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ON A CERTAIN CONVERSE OF HÖLDER'S INEQUALITY FOR LORENTZ SPACES

In this talk I shall discuss some results obtained jointly with T. S. Quek. Our work is motivated by our interest in the Fourier-Stieltjes transforms of measures μ in M(G), the measure algebra defined on a locally compact Abelian group G. Since every non-discrete locally compact Abelian group G that is not σ-compact contains a locally null non-null subset, it is *not* sufficient, for our purposes, to consider only σ-finite measure spaces $(X,\mathcal{A},\mu)$ in our formulation of the converse of Hölder's inequality for Lorentz spaces.

We recall the following definitions.

Definition 1. Let $(X,\mathcal{A},\mu)$ be a measure space. A set $A \in \mathcal{A}$ is said to be a *locally null* set if $A \cap F$ is a null set (i.e., $\mu(A \cap F) = 0$) for every measurable set F with $\mu(F) < \infty$.

Definition 2. Let f be a measurable function defined on a measure space $(X,\mathcal{A},\mu)$. For $y \geq 0$, we define

$$\mu(f,y) = \mu\{x \in X : |f(x)| > y\}.$$

Note that $\mu(f,\cdot)$ is a non-increasing, right-continuous function. For $x \geq 0$, we define

$$f^*(x) = \inf\{y : y > 0 \text{ and } \mu(f,y) \leq x\}$$
$$= \sup\{y : y > 0 \text{ and } \mu(f,y) > x\},$$

with the conventions $\inf \emptyset = \infty$ and $\sup \emptyset = 0$. We note that f^* is a non-increasing, right-continuous function and it is called the *non-increasing rearrangement* of f. For $1 < p < \infty$ and $1 \leq q \leq \infty$, we define

$$\|f\|_{pq} = \begin{cases} \left[\int_o^\infty (f^*(t)t^{1/p})^q \frac{dt}{t} \right]^{1/q} & \text{if } 1 \leq q < \infty; \\ \sup\limits_{t>o} f^*(t)t^{1/p} & \text{if } q = \infty. \end{cases}$$

By the *Lorentz space* $L_{pq}(X)$ we mean the space $\{ f : \|f\|_{pq} < \infty \}$ endowed with the norm $\|\cdot\|_{pq}$.

For $1 < r < \infty$, let r' denote the number such that $1/r + 1/r' = 1$. For $1 < p,\ q < \infty$, let

$$K_{pq} = \begin{cases} 1/p & \text{if } q < p, \\ 1 & \text{if } p = q, \\ 1/p' & \text{if } p < q. \end{cases}$$

<u>Theorem 1</u>. Let $(X,\mathcal{A},\mu)$ be a measure space and let $1 < p,\ q < \infty$.

(a) The following statements are equivalent:

(i) $(X,\mathcal{A},\mu)$ has no locally null non-null sets;

(ii) if f is a measurable function on $(X,\mathcal{A})$ such that

(*) $fg \in L_1(X)$ for every $g \in L_{p'q'}(X)$,

then $f \in L_{pq}(X)$.

(b) If $(X,\mathscr{A},\mu)$ has no locally null non-null sets and f is a measurable function on $(X,\mathscr{A})$ satisfying condition (*) above, then

$$K_p\|f\|_{pq} \le \sup\left|\int_X fgd\mu\right| \le \|f\|_{pq},$$

where the supremum is taken over all measurable functions g such that $\|g\|_{p'q'} \le 1$.

Remark 1. The case $p = q$ in Theorem 1(a) is proved in Leach [1]; part (b) of Theorem 1 is suggested by O'Neil [2, Theorem 6.13], but it should be noted here that O'Neil's theorem fails to hold whenever the measure space $(X,\mathscr{A},\mu)$ contains a locally null non-null set.

Theorem 2. Let $(X,\mathscr{A},\mu)$ be any measure space and let $1 < p,\ q < \infty$. Let f be a measurable function defined on X such that

$$\left|\int_X fg\right| \le C\,\|g\|_{p'q'}$$

for all $g \in L_{p'q'}(X)$, where C is a constant. Then there exists $h \in L_{pq}(X)$ such that $\|h\|_{pq} \le CK_{pq}$ and $h = f$ locally almost everywhere.

The proofs of Theorems 1 and 2 together with some applications will appear in a joint paper with T. S. Quek.

REFERENCES

1. E. B. Leach, On a converse of the Hölder inequality, Proc. Amer. Math. Soc. 7(1956), 607-608.

2. R. O'Neil, Integral transforms and tensor products on Orlicz spaces and L(p,q) spaces, J.D'Anal. Math. 21 (1968), 1-276.