.3, and 5.11 that also $F(B) = I(f,B)$. Since $A \in \mathrm{cl}^\star[\mathfrak{A}_S(A)]$ by Lemma 5.11, we see that F is the indefinite $\star$–integral of f in A. $\square$

Let $A \in \mathfrak{A}$ and $f\colon A \to \mathbb{R}$. We denote by $\mathfrak{I}^\star(A,f)$ the collection of all $B \in \mathfrak{A}(A)$ in which f is $\star$–integrable. By Proposition 5.9, the collection $\mathfrak{I}^\star(A,f)$ is an ideal in $\mathfrak{A}(A)$.

Replacing convergence by $\star$–convergence, the proof of the next proposition is identical to that of Proposition 4.25.

5.13. PROPOSITION. Let $A \in \mathfrak{A}$ and $f\colon A \to \mathbb{R}$. Then $f \in \mathcal{I}^\star(A)$ if and only if the following conditions are satisfied:

(i) $A \in \mathrm{cl}^\star[\mathfrak{I}^\star(A,f)]$;

(ii) a finite $\lim I^\star(f,B_n)$ exists for each sequence $\{B_n\}$ in $\mathfrak{I}^\star(A,f)$ $\star$–converging to A. $\square$

Thus the $\star$–integral $I^\star$ is closed with respect to the formation of *improper integrals* (by means of $\star$–convergence). We show next that $I^\star$ is actually obtained from I by adding all such improper integrals.

Let $A \in \mathfrak{A}$ and $f\colon A \to \mathbb{R}$. We denote by $\mathfrak{I}^\star$ the family of all $B \in \mathrm{cl}^\star[\mathfrak{I}(A,f)]$ such that a finite $\lim I(f,B_n)$ exists for each sequence $\{B_n\}$ in $\mathfrak{I}(A,f)$ with $\{B_n\} \overset{\star}{\to} B$. It is easy to verify that all such limits have the same value (see the proof of Proposition 4.25), denoted by $F^\star(B)$.

5.14. PROPOSITION. Let $A \in \mathfrak{A}$ and $f: A \to \mathbb{R}$. Then $\mathfrak{I}^\star = \mathfrak{I}^\star(A^-,f)$ and $I^\star(f,B) = F^\star(B)$ for each $B \in \mathfrak{I}^\star(A^-,f)$.

PROOF. In view of Remark 4.8, we may assume that A is closed. If $B \in \mathfrak{I}^\star(A,f)$, then $B \in cl^\star[\mathfrak{I}(B,f)]$, and consequently $B \in cl^\star[\mathfrak{I}(A,f)]$. If $\{B_n\}$ is a sequence in $\mathfrak{I}(A,f)$ which $\star$–converges to B, then by Lemma 5.3,

$$\lim I(f,B_n) = \lim I^\star(f,B_n) = I^\star(f,B) \neq \pm\infty .$$

Thus $B \in \mathfrak{I}^\star$ and $F^\star(B) = I^\star(f,B)$. On the other hand, $\mathfrak{I}^\star \subset \mathfrak{I}^\star(A,f)$ by Proposition 5.13. $\square$

5.15. LEMMA. Let $A \in \mathfrak{A}$, and let $\Phi: A \to \mathbb{R}^m$ be a regular map. Then the following statements are true.

(i) If $\{B_n\}$ is a sequence in $\mathfrak{A}(A^-)$ $\star$–converging to a set $B \in \mathfrak{A}(A^-)$, then we have $\{\Phi[B_n]\} \overset{\star}{\to} \Phi[B]$.

(ii) If $\mathfrak{C} \subset \mathfrak{A}(A^-)$, then $cl^\star(\{\Phi[E]: E \in \mathfrak{C}\}) = \{\Phi[E]: E \in cl^\star \mathfrak{C}\}$.

PROOF. (i) By [M, Theorem 50], there is a $c > 0$ such that $\|\Phi(C)\| \le c\,\|C\|$ for each $C \in \mathfrak{A}(A^-)$. Thus

$$0 \le \lim \|\Phi[B] - \Phi[B_n]\| = \lim \|\Phi[B - B_n]\| < c \lim \|B - B_n\| = 0 ,$$

and as $\Phi(B_n) \subset \Phi(B)$, we see that $\{\Phi[B_n]\} \overset{\star}{\to} \Phi[B]$.

(ii) In view of Lemma 5.1, this follows easily from (i). $\square$

The next theorem follows from Lemma 5.15 in the same manner in which Theorem 4.28 was obtained from Lemma 4.27.

5.16. THEOREM. Let $A \in \mathfrak{A}$, and let $\Phi: A \to \mathbb{R}^m$ be a regular map. If $f \in \mathcal{I}^\star(\Phi[A])$, then $f\circ\Phi \cdot |\det \Phi|$ belongs to $\mathcal{I}^\star(A)$ and

$$I^\star(f\circ\Phi \cdot |\det \Phi|, A) = I^\star(f, \Phi[A]) . \quad \square$$

5.17. EXAMPLE. We shall construct a function g in $\mathbb{R}^2$ which is $\star$–integrable, but neither integrable nor R–integrable (cf. Example 4.28).

Given a nondegenerate compact interval $C = [a,b]$, we let

$$f_C = \varphi_{C_+(1)} + \sum_{n=2}^{\infty} n^{-1}\left[\varphi_{C_+(n)} - \varphi_{C_-(n)}\right]$$

where $C_\pm(n)$ and $\varphi_{C_\pm(n)}$ are defined as in Example 4.27. We set

$$F_C(s) = -\int_s^b f_C \, d\lambda$$

if $s \in (a,b)$, $F_C(s) = 0$ if $s \geq b$, and $F_C(s) = -1$ if $s \leq a$. Then F_C is continuous on $\mathbb{R}$, and $F'_C(s) = f_C(s)$ for each $s \in \mathbb{R} - \{a\}$; for f_C is continuous in $\mathbb{R} - \{a\}$ and vanishes outside (a,b). Furthermore,

$$|F_C(s)| \leq -F_C(a + 2^{-1}|C|) = 1$$

for each $s \in \mathbb{R}$.

For $k = 1,2,\dots$, let

$$C_k = [(k+1)^{-1}, k^{-1}] , \qquad f_k = f_{C_{2k-1}} - f_{C_{2k}} , \qquad F_k = F_{C_{2k-1}} - F_{C_{2k}} .$$

Each F_k is continuous in $\mathbb{R}$, vanishes outside $(C_{2k} \cup C_{2k-1})^\circ$, and $|F_k| \le 1$. Moreover, $F_k'(s) = f_k(s)$ for all $s \in \mathbb{R} - \{(2k+1)^{-1}, (2k)^{-1}\}$. If

$$f = \sum_{k=1}^{\infty} f_k \qquad \text{and} \qquad F = \sum_{k=1}^{\infty} F_k ,$$

then F is continuous in $\mathbb{R} - \{0\}$, vanishes outside $(0,1)$, $|F| \le 1$, and $F'(s) = f(s)$ for each $s \in \mathbb{R} - S$ where $S = \{0\} \cup \{1/k : k = 2,3,\dots\}$. The function F is not continuous at 0 , for

$$F\left[\frac{1}{2k} + \frac{1}{4k(2k-1)}\right] = F_{C_{2k-1}}\left[\frac{1}{2k} + \frac{|C_{2k-1}|}{2}\right] = -1 ,$$

$k = 1,2,\dots$, and $F(0) = 0$.

For $(s,t) \in \mathbb{R}^2$, let $g(s,t) = f(s)$ and $v(s,t) = [F(s),0]$. Then v is a bounded vector field in $\mathbb{R}^2$, which is continuous in $\mathbb{R}^2 - (\{0\} \times \mathbb{R})$, and differentiable in $\mathbb{R}^2 - (S \times R)$. Moreover, $g = \operatorname{div} v$ in $\mathbb{R}^2$.

If $A = \{(s,t) \in \mathbb{R}^2 : 0 \le t \le s \le 1\}$, then $A \cap (S \times \mathbb{R})$ is a σ–thin set, and $A \cap (\{0\} \times \mathbb{R}) = \{(0,0)\}$ is a slight set. It follows from Theorem 5.12 that g is

$\star$–integrable in A. An argument analogous to that employed in Example 4.28 shows that g is neither v– nor R–integrable in A. We show next that g is not integrable in A either.

Let $B_k = C_{2k-1} \times [0,(2k)^{-1}]$ and $D_n = \cup_{k=n}^{2n} B_k$. We have

$$|B_k| = \left[\frac{1}{2k-1} - \frac{1}{2k}\right] \cdot \frac{1}{2k} = \frac{1}{4k^2(2k-1)},$$

$$\|B_k\| = 2\left[\frac{1}{2k-1} - \frac{1}{2k}\right] + 2\frac{1}{2k} = \frac{2}{2k-1} \leq 2/k,$$

for $k = 1,2,\ldots$, and so $\lim |D_n| = 0$, and since

$$\ln 2 = \int_n^{2n} t^{-1}\,dt < \sum_{k=n}^{2n} (1/k) < \int_{n-1}^{2n-1} t^{-1}\,dt = \ln \frac{2n-1}{n-1},$$

we see that $\lim \|D_n\| \leq 2 \cdot \ln 2$.

If g is integrable in A, and G is the indefinite integral of g in A, then using Theorem 3.12, a simple calculation shows that

$$G(D_n) = \sum_{k=n}^{2n} G(B_k) = \sum_{k=n}^{2n} \int_{B_k^{\cdot}} v \cdot n_{B_k}\, d\mathscr{H} = \sum_{k=n}^{2n} (1/2k) > (\ln 2)/2$$

for $n = 1,2,\ldots$. Letting $\epsilon = 1/\ln 8$, for each $\delta > 0$, we can find an integer $n \geq 1$ with $|D_n| < \delta$ and $\|D_n\| < 1/\epsilon$. As $G(D_n) > \epsilon$, this contradicts the continuity of G.

NOTE. The previous example is modeled on [MM, Section 34, Example 1]. The added complexity is due to the extra requirement that g must not be R–integrable.

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