Togo Nishiura, Department of Mathematics, Wayne State University,
Detroit, MI 48202

RADIAL CLUSTER SET AND INTERPOLATION

Let D be the open unit disk in the complex plane C and f be an analytic function from D into C. Denote by S the unit circle in the complex plane. For each σ in S, denote the radius from 0 to σ by $R(\sigma)$. That is, $R(\sigma) = \{t\sigma : 0 \leq t < 1\}$. We are interested in the behavior of f restricted to $R(\sigma)$. In general, the limit of $f(t\sigma)$ as t tends to 1 does not exist. But, as a function into the extended complex plane C^* (that is, the Riemann sphere), the radial cluster set $C(f,\sigma)$ does exist, where $C(f,\sigma)$ is the subset of C^* given by

$$C(f,\sigma) = \cap_{1>t\geq 0} \mathrm{Cl}(\{f(\tau\sigma) : t \leq \tau < 1\}),$$

where Cl(E) is the closure in C^* of the subset E of C^*. The continuity of f assures that $C(f,\sigma)$ is a nonempty subcontinuum of C^*. The collection of all nonempty subcontinua of C^* will be denoted by $C(C^*)$. The function f^* defined on S into $C(C^*)$ given by $f^*(\sigma) = C(f,\sigma)$ is called the *radial cluster set function of f*.

Corresponding to the metric on C^*, there is a metric on the collection $C(C^*)$ called the Hausdorff metric. By 1938, it was known that $C(C^*)$ with this metric is a Peano continuum (a connected, locally connected, compact metric space). In 1974, Curtis and Schori ([1] and [2]) proved a long standing conjecture of Wojdyslawski [3] on C(X), where X is a Peano continuum. As a result of this theorem of Curtis and Schori, we now know that $C(C^*)$ is homeomorphic to the Hilbert cube.

In 1987, there appeared a paper by Brinn [4] concerning the radial cluster set function f^* of an analytic function f defined on D. (Actually, her theorem concerned a technical modification of radial cluster sets.) Her theorem asserts that *for each nonempty, nowhere dense, perfect subset* K *of* S *and each nonempty closed subset* A *of* $C(C^*)$ *there is an analytic function* f *on* D *such that* $f^*[K] = A$, f *has an analytic extension to each point of* $S \setminus K$, *and* $\{ \sigma : f^*(\sigma) = p \}$ *is uncountable for each* p *in* A. *Moreover*, *this* f *satisfies the condition that the set-valued function* g *defined on* $S \times [0,1)$ *by* $g(\sigma,t) = Cl(\{f(\tau\sigma) : t \le \tau < 1\})$ $(\sigma \in S,\ 0 \le t < 1)$ *converges to* f^* *uniformly on* K. This theorem was a response to the following 1954 conjecture of Bagemihl and Seidel (stated only for the radial cluster set version) [5].

Conjecture: *Let* K *be a nonempty*, *nowhere dense*, *perfect subset of* S. *For a nonempty subset* A *of* $C(C^*)$, *there is an analytic function* f *on* D *such that its radial cluster set function* f^* *satisfies* $f^*[K] = A$ *and* f *has an analytic extension at each point of* $S \setminus K$ *if and only if* A *is an analytic subset of* $C(C^*)$.

In their paper, Bagemihl and Seidel established the necessity of the condition and proved the sufficiency for the collection of all locally connected subcontinua of C^*.

We consider the above existence statements to be interpolation theorems. Recent developments in analysis and topology permit us to prove the conjecture of Bagemihl and Seidel. We have already mentioned the results of Curtis and Schori. The Borel measurable selection theorems proved in 1985 by Himmelberg, Van Vleck and Prikry [6] and the results of Rogers [7] proved in 1988 concerning the coincidence of the Borel class 1 and the Baire class 1 functions play important roles in the proof. Also, results of dimension theory are used to advantage. We also prove that the analytic function f can be constructed in such a way that the set $\{ \sigma : f^*(\sigma) = p \}$ is uncountable for each p in A.

The above results are derived from research done jointly with Robert D. Berman.

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